Eighth Edition

Automatic Control Systems

CD INCLUDED

Benjamin C. Kuo
Farid Golnaraghi

Chapter 2 MATHEMATICAL FOUNDATION

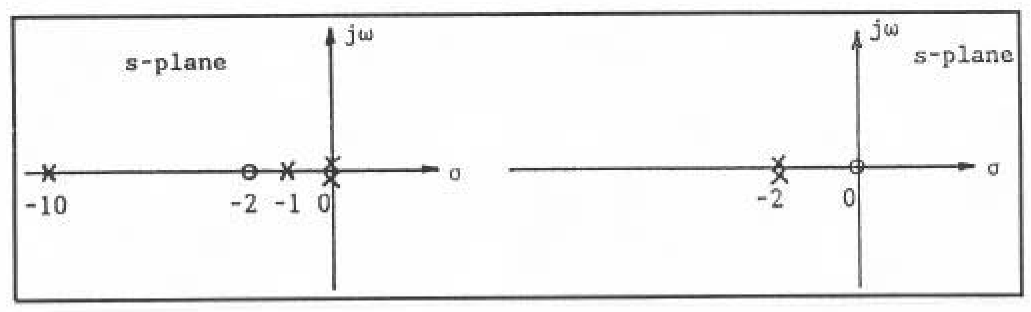
2-1 (a) Poles: $s = 0, 0, -1, -10$;

Zeros: $s = -2, \infty, \infty, \infty$.

(b) Poles: $s = -2, -2$;

Zeros: $s = 0$.

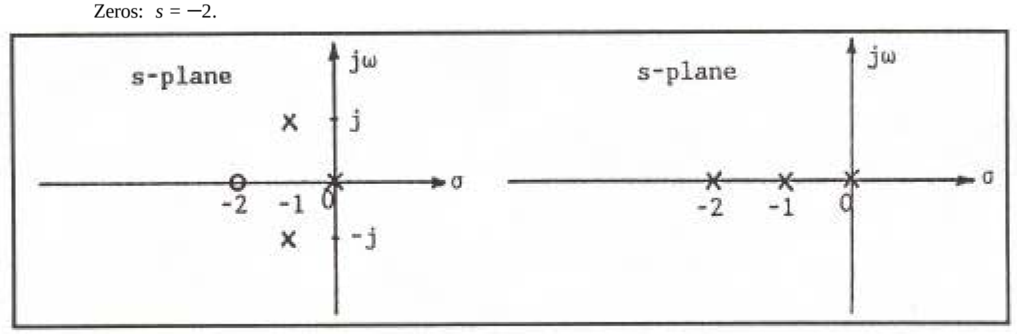
The pole and zero at $s = -1$ cancel each other.



(c) Poles: $s = 0, -1 + j, -1 - j$;

Zeros: $s = -2$.

(d) Poles: $s = 0, -1, -2, \infty$.



2-2 (a)

$$G(s) = \frac{5}{(s+5)^2}$$

(b)

$$G(s) = \frac{4s}{(s^2+4)} + \frac{1}{s+2}$$

(c)

$$G(s) = \frac{4}{s^2 + 4s + 8}$$

(d)

$$G(s) = \frac{1}{s^2 + 4}$$

(e)

$$G(s) = \sum_{k=0}^{\infty} e^{kT(s+5)} = \frac{1}{1 - e^{-T(s+5)}}$$

2-3 (a)

$$g(t) = u_s(t) - 2u_s(t-1) + 2u_s(t-2) - 2u_s(t-3) + \cdots$$

$$G(s) = \frac{1}{s}\left(1 - 2e^{-s} + 2e^{-2s} - 2e^{-3s} + \cdots\right) = \frac{1 - e^{-s}}{s\left(1 + e^{-s}\right)}$$

$$g_T(t) = u_s(t) - 2u_s(t-1) + u_s(t-2) \qquad 0 \le t \le 2$$

$$G_T(s) = \frac{1}{s}\left(1 - 2e^{-s} + e^{-2s}\right) = \frac{1}{s}\left(1 - e^{-s}\right)^2$$

$$g(t) = \sum_{k=0}^{\infty} g_T(t-2k)u_s(t-2k) \qquad G(s) = \sum_{k=0}^{\infty} \frac{1}{s}(1-e^{-s})^2 e^{-2ks} = \frac{1-e^{-s}}{s(1+e^{-s})}$$

(b)

$$g(t) = 2tu_s(t) - 4(t-0.5)u_s(t-0.5) + 4(t-1)u_s(t-1) - 4(t-1.5)u_s(t-1.5) + \cdots$$

$$G(s) = \frac{2}{s^2}\left(1 - 2e^{-0.5s} + 2e^{-s} - 2e^{-1.5s} + \cdots\right) = \frac{2\left(1-e^{-0.5s}\right)}{s^2\left(1+e^{-0.5s}\right)}$$

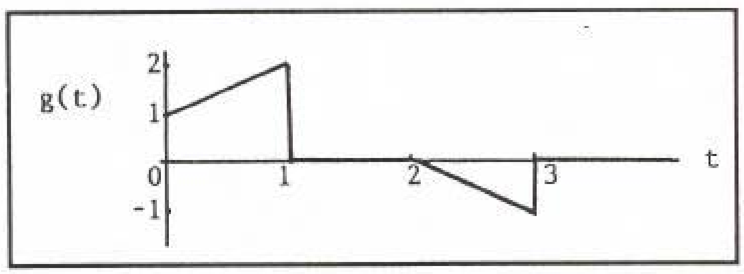
$$g_T(t) = 2tu_s(t) - 4(t-0.5)u_s(t-0.5) + 2(t-1)u_s(t-1) \qquad 0 \le t \le 1$$

$$G_T(s) = \frac{2}{s^2}\left(1 - 2e^{-0.5s} + e^{-s}\right) = \frac{2}{s^2}\left(1 - e^{-0.5s}\right)^2$$

$$g(t) = \sum_{k=0}^{\infty} g_T(t-k)u_s(t-k) \qquad G(s) = \sum_{k=0}^{\infty} \frac{2}{s^2}\left(1-e^{-0.5s}\right)^2 e^{-ks} = \frac{2\left(1-e^{-0.5s}\right)}{s^2\left(1+e^{-0.5s}\right)}$$

2-4

$$g(t) = (t+1)u_s(t) - (t-1)u_s(t-1) - 2u_s(t-1) - (t-2)u_s(t-2) + (t-3)u_s(t-3) + u_s(t-3)$$



$$G(s) = \frac{1}{s^2}\left(1 - e^{-s} - e^{-2s} + e^{-3s}\right) + \frac{1}{s}\left(1 - 2e^{-s} + e^{-3s}\right)$$

2-5 **(a)** Taking the Laplace transform of the differential equation, we get

$$(s^2 + 5s + 4)F(s) = \frac{1}{s+2} \qquad F(s) = \frac{1}{(s+1)(s+2)(s+4)} = \frac{1}{6(s+4)} + \frac{1}{3(s+1)} - \frac{1}{2(s+2)}$$

$$f(t) = \frac{1}{6}e^{-4t} + \frac{1}{3}e^{-t} - \frac{1}{2}e^{-2t} \qquad t \ge 0$$

(b) $sX_1(s) - x_1(0) = X_2(s) \qquad x_1(0) = 1 \qquad sX_2(s) - x_2(0) = -2X_1(s) - 3X_2(s) + \frac{1}{s} \qquad x_2(0) = 0$

Solving for $X_1(s)$ and $X_2(s)$, we have

$$X_1(s) = \frac{s^2 + 3s + 1}{s(s+1)(s+2)} = \frac{1}{2s} + \frac{1}{s+1} - \frac{1}{2(s+2)}$$

$$X_2(s) = \frac{-1}{(s+1)(s+2)} = \frac{-1}{s+1} + \frac{1}{s+2}$$

Taking the inverse Laplace transform on both sides of the last equation, we get

$$x_1(t) = 0.5 + e^{-t} - 0.5e^{-2t} \qquad t \ge 0 \qquad x_2(t) = -e^{-t} + e^{-2t} \qquad t \ge 0$$

2-6 (a)

$$G(s) = \frac{1}{3s} - \frac{1}{2(s+2)} + \frac{1}{3(s+3)} \qquad g(t) = \frac{1}{3} - \frac{1}{2}e^{-2t} + \frac{1}{3}e^{-3t} \qquad t \geq 0$$

(b)

$$G(s) = \frac{-2.5}{s+1} + \frac{5}{(s+1)^2} + \frac{2.5}{s+3} \qquad g(t) = -2.5e^{-t} + 5te^{-t} + 2.5e^{-3t} \qquad t \geq 0$$

(c)

$$G(s) = \left(\frac{50}{s} - \frac{20}{s+1} - \frac{30s+20}{s^2+4}\right)e^{-s} \qquad g(t) = \left[50 - 20e^{-(t-1)} - 30\cos 2(t-1) - 5\sin 2(t-1)\right]u_s(t-1)$$

(d)

$$G(s) = \frac{1}{s} - \frac{s-1}{s^2+s+2} = \frac{1}{s} + \frac{1}{s^2+s+2} - \frac{s}{s^2+s+2}$$ Taking the inverse Laplace transform,

$$g(t) = 1 + 1.069e^{-0.5t}\left[\sin 1.323t + \sin\left(1.323t - 69.3^{o}\right)\right] = 1 + e^{-0.5t}\left(1.447\sin 1.323t - \cos 1.323t\right) \quad t \geq 0$$

(e) $g(t) = 0.5t^2e^{-t} \qquad t \geq 0$

2-7

$$\mathbf{A} = \begin{bmatrix} -1 & 2 & 0 \\ 0 & -2 & 3 \\ -1 & -3 & -1 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \mathbf{u}(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$

2-8 **(a)**

$$\frac{Y(s)}{R(s)} = \frac{3s+1}{s^3+2s^2+5s+6}$$

(b)

$$\frac{Y(s)}{R(s)} = \frac{5}{s^4+10s^2+s+5}$$

(c)

$$\frac{Y(s)}{R(s)} = \frac{s(s+2)}{s^4+10s^3+2s^2+s+2}$$

(d)

$$\frac{Y(s)}{R(s)} = \frac{1+2e^{-s}}{2s^2+s+5}$$

Chapter 3 TRANSFER FUNCTION, BLOCK DIAGRAM, AND SIGNAL FLOW GRAPHS

3-1 **(a) Controller transfer function:**

$$\frac{F(s)}{sE_c(s)}=\frac{100}{s}-\frac{30}{s+6}-\frac{70}{s+10}=\frac{880(s+6.818)}{s(s+6)(s+10)} \qquad G_c(s)=\frac{F(s)}{E_c(s)}=\frac{880(s+6.818)}{(s+6)(s+10)}$$

(b) Open-loop transfer function:

$$\frac{V(s)}{E(s)}=\frac{K}{Ms}G_c(s)=\frac{880K(s+6.818)}{30000s(s+6)(s+10)}=\frac{0.0293K(s+6.818)}{s(s+6)(s+10)}$$

(c) System transfer function:

$$\frac{V(s)}{E_r(s)}=\frac{KG_c(s)/Ms}{1+KK_tG_c(s)/Ms}=\frac{KG_c(s)}{Ms+KK_tG_c(s)}=\frac{0.0293K(s+6.818)}{s^3+16s^2+(0.0044K+60)s+0.03K}$$

(d) Steady-state speed: $E_r=1\text{V},\ E_r(s)=E_r/s=1/s$

$$\lim_{t\to\infty}v(t)=\lim_{s\to 0}sV(s)=\lim_{s\to 0}\frac{0.0293K(s+6.818)}{s^3+16s^2+(0.0044K+60)s+0.03K}=6.66\ \text{ft/sec}$$

3-2 **(a) Controller transfer function:**

$$\frac{F(s)}{sE_c(s)}=\left(\frac{100}{s}-\frac{30}{s+6}\right)e^{-0.5s}=\frac{70(s+8.5714)}{s(s+6)}e^{-0.5s}$$

$$G_c(s)=\frac{F(s)}{E_c(s)}=\frac{70(s+8.5714)}{s+6}e^{-0.5s}$$

(b) Open-loop transfer function:

$$\frac{V(s)}{E_r(s)}=\frac{K}{Ms}G_c(s)=\frac{70K(s+8.5714)}{30000s(s+6)}e^{-0.5s}=\frac{0.002333K(s+8.5714)}{s(s+6)}e^{-0.5s}$$

(c) System transfer function:

$$\frac{V(s)}{E_r(s)}=\frac{KG_c(s)/Ms}{1+KK_tG_c(s)/Ms}=\frac{0.002333K(s+8.5714)e^{-0.5s}}{s^2+6s+0.00035K(s+8.5714)e^{-0.5s}}$$

(d) Steady-state speed: $E_r=1\text{V},\ E_r(s)=E_r/s=1/s$

$$\lim_{t\to\infty}v(t)=\lim_{s\to 0}sV(s)=\lim_{s\to 0}\frac{0.002333K(s+8.5714)e^{-0.5s}}{s^2+6s+0.00035K(s+8.5714)e^{-0.5s}}=6.66\ \text{ft/sec}$$

3-3 Taking the Laplace transform of the differential equations and expressing in matrix form, the following matrix equations are obtained. All the initial conditions are set to zero.

$$\begin{bmatrix} s(s+2) & 3 \\ 3s+1 & s^2-1\end{bmatrix}\begin{bmatrix} Y_1(s) \\ Y_2(s)\end{bmatrix}=\begin{bmatrix} 1 & 1 \\ s & 1\end{bmatrix}\begin{bmatrix} R_1(s) \\ R_2(s)\end{bmatrix} \qquad \begin{bmatrix} Y_1(s) \\ Y_2(s)\end{bmatrix}=\frac{1}{\Delta}\begin{bmatrix} s^2-3s-1 & s^2-4 \\ s^3+2s^2-3s-1 & s^2-s-1\end{bmatrix}\begin{bmatrix} R_1(s) \\ R_2(s)\end{bmatrix}$$

$$\Delta(s)=s^4+2s^3-s^2-11s-3$$

$$\left.\frac{Y_1(s)}{R_1(s)}\right|_{R_2=0}=\frac{s^2-3s-1}{\Delta} \qquad \left.\frac{Y_2(s)}{R_1(s)}\right|_{R_2=0}=\frac{s^3+2s^2-3s-1}{\Delta} \qquad \left.\frac{Y_1(s)}{R_2(s)}\right|_{R_1=0}=\frac{s^2-4}{\Delta} \qquad \left.\frac{Y_2(s)}{R_2(s)}\right|_{R_1=0}=\frac{s^2-s-1}{\Delta}$$

3-4

$$\mathbf{Y}(s) = \left[\mathbf{I} + \mathbf{G}(s)\mathbf{H}(s)\right]^{-1}\mathbf{G}(s)\mathbf{R}(s) = \mathbf{M}(s)\mathbf{R}(s)$$

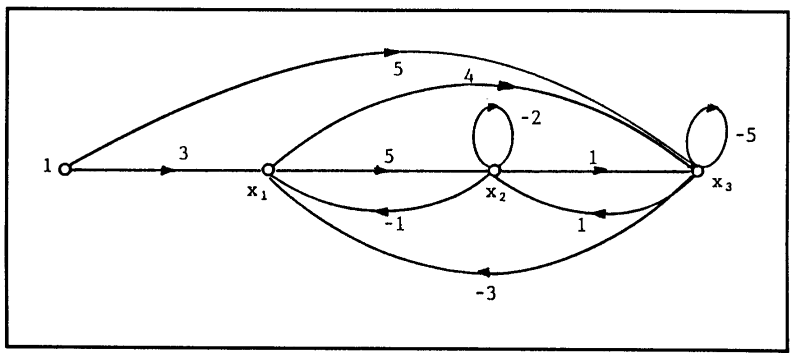
$$\mathbf{I} + \mathbf{G}(s)\mathbf{H}(s) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} \dfrac{2}{s(s+2)} & 10 \\ \dfrac{5}{s} & \dfrac{1}{s+1} \end{bmatrix} = \begin{bmatrix} \dfrac{s^2+2s+2}{s(s+2)} & 10 \\ \dfrac{5}{s} & \dfrac{s+2}{s+1} \end{bmatrix}$$

$$\left[\mathbf{I} + \mathbf{G}(s)\mathbf{H}(s)\right]^{-1} = \frac{1}{\Delta}\begin{bmatrix} \dfrac{s+2}{s+1} & -10 \\ \dfrac{-5}{s} & \dfrac{s^2+2s+2}{s(s+2)} \end{bmatrix} \qquad \Delta(s) = \frac{s^2-48s-48}{s(s+1)}$$

$$\mathbf{M}(s) = \left[\mathbf{I} + \mathbf{G}(s)\mathbf{H}(s)\right]^{-1}\mathbf{G}(s) = \frac{1}{\Delta}\begin{bmatrix} \dfrac{s+2}{s+1} & -10 \\ \dfrac{-5}{s} & \dfrac{s^2+2s+2}{s(s+2)} \end{bmatrix}\begin{bmatrix} \dfrac{2}{s(s+2)} & 10 \\ \dfrac{5}{s} & \dfrac{1}{s+1} \end{bmatrix}$$

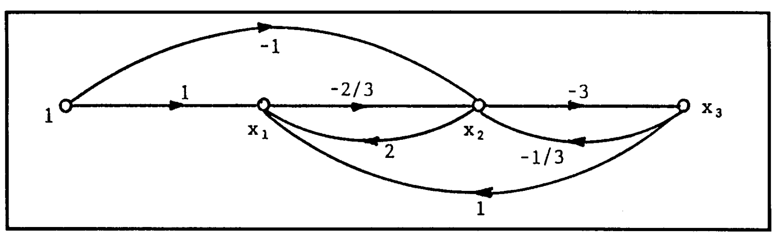
$$\mathbf{M}(s) = \frac{1}{\Delta}\begin{bmatrix} \dfrac{-50s-48}{s(s+1)} & 10 \\ \dfrac{5}{s} & \dfrac{-49s^2-148s-98}{s(s+1)(s+2)} \end{bmatrix}$$

3-5 (a)

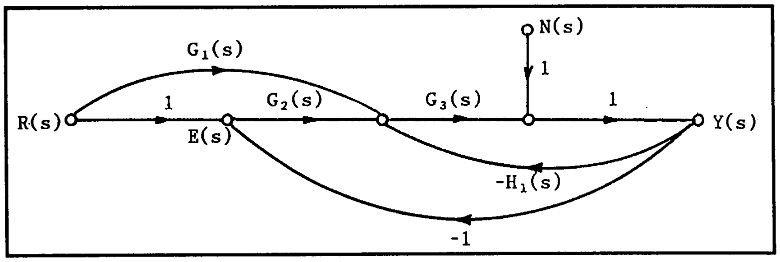


(b) Rewrite the equations as (This is not unique):

$x_1 = 2x_2 + x_3 + 1 \qquad x_2 = (-2/3)x_1 - (1/3)x_3 - 1 \qquad x_3 = -3x_2$



3-6



$$\left.\frac{Y(s)}{R(s)}\right|_{N=0} = \frac{G_1(s)G_3(s)+G_2(s)G_3(s)}{\Delta} \qquad \left.\frac{Y(s)}{N(s)}\right|_{R=0} = \frac{1}{\Delta} \qquad \left.\frac{E(s)}{N(s)}\right|_{R=0} = \frac{-1}{\Delta}$$

$$\left.\frac{E(s)}{R(s)}\right|_{N=0} = \frac{1+G_3(s)H_1(s)-G_1(s)G_3(s)}{\Delta} \qquad \Delta = 1+G_2(s)G_3(s)+G_3(s)H_1(s)$$

3-7 **(a)**

$$\frac{Y_5}{Y_1} = \frac{G_1G_2G_3+G_3G_4}{\Delta} \qquad \frac{Y_2}{Y_1} = \frac{1+G_3H_2}{\Delta} \qquad \frac{Y_5}{Y_2} = \frac{Y_5/Y_1}{Y_2/Y_1} = \frac{G_1G_2G_3+G_3G_4}{1+G_3H_2}$$

$$\Delta = 1+G_1H_1+G_3H_2+G_3G_4H_3+G_1G_2G_3H_3+G_1G_3H_1H_2$$

(b)

$$\frac{Y_5}{Y_1} = \frac{G_1G_2G_3+G_3G_4}{\Delta} \qquad \frac{Y_2}{Y_1} = \frac{1+G_3H_2+H_4}{\Delta} \qquad \frac{Y_5}{Y_2} = \frac{Y_5/Y_1}{Y_2/Y_1} = \frac{G_1G_2G_3+G_3G_4}{1+G_3H_2+H_4}$$

$$\Delta = 1+G_1H_1+G_3H_2+G_3G_4H_3+G_1G_2G_3H_3+H_4+G_1G_3H_1H_2+G_1H_1H_4$$

(c)

$$\frac{Y_5}{Y_1} = \frac{G_1G_2G_3+G_4}{\Delta} \qquad \frac{Y_2}{Y_1} = \frac{1+G_2G_3H_3}{\Delta} \qquad \frac{Y_5}{Y_2} = \frac{Y_5/Y_1}{Y_2/Y_1} = \frac{G_1G_2G_3+G_3G_4}{1+G_2G_3H_3}$$

$$\Delta = 1+G_1H_1+G_2G_3H_3+G_1G_2H_2-G_2G_4H_2H_3$$

(d)

$$\frac{Y_5}{Y_1} = \frac{G_3G_4+G_1G_2G_3}{\Delta} \qquad \frac{Y_2}{Y_1} = \frac{1+G_2H_2}{\Delta} \qquad \frac{Y_5}{Y_2} = \frac{Y_5/Y_1}{Y_2/Y_1} = \frac{G_3G_4+G_1G_2G_3}{1+G_2H_2}$$

$$\Delta = 1+G_1H_1+G_2H_2+G_3G_4H_3+G_1G_2G_3H_3-G_4H_1H_2$$

(e)

$$\frac{Y_5}{Y_1} = \frac{G_1G_2G_3(1+H_4)+G_4G_5(1+G_2H_1)}{\Delta} \qquad \frac{Y_2}{Y_1} = \frac{1+G_2H_1+G_3H_2+H_4+G_2H_1H_4+G_3H_2H_4}{\Delta}$$

$$\frac{Y_5}{Y^2} = \frac{Y_5/Y_1}{Y_2/Y_1} = \frac{G_1G_2G_3(1+H_4)+G_4G_5(1+G_2H_1)}{1+G_2H_1+G_3H_2+H_4+G_2H_1H_4+G_3H_2H_4}$$

$$\Delta = 1+G_2H_1+G_3H_2+H_4+G_4G_5H_3+G_1G_2G_3H_3+G_2H_1H_4+G_3H_2H_4+G_1G_2G_3H_3H_4+G_2G_4H_1H_3$$

3-8 **(a)**

$$\frac{Y_7}{Y_1} = \frac{G_1G_2G_3G_4G_5 + G_5G_6(1+G_2H_2+G_3H_3)}{\Delta}$$

$$\frac{Y_2}{Y_1} = \frac{1+G_2H_2+G_3H_3+G_4G_5H_4+H_6+G_2G_3G_4G_5H_5+G_2H_2G_4G_5H_4+G_2H_2H_6+G_2H_3H_6}{\Delta}$$

$$\Delta = 1+G_1H_1+G_2H_2+G_3H_3+G_4G_5H_4+H_6+G_2G_3G_4G_5H_5-G_5G_6H_1H_5-G_5G_6H_1H_2H_3H_4+G_1G_3H_1H_3$$
$$+G_1G_4G_5H_1H_4+G_1H_1H_6+G_2G_4G_5H_2H_4+G_2H_2H_6+G_3H_3H_6-G_3G_5G_6H_1H_3H_5+G_1G_3H_1H_3H_6$$

(b)

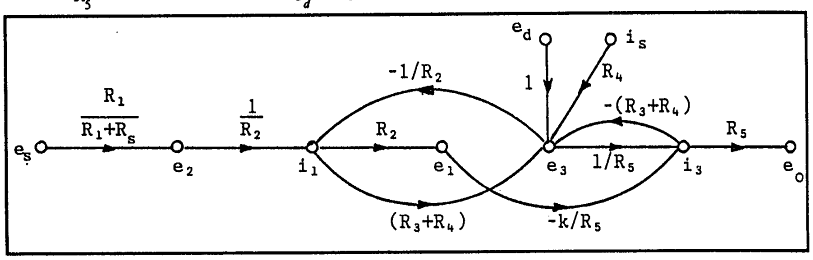
$$\frac{Y_7}{Y_1} = \frac{G_1G_2G_3G_4G_5 + G_6\left(1+G_3H_2+G_4H_3\right)}{\Delta} \qquad \frac{Y_2}{Y_1} = \frac{1+G_3H_2+G_4H_3+G_2G_3G_4G_5H_4}{\Delta}$$

$$\Delta = 1+G_1G_2H_1+G_3H_2+G_4H_3+G_2G_3G_4G_5H_4-G_2G_6H_1H_4+G_1G_2G_4H_1H_3-G_2G_4G_6H_1H_3H_4$$

3-9

$$e_2 = \frac{R_1}{R_1+R_s}e_s \qquad i_1 = \frac{e_2-e_3}{R_2} \qquad e_1 = R_2 i_1 \qquad e_3 = e_d + R_3(i_1-i_3)+(i_s+i_1-i_3)R_4$$

$$i_3 = \frac{e_3 - ke_1}{R_5} \qquad e_o = R_5 i_3 \qquad \frac{e_o}{e_d} = \frac{1+k}{\Delta} = 0 \qquad k = -1$$



3-10 **(a)**

$$\frac{Y_3}{Y_1} = \frac{G}{1+GH}$$

(b)

$$\frac{Y_3}{Y_1} = \frac{G}{1+GH}$$

3-11 **(a)** The three loops are not in touch.

$$\Delta = 1+G_1H_1+G_2H_2+G_3H_3+G_1G_2H_1H_2+G_2G_3H_2H_3+G_1G_3H_1H_3+G_1G_2G_3H_1H_2H_3$$

(b) The three loops are in touch. $\Delta = 1+G_1H_1+G_2H_2+G_3H_3+G_1G_3H_1H_3$

3-12 **(a)**

$$\left.\frac{Y_6}{Y_1}\right|_{Y_7=0} = \frac{G_1G_2G_3G_4+G_3G_4G_5}{\Delta} \qquad \left.\frac{Y_6}{Y_7}\right|_{Y_1=0} = \frac{1+G_2H_1}{\Delta}$$

$$\Delta = 1+G_2H_1+G_4H_2+G_1G_2G_3G_4H_3+G_3G_4G_5H_3+G_2G_4H_1H_2$$

(b)

$$\left.\frac{Y_6}{Y_1}\right|_{Y_7=0} = \frac{G_1G_2G_3G_4 + G_3G_4G_5}{\Delta} \qquad \left.\frac{Y_6}{Y_7}\right|_{Y_1=0} = \frac{1+G_1H_1+G_3H_2+G_1G_3H_1H_2}{\Delta}$$

$$\Delta = 1+G_1H_1+G_3H_2+G_3G_4G_5H_4+G_1G_2G_3G_4H_4+G_1G_3H_1H_2+G_1G_3G_4H_1H_3$$

3-13 **(a)**

$$\left.\frac{Y_7}{Y_1}\right|_{Y_8=0} = \frac{G_1G_2G_3G_4G_5 + G_3G_4G_5G_6}{\Delta}$$

$$\Delta = 1+G_2H_1+G_5H_2+G_1G_2G_3G_4G_5H_3+G_3G_4G_5G_6H_3+G_2G_5H_1H_2$$

(b)

$$\left.\frac{Y_7}{Y_8}\right|_{Y_1=0} = \frac{G_4G_5(1+G_2H_1)}{\Delta}$$

(c)

$$\left.\frac{Y_7}{Y_4}\right|_{Y_8=0} = \left.\frac{Y_7/Y_1}{Y_4/Y_1}\right|_{Y_8=0} = \frac{G_1G_2G_3G_4G_5+G_3G_4G_5G_6}{(G_1G_2+G_6)(1+G_5H_2)}$$

(d)

$$\left.\frac{Y_7}{Y_4}\right|_{Y_1=0} = \left.\frac{Y_7/Y_8}{Y_4/Y_8}\right|_{Y_1=0} = \frac{-G_4G_5(1+G_2H_1)}{G_4G_5H_3(G_6+G_1G_2)}$$

The results in (c) and (d) are different because different inputs are used.

3-14 **(a)**

$$\left.\frac{Y(s)}{R(s)}\right|_{N=0} = \frac{\dfrac{10(s+4)}{s(s+1)}}{1+\dfrac{5s}{s(s+1)}+\dfrac{10(s+2)}{s(s+1)}} = \frac{10(s+4)}{s^2+16s+20}$$

(b)

$$\left.\frac{Y(s)}{E(s)}\right|_{N=0} = \left.\frac{Y(s)/R(s)}{E(s)/R(s)}\right|_{N=0} = \frac{\dfrac{10(s+4)}{s(s+1)}}{1+\dfrac{5s}{s(s+1)}-\dfrac{20}{s(s+1)}} = \frac{10(s+4)}{s^2+6s-20}$$

(c)

$$\left.\frac{Y(s)}{N(s)}\right|_{R=0} = \frac{1}{1+\dfrac{5s}{s(s+1)}+\dfrac{10(s+2)}{s(s+1)}} = \frac{s(s+1)}{s^2+16s+20}$$

(d)

$$Y(s) = \left.\frac{Y(s)}{R(s)}\right|_{N=0} R(s) + \left.\frac{Y(s)}{N(s)}\right|_{R=0} N(s)$$

3-15 **(a)**

$$\left.\frac{Y(s)}{R(s)}\right|_{N=0} = \frac{G_1(s)G_2(s)G_3(s)+G_4(s)}{\Delta} \qquad \left.\frac{Y(s)}{N(s)}\right|_{R=0} = \frac{1+G_1(s)G_2(s)H_1(s)}{\Delta}$$

$$\Delta = 1+G_1(s)G_2(s)H_1(s)+G_2(s)G_3(s)H_2(s)+G_4(s)-G_2(s)G_4(s)H_1(s)H_2(s)$$

$$Y(s) = \left.\frac{Y(s)}{R(s)}\right|_{N=0} R(s) + \left.\frac{Y(s)}{N(s)}\right|_{R=0} N(s)$$

(b) When $1+G_1(s)G_2(s)H_1(s)=0$ $Y(s)$ is not affected by $N(s)$.

3-16

Set $$\left.\frac{Y(s)}{N(s)}\right|_{R=0} = \frac{1-\dfrac{10(s+5)}{s(s+5)(s+10)}G_d(s)}{\Delta} = 0 \qquad \text{Then, } G_d(s) = \frac{s(s+10)}{10}$$

3-17 **(a)**

$$\left.\frac{Y(s)}{N(s)}\right|_{R=0} = \frac{1+G(s)H(s)}{\Delta} = 0 \qquad H(s) = \frac{-1}{G(s)} = -\frac{s(s+1)(s+2)}{K(s+3)}$$

(b)

$N=0.$ $$E(s) = \frac{R(s)}{1+G(s)+G(s)H(s)} = \frac{R(s)}{G(s)} = \frac{s(s+1)(s+2)}{K(s+3)}R(s) \qquad R(s)=\frac{1}{s^2}$$

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0}\frac{s(s+1)(s+2)}{Ks(s+3)} = \frac{2}{3K} = 0.1 \qquad K = 6.67$$

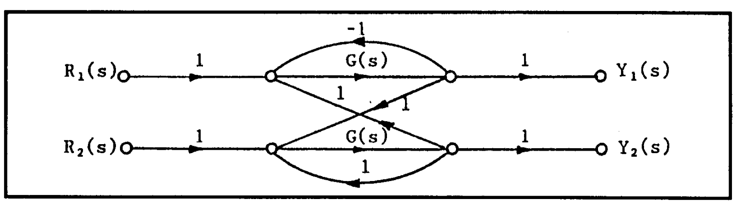
3-18 **(a) Open-loop transfer function:**

$$\frac{\Theta_o(s)}{\Theta_e(s)} = \frac{KK_sK_1K_iN}{s\left[L_aJ_ts^2+\left(L_aB_t+R_aJ_t+K_1K_2J_t\right)s+R_aB_t+K_iK_b+KK_1K_iK_t+K_1K_2B_t\right]}$$

(b) System transfer function:

$$\frac{\Theta_o(s)}{\Theta_r(s)} = \frac{KK_sK_1K_iN}{\left[L_aJ_ts^3+\left(L_aB_t+R_aJ_t+K_1K_2J_t\right)s^2+\left(R_aB_t+K_iK_b+KK_1K_iK_t+K_1K_2B_t\right)s+KK_sK_1K_iN\right]}$$

3-19 **(a) Equivalent SFG:**



(b) $\Delta = 1-2[G(s)]^2$

(c)

$$\left.\frac{Y_1(s)}{R_1(s)}\right|_{R_2=0} = \frac{G(s)[1-G(s)]}{\Delta} \qquad \left.\frac{Y_1(s)}{R_2(s)}\right|_{R_1=0} = \frac{[G(s)]^2}{\Delta}$$

$$\left.\frac{Y_2(s)}{R_1(s)}\right|_{R_2=0} = \frac{[G(s)]^2}{\Delta} \qquad \left.\frac{Y_2(s)}{R_2(s)}\right|_{R_1=0} = \frac{G(s)[1+G(s)]}{\Delta}$$

(d) Transfer function in matrix form: $\mathbf{Y}(s) = \mathbf{G}(s)\mathbf{R}(s)$

$$\mathbf{G}(s) = \frac{1}{\Delta}\begin{bmatrix} G(s)[1-G(s)] & [G(s)]^2 \\ [G(s)]^2 & G(s)[1+G(s)] \end{bmatrix}$$

3-20 (a)

$$\left.\frac{Y(s)}{R(s)}\right|_{N=0} = \frac{G_p(s)\left[1+G_c(s)H(s)\right]}{1+G_p(s)H(s)} \qquad \left.\frac{Y(s)}{N(s)}\right|_{R=0} = \frac{G_p(s)}{1+G_p(s)H(s)}$$

When $G_c(s) = G_p(s)$ $\quad \left.\dfrac{Y(s)}{R(s)}\right|_{N=0} = G_p(s)$

(b)

$$G_p(s) = G_c(s) = \frac{100}{(s+1)(s+5)} \qquad \left.\frac{Y(s)}{R(s)}\right|_{N=0} = G_p(s) = \frac{100}{(s+1)(s+5)}$$

$$R(s) = \frac{1}{s} \qquad Y(s) = \frac{100}{s(s+1)(s+5)} = \frac{20}{s} - \frac{25}{s+1} + \frac{5}{s+5} \qquad y(t) = \left(20 - 25e^{-t} + 5e^{-5t}\right)u_s(t)$$

(c)

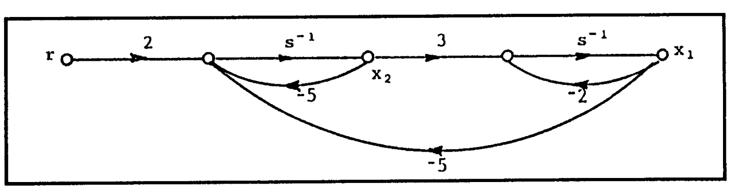
$$\left.\frac{Y(s)}{N(s)}\right|_{R=0} = \frac{G_p(s)}{1+G_p(s)H(s)} = \frac{100}{(s+1)(s+5)+100H(s)} \qquad N(s) = \frac{1}{s} \qquad G(s)\big|_{R=0} = \frac{100}{s(s+1)(s+5)+100s}$$

$H(s)$ must have a pole at $s = 0$, but the system must be stable.

$$H(s) = \frac{K}{s}. \qquad \Delta = s(s+1)(s+5) + 100K$$

K must be selected so that the system is stable.

3-21 (a) State diagram:



(b) **Characteristic equation:** $\Delta = 1+2s^{-1}+5s^{-1}+15s^{-1}+10s^{-2} = 0 \qquad s^2+7s+25=0$

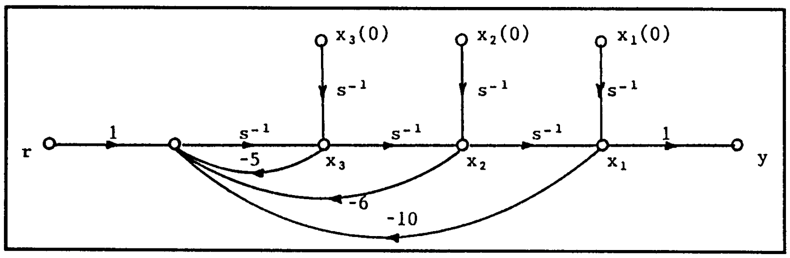
(c) **Transfer functions:**

$$\frac{X_1(s)}{R(s)} = \frac{6s^{-2}}{\Delta} = \frac{6}{s^2+7s+25} \qquad \frac{X_2(s)}{R(s)} = \frac{2s^{-1}(1+2s^{-1})}{\Delta} = \frac{2(s+2)}{s^2+7s+25}$$

3-22 **(a)** Write the differential equation as

$$\frac{d^3y(t)}{dt^3} = r(t) - 5\frac{d^2y(t)}{dt^2} - 6\frac{dy(t)}{dt} - 10y(t)$$

State diagram:



(b) **State equations:**

$$\frac{dx_1(t)}{dt} = x_2(t) \qquad \frac{dx_2(t)}{dt} = x_3(t) \qquad \frac{dx_3(t)}{dt} = -10x_1(t) - 6x_2(t) - 5x_3(t) + r(t)$$

(c) **Characteristic equation:**

$$\Delta = 1+5s^{-1}+6s^{-2}+10s^{-3} = 0 \qquad s^3+5s^2+6s+10=0$$

Characteristic equation roots:

$-4.1337, \qquad -0.43313+j1.4938, \qquad -0.43313-j1.4938$

(d) **Transfer function:**

$$\frac{Y(s)}{R(s)} = \frac{s^{-3}}{1+5s^{-1}+6s^{-2}+10s^{-3}} = \frac{1}{s^3+5s^2+6s+10}$$

(e) $R(s) = 1/s$.

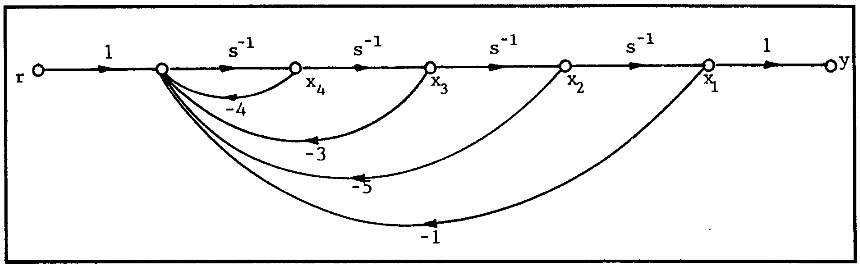
$$Y(s) = \frac{1}{s(s^3+5s^2+6s+10)} = \frac{0.1}{s} - \frac{0.01519}{s+4.1337} - \frac{0.08481(s+0.4331)}{(s+0.4331)^2+2.232} - \frac{0.09953}{(s+0.4331)^2+2.232}$$

$$y(t) = \left[0.1 - 0.01519e^{-4.1337t} - 0.08481e^{-0.4331t}\cos(1.494t) - 0.06662e^{-0.4331t}\sin(1.494t)\right]u_s(t)$$

3-23 **(a)** Write the differential equation as

$$\frac{d^4y(t)}{dt^4} = r(t) - 4\frac{d^3y(t)}{dt^3} - 3\frac{d^2y(t)}{dt^2} - 5\frac{dy(t)}{dt} - y(t)$$

State diagram:



(b) State equations:

$$\frac{dx_1(t)}{dt} = x_2(t) \qquad \frac{dx_2(t)}{dt} = x_3(t) \qquad \frac{dx_3(t)}{dt} = x_4(t) \qquad \frac{dx_4(t)}{dt} = -x_1(t) - 5x_2(t) - 3x_3(t) - 4x_4(t) + r(t)$$

(c) Characteristic equation:

$$s^4 + 4s^3 + 3s^2 + 5s + 1 = 0$$

Characteristic equation roots:

$$-3.5286, \qquad -0.2212, \qquad -0.1251 + j1.125 \qquad -0.1251 - j1.125$$

(d) Transfer function:

$$\frac{Y(s)}{R(s)} = \frac{1}{s^4 + 4s^3 + 3s^2 + 5s + 1}$$

(e) $R(s)=1/s$.

$$Y(s) = \frac{1}{s(s^4 + 4s^3 + 3s^2 + 5s + 1)} = \frac{1}{s} - \frac{1.072}{s + 0.2212} + \frac{0.006668}{s + 3.5286} + \frac{0.06558(s + 0.1251)}{(s + 0.1251)^2 + 1.2656} - \frac{0.2054}{(s + 0.1251)^2 + 1.2656}$$

$$y(t) = [1 - 1.072e^{-0.2212t} + 0.006668e^{-3.5286t} + 0.06558e^{-0.125t}\cos(1.125t) - 0.1826e^{-0.125t}\sin(1.125t)]u_s(t)$$

3-24 **(a)**

$$\left.\frac{Y(s)}{R(s)}\right|_{N=0} = \frac{G_1G_2G_3}{1 + G_1G_2H_1 + G_2G_3H_2 + G_1G_2G_3} = \frac{100(s+1)}{101s^3 + 2122s^2 + 3050s + 1010}$$

$$\left.\frac{Y(s)}{N(s)}\right|_{R=0} = \frac{(1 + G_1G_2H_1) - G_2G_3G_4}{1 + G_1G_2H_1 + G_2G_3H_2 + G_1G_2G_3} = \frac{(101s^3 + 2122s^2 + 2040s) - 10(s+1)G_4}{101s^3 + 2122s^2 + 3050s + 1010}$$

$$\left.\frac{E(s)}{R(s)}\right|_{N=0} = \frac{1 + G_2G_3H_2}{1 + G_1G_2H_1 + G_2G_3H_2 + G_1G_2G_3} = \frac{s^3 + 22s^2 + 50s + 10}{101s^3 + 2122s^2 + 3050s + 1010}$$

(b)

$$G_4(s) = \frac{1+G_1(s)G_2(s)H_1(s)}{G_2(s)G_3(s)} = \frac{101s^3 + 2122s^2 + 2040s}{10(s+1)}$$

(c) Characteristic equation: $101s^3 + 2122s^2 + 3050s + 1010 = 0$ $\quad s^3 + 21.01s^2 + 30.198s + 10 = 0$

Characteristic equation roots: $-0.5029, \quad -1.0205, \quad -19.4867$

(d) $R(s) = 1/s$.

$$E(s) = \frac{s^3 + 22s^2 + 50s + 10}{s(101s^3 + 2122s^2 + 2050s + 1010)} \qquad \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = 0.0099$$

(e) $$Y(s) = \frac{100(s+1)}{s(101s^3 + 2122s^2 + 3050s + 1010)} = \frac{0.099}{s} + \frac{0.002679}{s+19.49} - \frac{0.002078}{s+1.02} - \frac{0.00996}{s+0.5029}$$

$$y(t) = \left(0.099 + 0.002679e^{-19.49t} - 0.002078e^{-1.02t} - 0.00996e^{-0.5029t}\right)u_s(t)$$

Chapter 4 MATHEMATICAL MODELING OF PHYSICAL SYSTEMS

4-1 (a) Force equations:

$$f(t) = M_1 \frac{d^2 y_1}{dt^2} + B_1 \frac{dy_1}{dt} + B_3 \left(\frac{dy_1}{dt} - \frac{dy_2}{dt} \right) + K\left(y_1 - y_2 \right)$$

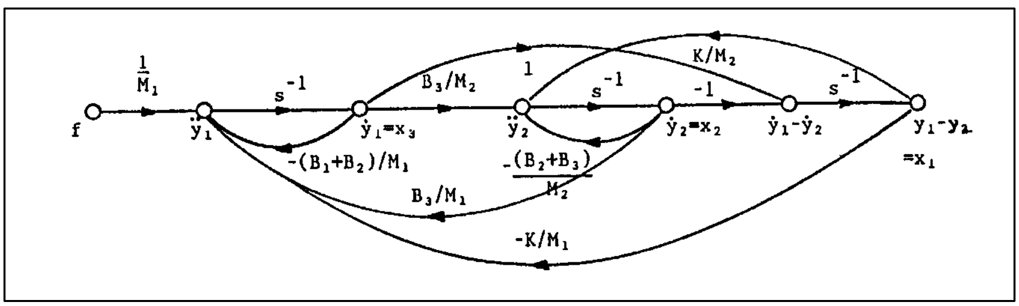
$$B_3 \left(\frac{dy_1}{dt} - \frac{dy_2}{dt} \right) + K\left(y_1 - y_2 \right) + M_2 \frac{d^2 y_2}{dt^2} + B_2 \frac{dy_2}{dt}$$

Rearrange the equations as follows:

$$\frac{d^2 y_1}{dt^2} = -\frac{\left(B_1 + B_3\right)}{M_1}\frac{dy_1}{dt} + \frac{B_3}{M_1}\frac{dy_2}{dt} - \frac{K}{M_1}\left(y_1 - y_2\right) + \frac{f}{M_1}$$

$$\frac{d^2 y_2}{dt^2} = \frac{B_3}{M_2}\frac{dy_1}{dt} - \frac{\left(B_2 + B_3\right)}{M_2}\frac{dy_2}{dt} + \frac{K}{M_2}\left(y_1 - y_2\right)$$

(i) State diagram: Since $y_1 - y_2$ appears as one unit, the minimum number of integrators is three.



State equations: Define the state variables as $x_1 = y_1 - y_2$, $x_2 = \frac{dy_2}{dt}$, $x_3 = \frac{dy_1}{dt}$.

$$\frac{dx_1}{dt} = -x_2 + x_3 \qquad \frac{dx_2}{dt} = \frac{K}{M_2}x_1 - \frac{\left(B_2 + B_3\right)}{M_2}x_2 + \frac{B_3}{M_2}x_3 \qquad \frac{dx_3}{dt} = -\frac{K}{M_1}x_1 + \frac{B_3}{M_1}x_2 - \frac{\left(B_1 + B_3\right)}{M_1}x_3 + \frac{1}{M}f$$

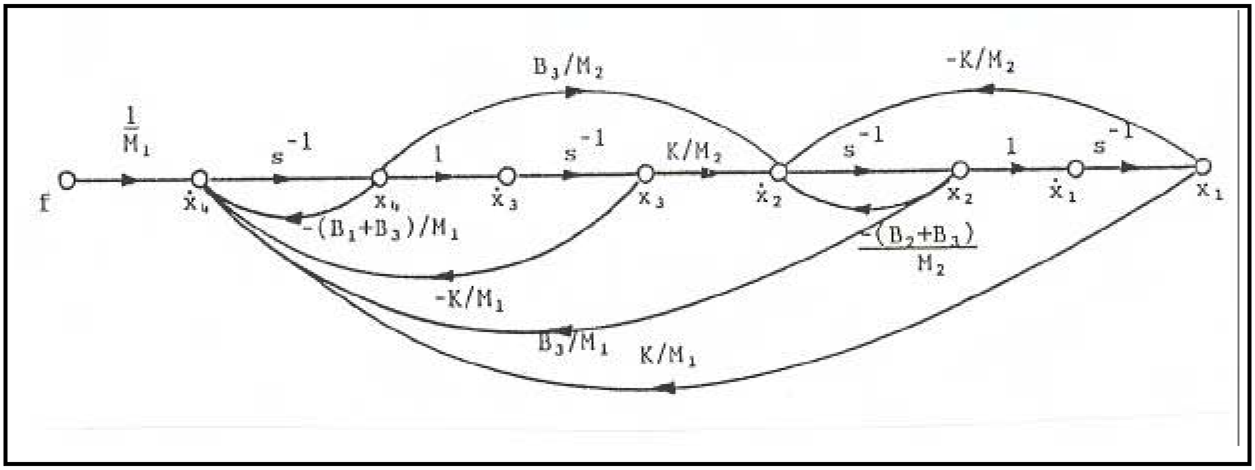
(ii) State variables: $x_1 = y_2$, $x_2 = \frac{dy_2}{dt}$, $x_3 = y_1$, $x_4 = \frac{dy_1}{dt}$.

State equations:

$$\frac{dx_1}{dt} = x_2 \qquad \frac{dx_2}{dt} = -\frac{K}{M_2}x_1 - \frac{B_2 + B_3}{M_2}x_2 + \frac{K}{M_2}x_3 + \frac{B_3}{M_2}x_4$$

$$\frac{dx_3}{dt} = x_4 \qquad \frac{dx_4}{dt} = \frac{K}{M_1}x_1 + \frac{B_3}{M_1}x_2 - \frac{K}{M_1}x_3 - \frac{B_1 + B_3}{M_1}x_4 + \frac{1}{M_1}f$$

State diagram:



Transfer functions:

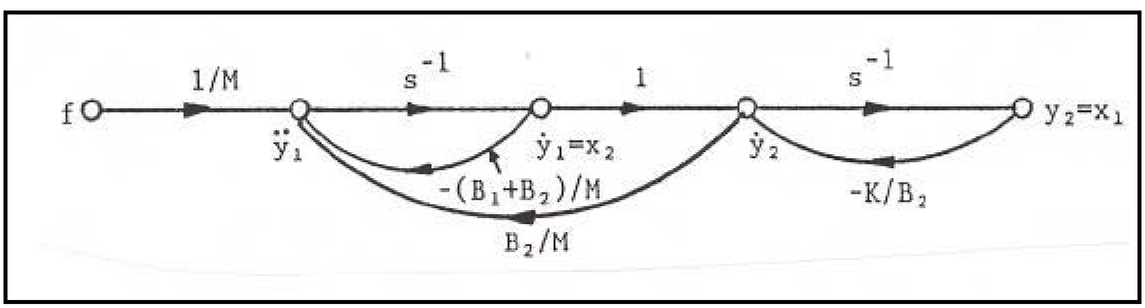
$$\frac{Y_1(s)}{F(s)}=\frac{M_2s^2+\left(B_2+B_3\right)s+K}{s\left\{M_1M_2s^3+\left[\left(B_1+B_3\right)M_2+\left(B_2+B_3\right)M_1\right]s^2+\left[K\left(M_1+M_2\right)+B_1B_2+B_2B_3+B_1B_3\right]s+\left(B_1+B_2\right)K\right\}}$$

$$\frac{Y_2(s)}{F(s)}=\frac{B_3s+K}{s\left\{M_1M_2s^3+\left[\left(B_1+B_3\right)M_2+\left(B_2+B_3\right)M_1\right]s^2+\left[K\left(M_1+M_2\right)+B_1B_2+B_2B_3+B_1B_3\right]s+\left(B_1+B_2\right)K\right\}}$$

(b) Force equations:

$$\frac{d^2y_1}{dt^2}=-\frac{\left(B_1+B_2\right)}{M}\frac{dy_1}{dt}+\frac{B_2}{M}\frac{dy_2}{dt}+\frac{1}{M}f \qquad \frac{dy_2}{dt}=\frac{dy_1}{dt}-\frac{K}{B_2}y_2$$

(i) State diagram:



Define the outputs of the integrators as state variables, $x_1=y_2$, $x_2=\frac{dy_1}{dt}$.

State equations:

$$\frac{dx_1}{dt}=-\frac{K}{B_2}x_1+x_2 \qquad \frac{dx_2}{dt}=-\frac{K}{M}x_1-\frac{B_1}{M}x_2+\frac{1}{M}f$$

(ii) State equations: State variables: $x_1=y_2$, $x_2=y_1$, $x_3=\frac{dy_1}{dt}$.

$$\frac{dx_1}{dt}=-\frac{K}{B_2}x_1+x_3 \qquad \frac{dx_2}{dt}=x_3 \qquad \frac{dx_3}{dt}=-\frac{K}{M}x_1-\frac{B_1}{M}x_3+\frac{1}{M}f$$

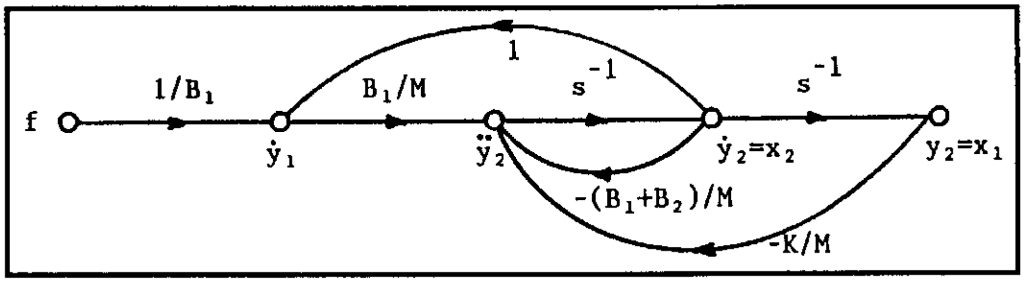
Transfer functions:

$$\frac{Y_1(s)}{F(s)} = \frac{B_2 s + K}{s\left[MB_2 s^2 + \left(B_1 B_2 + KM\right)s + \left(B_1 + B_2\right)K\right]} \qquad \frac{Y_2(s)}{F(s)} = \frac{B_2}{MB_2 s^2 + (B_1 B_2 + KM)s + (B_1 + B_2)K}$$

(c) Force equations:

$$\frac{dy_1}{dt} = \frac{dy_2}{dt} + \frac{1}{B_1} f \qquad \frac{d^2 y_2}{dt^2} = -\frac{\left(B_1 + B_2\right)}{M}\frac{dy_2}{dt} + \frac{B_1}{M}\frac{dy_2}{dt} + \frac{B_1}{M}\frac{dy_1}{dt} - \frac{K}{M} y_2$$

(i) State diagram:



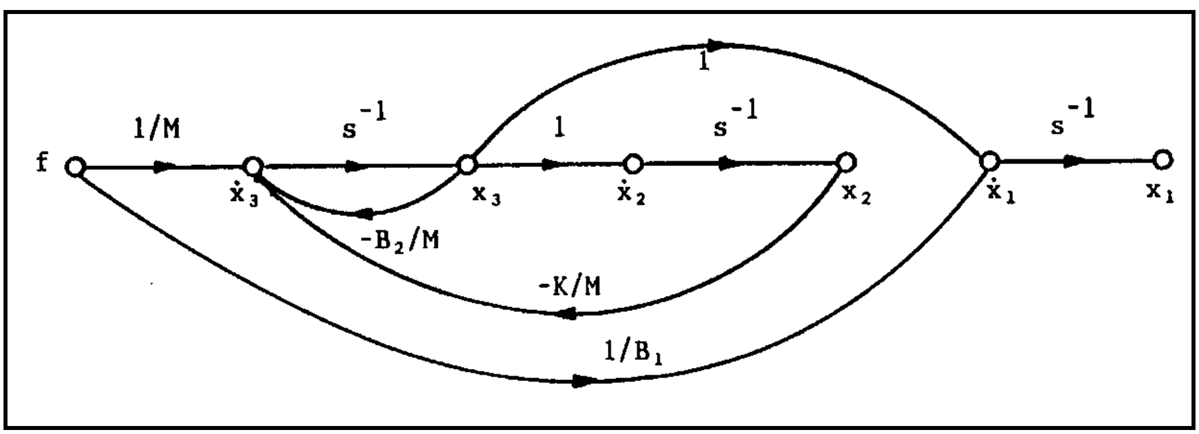
State equations: Define the outputs of integrators as state variables.

$$\frac{dx_1}{dt} = x_2 \qquad \frac{dx_2}{dt} = -\frac{K}{M}x_1 - \frac{B_2}{M}x_2 + \frac{1}{M} f$$

(ii) State equations: state variables: $x_1 = y_1, \quad x_2 = y_2, \quad x_3 = \frac{dy_2}{dt}$.

$$\frac{dx_1}{dt} = x_3 + \frac{1}{B_1} f \qquad \frac{dx_2}{dt} = x_3 \qquad \frac{dx_3}{dt} = -\frac{K}{M}x_2 - \frac{B_2}{M}x_3 + \frac{1}{M} f$$

State diagram:



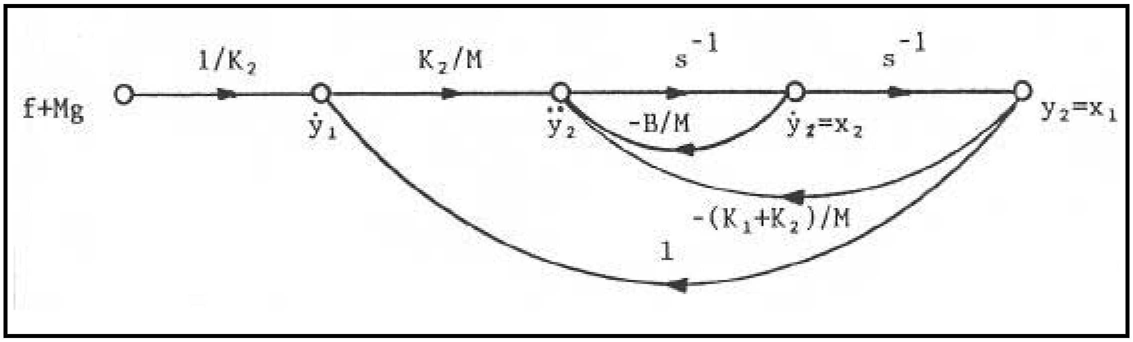
Transfer functions:

$$\frac{Y_1(s)}{F(s)} = \frac{Ms^2 + \left(B_1 + B_2\right)s + K}{B_1 s\left(Ms^2 + B_2 s + K\right)} \qquad \frac{Y_2(s)}{F(s)} = \frac{1}{Ms^2 + B_2 s + K}$$

4-2 (a) Force equations:

$$y_1 = \frac{1}{K_2}(f + Mg) + y_2 \qquad \frac{d^2 y_2}{dt^2} = -\frac{B}{M}\frac{dy_2}{dt} - \frac{K_1 + K_2}{M} y_2 + \frac{K_2}{M} y_1$$

State diagram:



State equations: Define the state variables as: $x_1 = y_2$, $x_2 = \frac{dy_2}{dt}$.

$$\frac{dx_1}{dt} = x_2 \qquad \frac{dx_2}{dt} = -\frac{K_1}{M}x_1 - \frac{B}{M}x_2 + \frac{1}{M}(f + Mg)$$

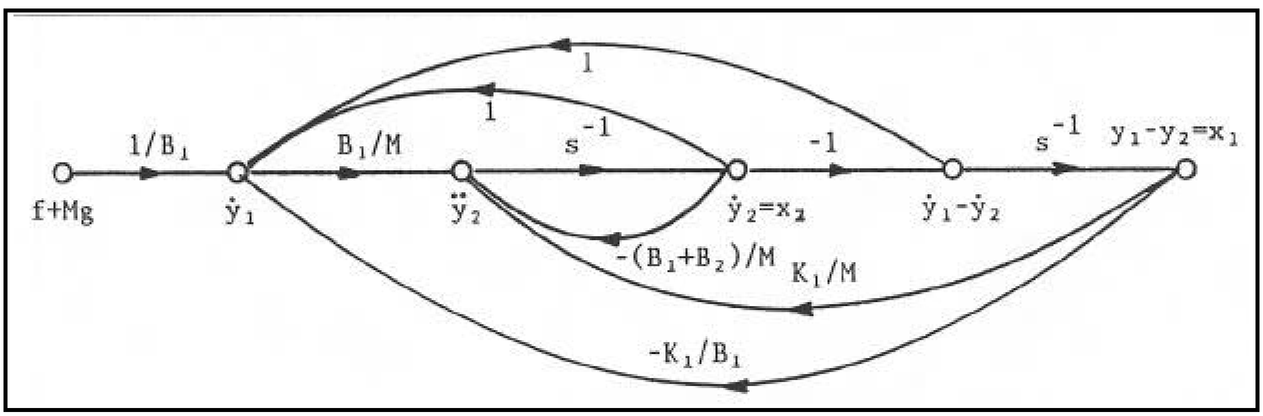
Transfer functions:

$$\frac{Y_1(s)}{F(s)} = \frac{s^2 + Bs + K_1 + K_2}{K_2(Ms^2 + Bs + K_1)} \qquad \frac{Y_2(s)}{F(s)} = \frac{1}{Ms^2 + Bs + K_1}$$

(b) Force equations:

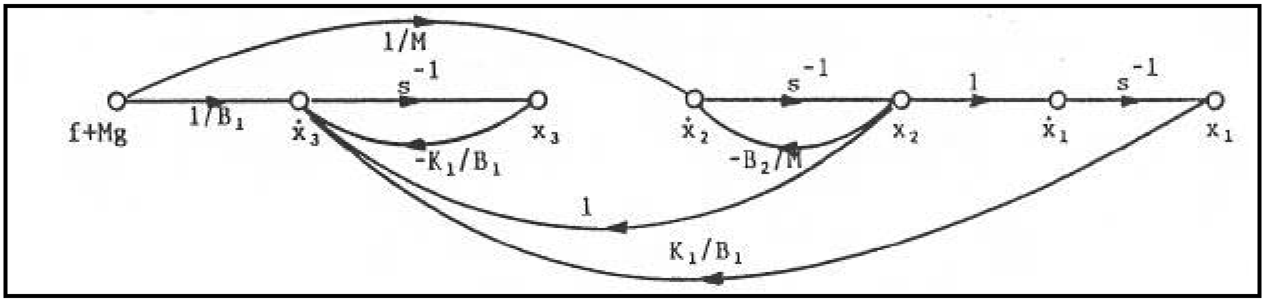
$$\frac{dy_1}{dt} = \frac{1}{B_1}[f(t) + Mg] + \frac{dy_2}{dt} - \frac{K_1}{B_1}(y_1 - y_2) \qquad \frac{d^2y_2}{dt^2} = \frac{B_1}{M}\left(\frac{dy_1}{dt} - \frac{dy_2}{dt}\right) + \frac{K_1}{M}(y_1 - y_2) - \frac{B_2}{M}(y_1 - y_2) - \frac{B_2}{M}\frac{dy_2}{dt}$$

State diagram: (With minimum number of integrators)



To obtain the transfer functions $Y_1(s)/F(s)$ and $Y_2(s)/F(s)$, we need to redefine the state variables as: $x_1 = y_2$, $x_2 = dy_2/dt$, and $x_3 = y_1$.

State diagram:



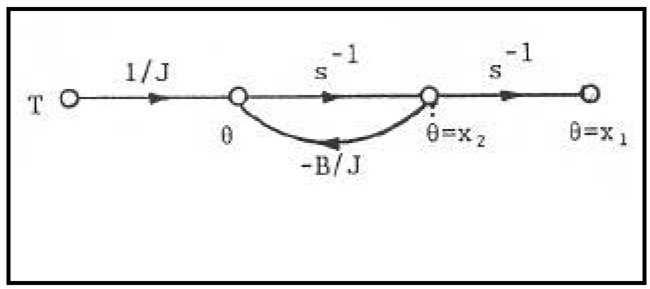
Transfer functions:

$$\frac{Y_1(s)}{F(s)} = \frac{Ms^2 + (B_1 + B_2)s + K_1}{s^2\left[MB_1 s + (B_1 B_2 + MK_1)\right]} \qquad \frac{Y_2(s)}{F(s)} = \frac{Bs + K_1}{s^2\left[MB_1 s + (B_1 B_2 + MK_1)\right]}$$

4-3 (a) Torque equation:

$$\frac{d^2 q}{dt^2} = -\frac{B}{J}\frac{dq}{dt} + \frac{1}{J}T(t)$$

State diagram:



State equations:

$$\frac{dx_1}{dt} = x_2 \qquad \frac{dx_2}{dt} = -\frac{B}{J}x_2 + \frac{1}{J}T$$

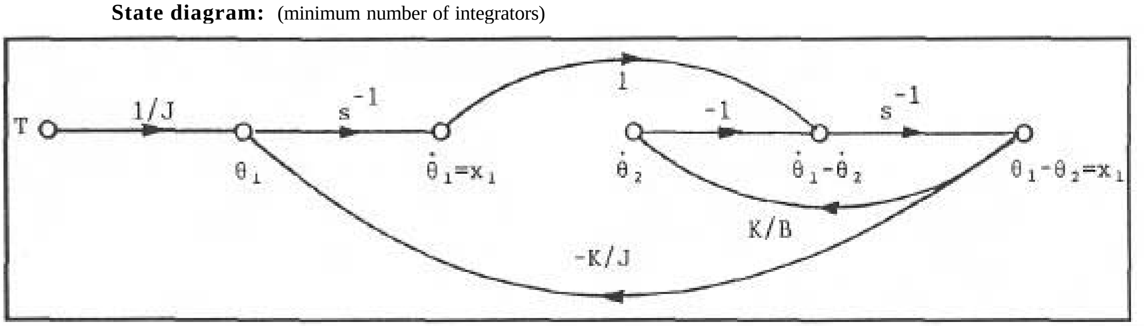
Transfer function:

$$\frac{\Theta(s)}{T(s)} = \frac{1}{s(Js + B)}$$

(b) Torque equations:

$$\frac{d^2 q_1}{dt^2} = -\frac{K}{J}(q_1 - q_2) + \frac{1}{J}T \qquad K(q_1 - q_2) = B\frac{dq_2}{dt}$$

State diagram: (minimum number of integrators)



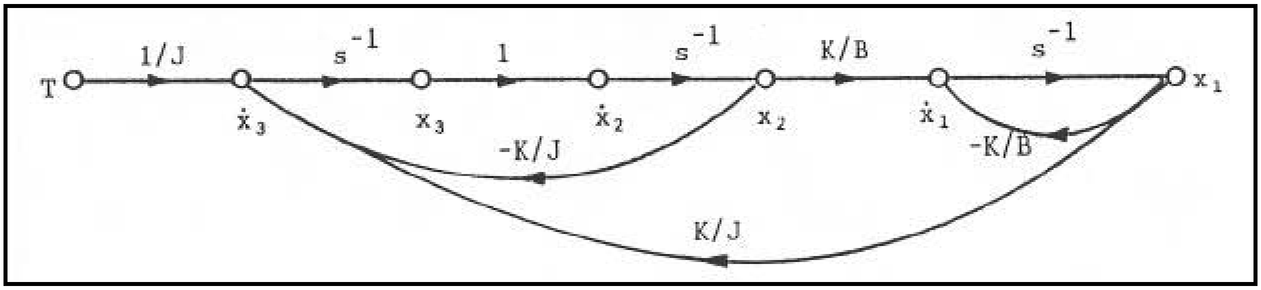
State equations:

$$\frac{dx_1}{dt} = -\frac{K}{B}x_1 + x_2 \qquad \frac{dx_2}{dt} = -\frac{K}{J}x_1 + \frac{1}{J}T$$

State equations: Let $x_1 = q_2$, $x_2 = q_1$, and $x_3 = \frac{dq_1}{dt}$.

$$\frac{dx_1}{dt} = -\frac{K}{B}x_1 + \frac{K}{B}x_2 \qquad \frac{dx_2}{dt} = x_3 \qquad \frac{dx_3}{dt} = \frac{K}{J}x_1 - \frac{K}{J}x_2 + \frac{1}{J}T$$

State diagram:



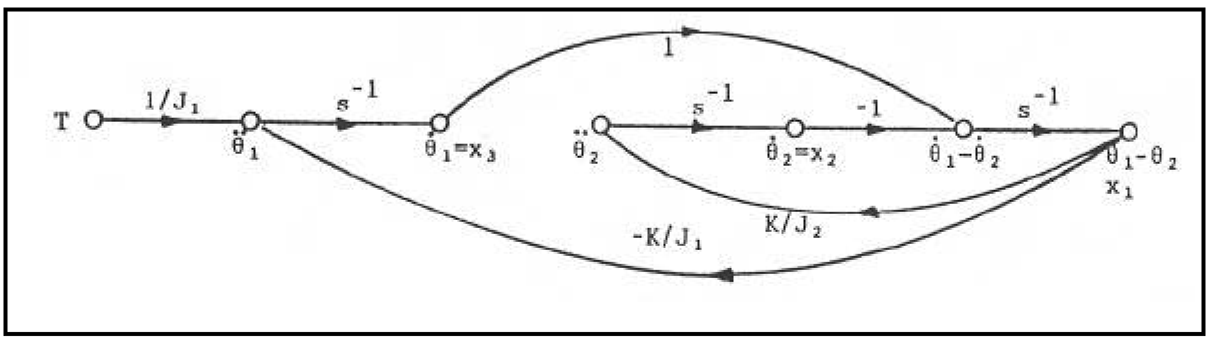
Transfer functions:

$$\frac{\Theta_1(s)}{T(s)} = \frac{Bs+K}{s\left(BJs^2 + JKs + BK\right)} \qquad \frac{\Theta_2(s)}{T(s)} = \frac{K}{s\left(BJs^2 + JKs + BK\right)}$$

(c) Torque equations:

$$T(t) = J_1 \frac{d^2 q_1}{dt^2} + K\left(q_1 - q_2\right) \qquad K\left(q_1 - q_2\right) = J_2 \frac{d^2 q_2}{dt^2}$$

State diagram:



State equations: state variables: $x_1 = q_2, \quad x_2 = \dfrac{dq_2}{dt}, \quad x_3 = q_1, \quad x_4 = \dfrac{dq_1}{dt}.$

$$\frac{dx_1}{dt} = x_2 \qquad \frac{dx_2}{dt} = -\frac{K}{J_2}x_1 + \frac{K}{J_2}x_3 \qquad \frac{dx_3}{dt} = x_4 \qquad \frac{dx_4}{dt} = \frac{K}{J_1}x_1 - \frac{K}{J_1}x_3 + \frac{1}{J_1}T$$

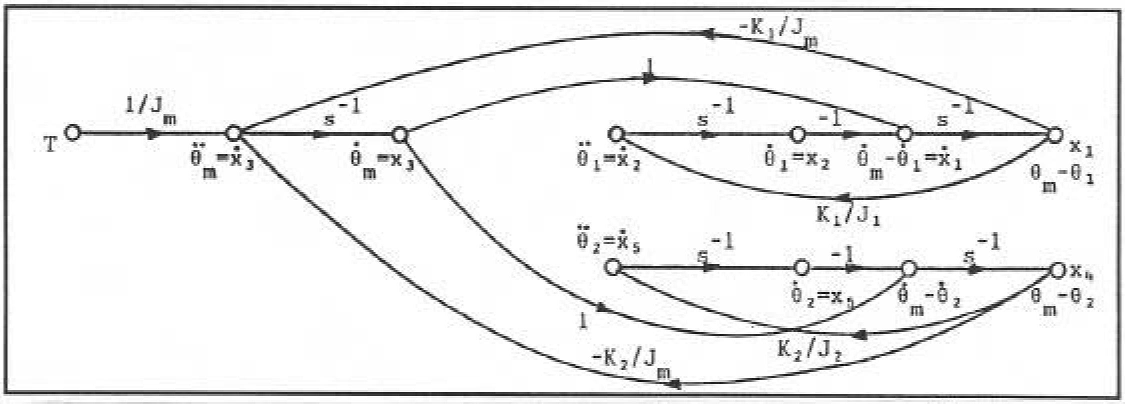
Transfer functions:

$$\frac{\Theta_1(s)}{T(s)} = \frac{J_2 s^2 + K}{s^2\left[J_1 J_2 s^2 + K\left(J_1 + J_2\right)\right]} \qquad \frac{\Theta_2(s)}{T(s)} = \frac{K}{s^2\left[J_1 J_2 s^2 + K\left(J_1 + J_2\right)\right]}$$

(d) Torque equations:

$$T(t) = J_m \frac{d^2 q_m}{dt^2} + K_1\left(q_m - q_1\right) + K_2\left(q_m - q_2\right) \qquad K_1\left(q_m - q_1\right) = J_1 \frac{d^2 q_1}{dt^2} \qquad K_2\left(q_m - q_2\right) = J_2 \frac{d^2 q_2}{dt^2}$$

State diagram:



State equations: $x_1 = q_m - q_1, \quad x_2 = \dfrac{dq_1}{dt}, \quad x_3 = \dfrac{dq_m}{dt}, \quad x_4 = q_m - q_2, \quad x_5 = \dfrac{dq_2}{dt}.$

$$\frac{dx_1}{dt} = -x_2 + x_3 \quad \frac{dx_2}{dt} = \frac{K_1}{J_1}x_1 \quad \frac{dx_3}{dt} = -\frac{K_1}{J_m}x_1 - \frac{K_2}{J_m}x_4 + \frac{1}{J_m}T \quad \frac{dx_4}{dt} = x_3 - x_5 \quad \frac{dx_5}{dt} = \frac{K_2}{J_2}x_4$$

Transfer functions:

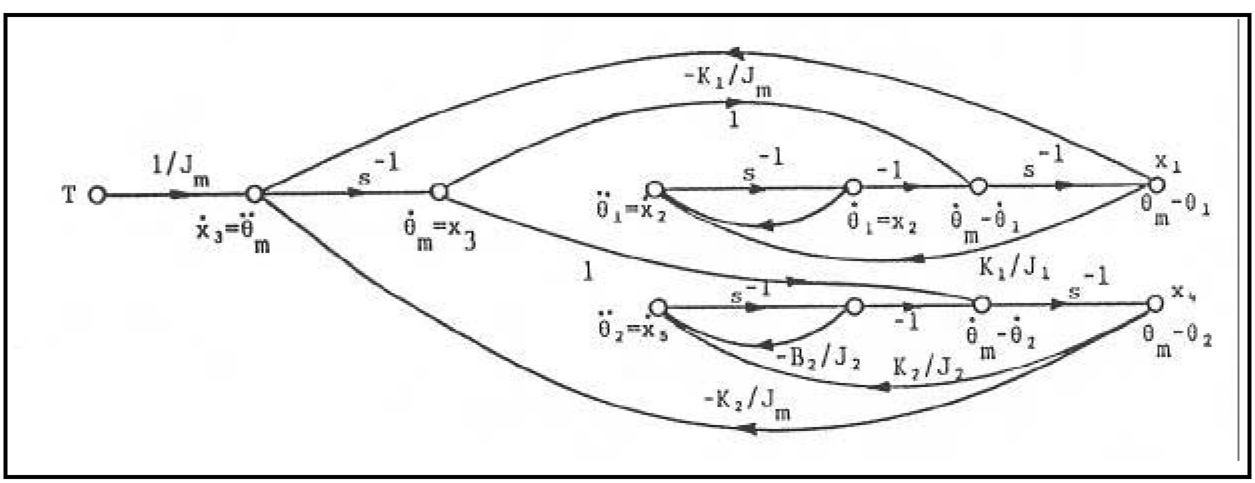
$$\frac{\Theta_1(s)}{T(s)} = \frac{K_1(J_2 s^2 + K_2)}{s^2\left[s^4 + \left(K_1J_2J_m + K_2J_1J_m + K_1J_1J_2 + K_2J_1J_2\right)s^2 + K_1K_2\left(J_m + J_1 + J_2\right)\right]}$$

$$\frac{\Theta_2(s)}{T(s)} = \frac{K_2(J_1 s^2 + K_1)}{s^2\left[s^4 + \left(K_1J_2J_m + K_2J_1J_m + K_1J_1J_2 + K_2J_1J_2\right)s^2 + K_1K_2\left(J_m + J_1 + J_2\right)\right]}$$

(e) Torque equations:

$$\frac{d^2q_m}{dt^2} = -\frac{K_1}{J_m}(q_m - q_1) - \frac{K_2}{J_m}(q_m - q_2) + \frac{1}{J_m}T \qquad \frac{d^2q_1}{dt^2} = \frac{K_1}{J_1}(q_m - q_1) - \frac{B_1}{J_1}\frac{dq_1}{dt} \qquad \frac{d^2q_2}{dt^2} = \frac{K_2}{J_2}(q_m - q_1) - \frac{B_2}{J_2}\frac{dq_2}{dt}$$

State diagram:



State variables: $x_1 = q_m - q_1, \quad x_2 = \dfrac{dq_1}{dt}, \quad x_3 = \dfrac{dq_m}{dt}, \quad x_4 = q_m - q_2, \quad x_5 = \dfrac{dq_2}{dt}.$

State equations:

$$\frac{dx_1}{dt} = -x_2 + x_3 \qquad \frac{dx_2}{dt} = \frac{K_1}{J_1}x_1 - \frac{B_1}{J_1}x_2 \qquad \frac{dx_3}{dt} = -\frac{K_1}{J_m}x_1 - \frac{K_2}{J_m}x_4 + \frac{1}{J_m}T \qquad \frac{dx_4}{dt} = x_3 - x_5 \qquad \frac{dx_5}{dt} = \frac{K_2}{J_2}x_4 - \frac{B_2}{J_2}x_5$$

Transfer functions:

$$\frac{\Theta_1(s)}{T(s)} = \frac{K_1\left(J_2 s^2 + B_2 s + K_2\right)}{\Delta(s)} \qquad \frac{\Theta_2(s)}{T(s)} = \frac{K_2\left(J_1 s^2 + B_1 s + K_1\right)}{\Delta(s)}$$

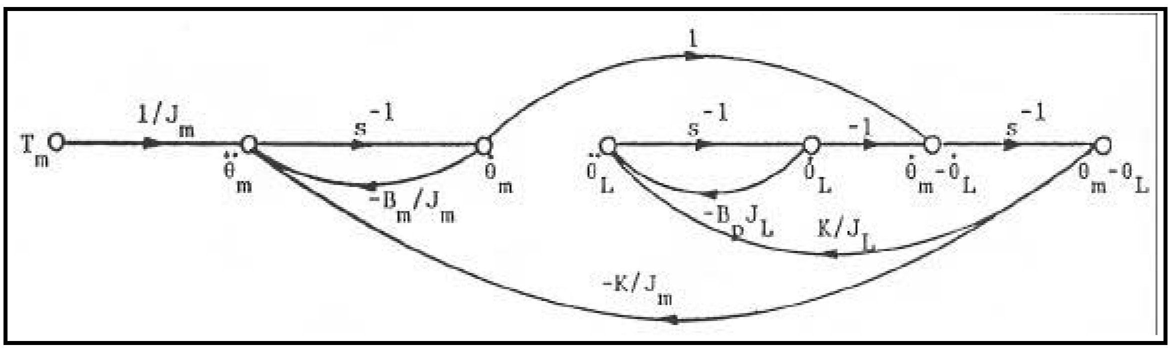
$$\Delta(s) = s^2\{J_1 J_2 J_m s^4 + J_m\left(B_1 + B_2\right)s^3 + \left[\left(K_1 J_2 + K_2 J_1\right)J_m + \left(K_1 + K_2\right)J_1 J_2 + B_1 B_2 J_m\right]s^2$$
$$+\left[\left(B_1 K_2 + B_2 K_1\right)J_m + B_1 K_2 J_2 + B_2 K_1 J_1\right]s + K_1 K_2\left(J_m + J_1 + J_2\right)\}$$

4-4 **System equations:**

$$T_m(t) = J_m\frac{d^2 q_m}{dt^2} + B_m\frac{dq_m}{dt} + K\left(q_m - q_L\right) \qquad K\left(q_m - q_L\right) = J_L\frac{d^2 q_L}{dt^2} + B_p\frac{dq_L}{dt}$$

Output equation: $e_o = \dfrac{E q_L}{20\,p}$

State diagram:



Transfer function:

$$\frac{\Theta_L(s)}{T_m(s)} = \frac{K}{s\left[J_m J_L s^3 + \left(B_m J_L + B_p J_m\right)s^2 + \left(J_m K + J_L K + B_m B_p\right)s + B_m K\right]}$$

$$\frac{E_o(s)}{T_m(s)} = \frac{KE/20p}{s\left[J_m J_L s^3 + \left(B_m J_L + B_p J_m\right)s^2 + \left(J_m K + J_L K + B_m B_p\right)s + B_m K\right]}$$

4-5 **(a)**

$$T_m(t) = J_m\frac{d^2 q_1}{dt^2} + T_1 \qquad T_1 = \frac{N_1}{N_2}T_2 \qquad T_3 = \frac{N_3}{N_4}T_4 \qquad T_4 = J_L\frac{d^2 q_3}{dt^2} \qquad T_2 = T_3 \qquad q_2 = \frac{N_1}{N_2}q_1$$

$$q_3 = \frac{N_1 N_3}{N_2 N_4}q_1 \qquad T_2 = \frac{N_3}{N_4}T_4 = \frac{N_3}{N_4}J_L\frac{d^2 q_3}{dt^2} \qquad T_m = J_m\frac{d^2 q_1}{dt^2} + \frac{N_1 N_3}{N_2 N_4}T_4 = \left[J_m + \left[\frac{N_1 N_3}{N_2 N_4}\right]^2 J_L\right]\frac{d^2 q_1}{dt^2}$$

(b)

$$T_m = J_m \frac{d^2 q_1}{dt^2} + T_1 \qquad T_2 = J_2 \frac{d^2 q_2}{dt^2} + T_3 \qquad T_4 = \left(J_3 + J_L\right)\frac{d^2 q_3}{dt^2} \qquad T_1 = \frac{N_1}{N_2} T_2 \qquad T_3 = \frac{N_3}{N_4} T_4$$

$$q_2 = \frac{N_1}{N_2} q_1 \qquad q_3 = \frac{N_1 N_3}{N_2 N_4} q_1 \qquad T_2 = J_2 \frac{d^2 q_2}{dt^2} + \frac{N_3}{N_4} T_4 = J_2 \frac{d^2 q_2}{dt^2} + \frac{N_3}{N_4}\left(J_3 + J_L\right)\frac{d^2 q_3}{dt^2}$$

$$T_m(t) = J_m \frac{d^2 q_1}{dt^2} + \frac{N_1}{N_2}\left(J_2 \frac{d^2 q_2}{dt^2} + \frac{N_3}{N_4}\left(J_3 + J_4\right)\frac{d^2 q_3}{dt^2}\right) = \left[J_m + \left(\frac{N_1}{N_2}\right)^2 J_2 + \left(\frac{N_1 N_3}{N_2 N_4}\right)^2 \left(J_3 + J_L\right)\right]\frac{d^2 q_1}{dt^2}$$

4-6 (a) Force equations:

$$f(t) = K_h\left(y_1 - y_2\right) + B_h\left(\frac{dy_1}{dt} - \frac{dy_2}{dt}\right) \qquad K_h\left(y_1 - y_2\right) + B_h\left(\frac{dy_1}{dt} - \frac{dy_2}{dt}\right) = M\frac{d^2 y_2}{dt^2} + B_t\frac{dy_2}{dt}$$

(b) State variables: $x_1 = y_1 - y_2, \quad x_2 = \frac{dy_2}{dt}$

State equations:

$$\frac{dx_1}{dt} = -\frac{K_h}{B_h} x_1 + \frac{1}{B_h} f(t) \qquad \frac{dx_2}{dt} = -\frac{B_t}{M} x_2 + \frac{1}{M} f(t)$$

4-7 (a)

$$T_m = J_m \frac{d^2 q_m}{dt^2} + T_1 \qquad T_2 = J_L \frac{d^2 q_L}{dt^2} + T_L \qquad T_1 = \frac{N_1}{N_2} T_2 = nT_2 \qquad q_m N_1 = q_L N_2$$

$$T_m = J_m \frac{d^2 q_m}{dt^2} + nJ_L \frac{d^2 q_L}{dt^2} + nT_L = \left(\frac{J_m}{n} + nJ_L\right) a_L + nT_L \qquad \text{Thus, } a_L = \frac{nT_m - n^2 T_L}{J_m + n^2 J_L}$$

Set $\frac{¶ a_L}{¶ n} = 0.$ $\quad \left(T_m - 2nT_L\right)\left(J_m + n^2 J_L\right) - 2nJ_L\left(nT_m - n^2 J_L\right) = 0$ Or, $n^2 + \frac{J_m T_L}{J_L T_m} n - \frac{J_m}{J_L} = 0$

Optimal gear ratio: $n^* = -\frac{J_m T_L}{2J_L T_m} + \frac{\sqrt{J_m^2 T_L^2 + 4 J_m J_L T_m^2}}{2J_L T_m}$ where the + sign has been chosen.

(b) When $T_L = 0$, the optimal gear ratio is

$$n^* = \sqrt{J_m / J_L}$$

4-8 (a) Torque equation about the motor shaft: Relation between linear and rotational displacements:

$$T_m = J_m \frac{d^2 q_m}{dt^2} + Mr^2 \frac{d^2 q_m}{dt^2} + B_m \frac{dq_m}{dt} \qquad\qquad y = rq_m$$

(b) Taking the Laplace transform of the equations in part (a), with zero initial conditions, we have

$$T_m(s) = \left(J_m + Mr^2\right) s^2 \Theta_m(s) + B_m s\Theta_m(s) \qquad\qquad Y(s) = r\Theta_m(s)$$

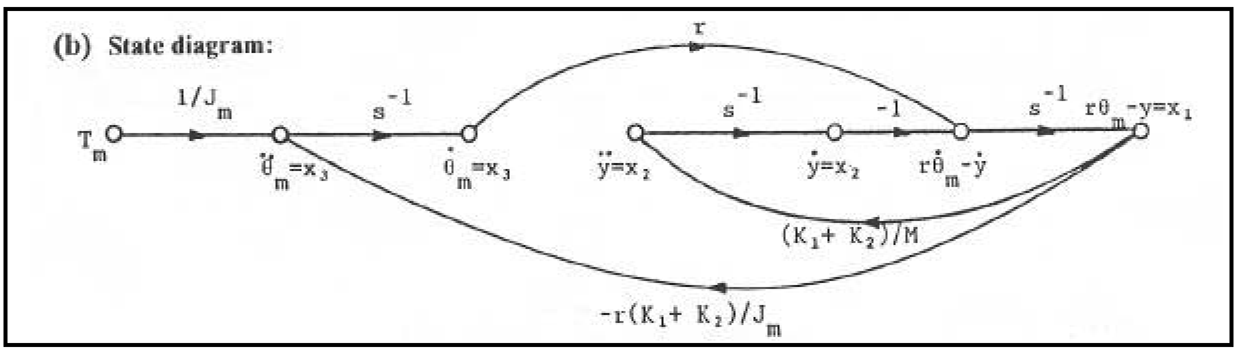
Transfer function:

$$\frac{Y(s)}{T_m(s)} = \frac{r}{s\left[\left(J_m + Mr^r\right)s + B_m\right]}$$

4-9 (a)

$$T_m = J_m \frac{d^2 q_m}{dt^2} + r\left(T_1 - T_2\right) \qquad T_1 = K_2\left(rq_m - rq_p\right) = K_2\left(rq_m - y\right) \qquad T_2 = K_1\left(y - rq_m\right)$$

$$T_1 - T_2 = M\frac{d^2 y}{dt^2} \qquad \text{Thus, } T_m = J_m \frac{d^2 q_m}{dt^2} + r\left(K_1 + K_2\right)\left(rq_m - y\right) \qquad M\frac{d^2 y}{dt^2} = \left(K_1 + K_2\right)\left(rq_m - y\right)$$



(c) State equations:

$$\frac{dx_1}{dt} = rx_3 - x_2 \qquad \frac{dx_2}{dt} = \frac{K_1 + K_2}{M}x_1 \qquad \frac{dx_3}{dt} = \frac{-r\left(K_1 + K_2\right)}{J_m}x_1 + \frac{1}{J_m}T_m$$

(d) Transfer function:

$$\frac{Y(s)}{T_m(s)} = \frac{r\left(K_1 + K_2\right)}{s^2\left[J_m M s^2 + \left(K_1 + K_2\right)\left(J_m + rM\right)\right]}$$

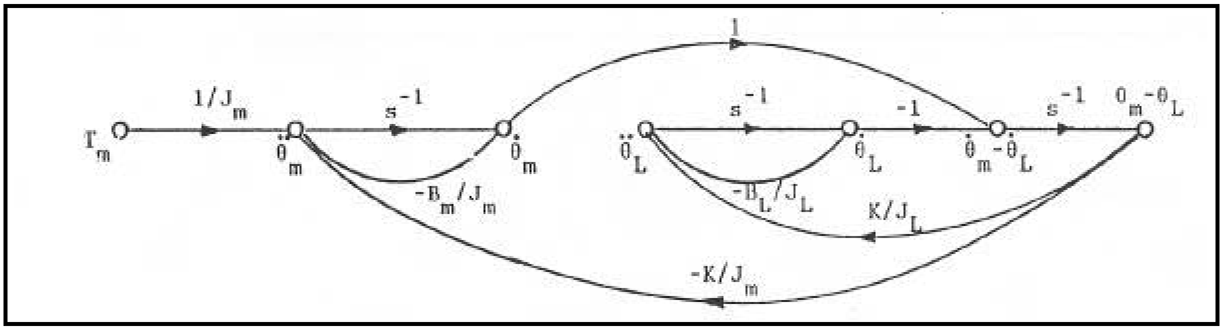
(e) Characteristic equation:

$$s^2\left[J_m M s^2 + \left(K_1 + K_2\right)\left(J_m + rM\right)\right] = 0$$

4-10 (a) Torque equations:

$$T_m(t) = J_m \frac{d^2 q_m}{dt^2} + B_m \frac{dq_m}{dt} + K\left(q_m - q_L\right) \qquad K\left(q_m - q_L\right) = J_L \frac{d^2 q_L}{dt^2} + B_L \frac{dq_L}{dt}$$

State diagram:



(b) **Transfer functions:**

$$\frac{\Theta_L(s)}{T_m(s)} = \frac{K}{\Delta(s)} \qquad \frac{\Theta_m(s)}{T_m(s)} = \frac{J_L s^2 + B_L s + K}{\Delta(s)} \qquad \Delta(s) = s\left[J_m J_L s^3 + \left(B_m J_L + B_L J_m\right)s^2 + \left(KJ_m + KJ_L + B_m B_L\right)s + B_m K\right]$$

(c) **Characteristic equation:** $\Delta(s) = 0$

(d) **Steady-state performance:** $T_m(t) = T_m =$ consta nt. $T_m(s) = \frac{T_m}{s}.$

$$\lim_{t\to\infty} w_m(t) = \lim_{s\to 0} s\Omega_m(s) = \lim_{s\to 0} \frac{J_L s^2 + B_L s + K}{J_m J_L s^3 + \left(B_m J_L + B_L J_m\right)s^2 + \left(KJ_m + KJ_L + B_m B_L\right)s + B_m K} = \frac{1}{B_m}$$

Thus, in the steady state, $w_m = w_L$.

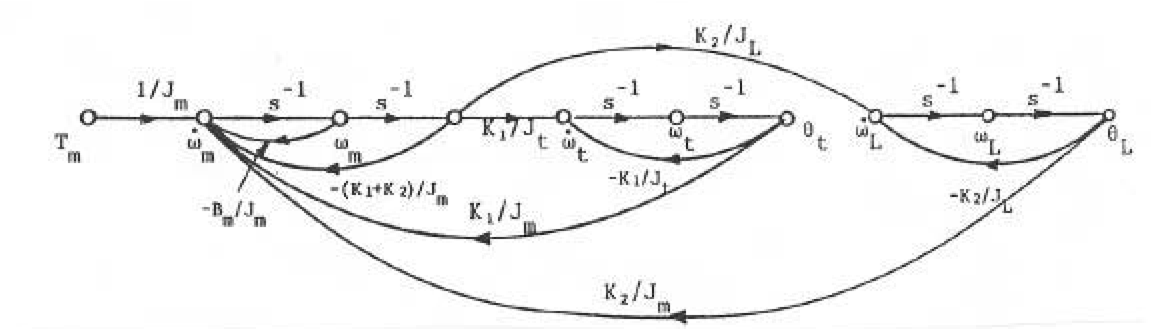
(e) The steady-state values of w_m and w_L do not depend on J_m and J_L.

4-11 **(a)** **State equations:**

$$\frac{dq_L}{dt} = w_L \qquad \frac{dw_L}{dt} = \frac{K_2}{J_L}q_m - \frac{K_2}{J_L}q_L \qquad \frac{dq_t}{dt} = w_t \qquad \frac{dw_t}{dt} = \frac{K_1}{J_t}q_m - \frac{K_1}{J_t}q_t$$

$$\frac{dq_m}{dt} = w_m \qquad \frac{dw_m}{dt} = -\frac{B_m}{J_m}w_m - \frac{\left(K_1 + K_2\right)}{J_m}q_m + \frac{K_1}{J_m}q_t + \frac{K_2}{J_m}q_L + \frac{1}{J_m}T_m$$

(b) **State diagram:**



(c) **Transfer functions:**

$$\frac{\Theta_L(s)}{T_m(s)} = \frac{K_2\left(J_t s^2 + K_1\right)}{\Delta(s)} \qquad \frac{\Theta_t(s)}{T_m(s)} = \frac{K_1\left(J_L s^2 + K_2\right)}{\Delta(s)} \qquad \frac{\Theta_m(s)}{T_m(s)} = \frac{J_t J_L s^4 + \left(K_1 J_L + K_2 J_t\right) s^2 + K_1 K_2}{\Delta(s)}$$

$$\Delta(s) = s[J_m J_L s^5 + B_m J_L J_t s^4 + \left(K_1 J_L J_t + K_2 J_L J_t + K_1 J_m J_L + K_2 J_m J_t\right) s^3$$
$$+ B_m J_L\left(K_1 + K_2\right) s^2 + K_1 K_2\left(J_L + J_t + J_m\right) s + B_m K_1 K_2] = 0$$

(d) Characteristic equation: $\Delta(s) = 0$.

4-12 (a)

$$\left.\frac{\Omega_m(s)}{T_L(s)}\right|_{w_r=0} = \frac{\dfrac{-1}{B+Js}\left(1 + K_1 H_e(s) + \dfrac{K_1 H_i(s)}{R_a + L_a s}\right)}{\Delta(s)} \cong \frac{\dfrac{-K_1}{B+Js}\left(H_e(s) + \dfrac{H_i(s)}{R_a + L_a s}\right)}{\Delta(s)} = 0$$

Thus,

$$H_e(s) = -\frac{H_i(s)}{R_a + L_a s} \qquad \frac{H_i(s)}{H_e(s)} = -\left(R_a + L_a s\right)$$

(b)

$$\left.\frac{\Omega_m(s)}{\Omega_r(s)}\right|_{T_L=0} = \frac{\dfrac{K_1 K_i}{\left(R_a + L_a s\right)\left(B + Js\right)}}{\Delta(s)}$$

$$\Delta(s) = 1 + K_1 H_e(s) + \frac{K_1 K_b}{\left(R_a + L_a s\right)\left(B + Js\right)} + \frac{K_1 H_i(s)}{R_a + L_a s} + \frac{K_1 K_i K_b H_e(s)}{\left(R_a + L_a s\right)\left((B + Js\right)}$$

$$= 1 + \frac{K_1 K_b}{\left(R_a + L_a s\right)\left(B + Js\right)} + \frac{K_1 K_i}{\left(R_a + L_a s\right)\left(B + Js\right)}$$

$$\left.\frac{\Omega_m(s)}{\Omega_r(s)}\right|_{T_L=0} = \frac{K_1 K_i}{\left(R_a + L_a s\right)\left(B + Js\right) + K_i K_b + K_1 K_i K_b H_e(s)} \cong \frac{1}{K_b H_e(s)}$$

4-13 (a) Torque equation: (About the center of gravity C)

$$J\frac{d^2 q}{dt^2} = T_s d_2 \sin d + F_d d_1 \qquad F_a d_1 = J_a a_1 = K_F d_1 q \qquad \sin d \cong d$$

Thus, $$J\frac{d^2 q}{dt^2} = T_s d_2 d + K_F d_1 q \qquad J\frac{d^2 q}{dt^2} - K_F d_1 q = T_s d_2 d$$

(b) $Js^2\Theta(s) - K_F d_1 \Theta(s) = T_s d_2 \Delta(s)$

(c) With C and P interchanged, the torque equation about C is:

$$T_s\left(d_1 + d_2\right) d + F_a d_2 = J\frac{d^2 q}{dt^2} \qquad T_s\left(d_1 + d_2\right) d + K_F d_2 q = J\frac{d^2 q}{dt^2}$$

$$Js^2\Theta(s) - K_F d_2 \Theta(s) = T_s\left(d_1 + d_2\right)\Delta(s) \qquad \frac{\Theta(s)}{\Delta(s)} = \frac{T_s\left(d_1 + d_2\right)}{Js^2 - K_F d_2}$$

4-14 (a) Cause-and-effect equations: $q_e = q_r - q_o \qquad e = K_s q_e \qquad e_a = Ke$

$$\frac{di_a}{dt} = -\frac{R_a}{L_a} i_a + \frac{1}{L_a}\left(e_a - e_b\right) \qquad T_m = K_i i_a$$

$$\frac{d^2 q_m}{dt^2} = -\frac{B_m}{J_m}\frac{dq_m}{dt} + \frac{1}{J}T_m - \frac{nK_L}{J_m}\left(nq_m - q_o\right) \qquad T_2 = \frac{T_m}{n} \qquad q_2 = nq_m$$

$$\frac{d^2 q_o}{dt^2} = \frac{K_L}{J_L}\left(q_2 - q_o\right)$$

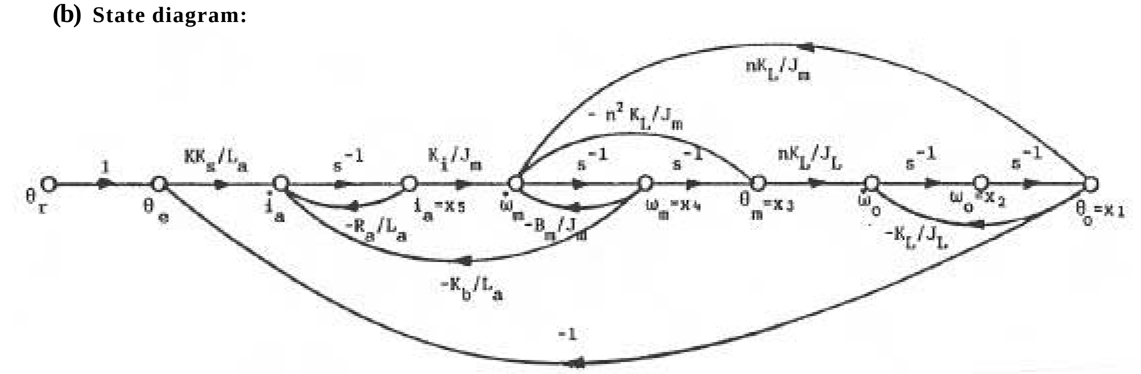
State variables: $x_1 = q_o, \quad x_2 = w_o, \quad x_3 = q_m, \quad x_4 = w_m, \quad x_5 = i_a$

State equations:

$$\frac{dx_1}{dt} = x_2 \qquad \frac{dx_2}{dt} = -\frac{K_L}{J_L}x_1 + \frac{nK_L}{J_L}x_3 \qquad \frac{dx_3}{dt} = x_4$$

$$\frac{dx_4}{dt} = -\frac{nK_L}{J_m}x_1 - \frac{n^2 K_L}{J_m}x_3 - \frac{B_m}{J_m}x_4 + \frac{K_i}{J_m}x_5 \qquad \frac{dx_5}{dt} = -\frac{KK_s}{L_a}x_1 - \frac{K_b}{L_a}x_4 - \frac{R_a}{L_a}x_5 + \frac{KK_s}{L_a}q_r$$

(b) State diagram:



(c) Forward-path transfer function:

$$\frac{\Theta_o(s)}{\Theta_e(s)} = \frac{KK_s K_i nK_L}{s\left[J_m J_L L_a s^4 + J_L\left(R_a J_m + B_m J_m + B_m L_a\right)s^3 + \left(n^2 K_L L_a J_L + K_L J_m L_a + B_m R_a J_L\right)s^2 + \left(n^2 R_a K_L J_L + R_a K_L J_m + B_m K_L L_a\right)s + K_i K_b K_L + R_a B_m K_L\right]}$$

Closed-loop transfer function:

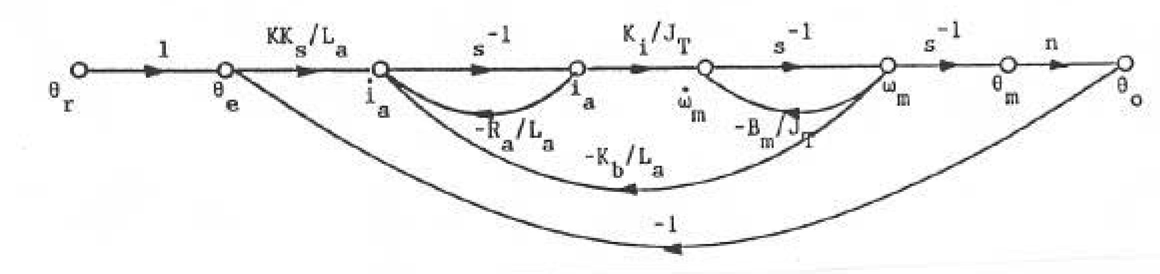
$$\frac{\Theta_o(s)}{\Theta_r(s)} = \frac{KK_s K_i nK_L}{J_m J_L L_a s^5 + J_L\left(R_a J_m + B_m J_m + B_m L_a\right)s^4 + \left(n^2 K_L L_a J_L + K_L J_m L_a + B_m R_a J_L\right)s^3 + \left(n^2 R_a K_L J_L + R_a K_L J_m + B_m K_L L_a\right)s^2 + \left(K_i K_b K_L + R_a B_m K_L\right)s + nKK_s K_i K_L}$$

(d) $K_L = \infty, \quad q_o = q_2 = nq_m$. J_L is reflected to motor side so $J_T = J_m + n^2 J_L$.

State equations:

$$\frac{d\omega_m}{dt} = -\frac{B_m}{J_T}\omega_m + \frac{K_i}{J_T} i_a \qquad \frac{d\theta_m}{dt} = \omega_m \qquad \frac{di_a}{dt} = -\frac{R_a}{L_a} i_a + \frac{KK_s}{L_a}\theta_r - \frac{KK_s}{L_a} n\theta_m - \frac{K_b}{L_a}\omega_m$$

State diagram:



Forward-path transfer function:

$$\frac{\Theta_o(s)}{\Theta_e(s)} = \frac{KK_sK_in}{s\left[J_TL_as^2 + \left(R_aJ_T + B_mL_a\right)s + R_aB_m + K_iK_b\right]}$$

Closed-loop transfer function:

$$\frac{\Theta_o(s)}{\Theta_r(s)} = \frac{KK_sK_in}{J_TL_as^3 + \left(R_aJ_T + B_mL_a\right)s^2 + \left(R_aB_m + K_iK_b\right)s + KK_sK_in}$$

From part (c), when

$K_L = \infty$, all the terms without K_L in $\Theta_o(s)/\Theta_e(s)$ and $\Theta_o(s)/\Theta_r(s)$ can be neglected.

The same results as above are obtained.

4-15 (a) System equations:

$$f = K_ii_a = M_T\frac{dv}{dt} + B_Tv \qquad e_a = R_ai_a + \left(L_a + L_{as}\right)\frac{di_a}{dt} - L_{as}\frac{di_s}{dt} + e_b \qquad 0 = R_si_s + \left(L_s + L_{as}\right)\frac{di_s}{dt} - L_{as}\frac{di_a}{dt}$$

(b) Take the Laplace transform on both sides of the last three equations, with zero initial conditions, we have

$$K_iI_a(s) = \left(M_Ts + B_T\right)V(s) \qquad E_a(s) = \left[R_a + \left(L_a + L_{as}\right)s\right]I_a(s) - L_{as}sI_s(s) + K_bV(s)$$

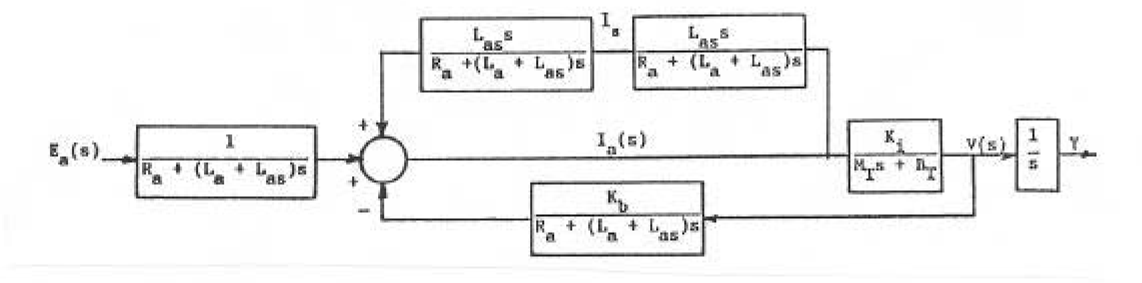
$$0 = -L_{as}sI_a(s) + \left[R_s + s\left(L_s + L_{as}\right)\right]I_s(s)$$

Rearranging these equations, we get

$$V(s) = \frac{K_i}{M_Ts + B_T}I_a(s) \qquad Y(s) = \frac{V(s)}{s} = \frac{K_i}{s\left(M_Ts + B_T\right)}I_a(s)$$

$$I_a(s) = \frac{1}{R_a + \left(L_a + L_{as}\right)s}\left[E_a(s) + L_{as}sI_s(s) - K_bV(s)\right] \qquad I_s(s) = \frac{L_{as}s}{R_a + \left(L_a + L_{as}\right)s}I_a(s)$$

Block diagram:



(c) Transfer function:

$$\frac{Y(s)}{E_a(s)}=\frac{K_i\left[R_s+\left(L_s+L_{as}\right)s\right]}{s\left[R_a+\left(L_a+L_{as}\right)s\right]\left[R_s+\left(L_s+L_{as}\right)s\right]\left(M_T s+B_T\right)+K_iK_b\left[R_s+\left(L_a+L_{as}\right)s\right]-L_{as}^2s^2\left(M_T s+B_T\right)}$$

4-16 (a) Cause-and-effect equations:

$$q_e=q_r-q_L \qquad e=K_sq_e \qquad K_s=1\text{ V/rad} \qquad e_a=Ke \qquad i_a=\frac{e_a-e_b}{R_a}$$

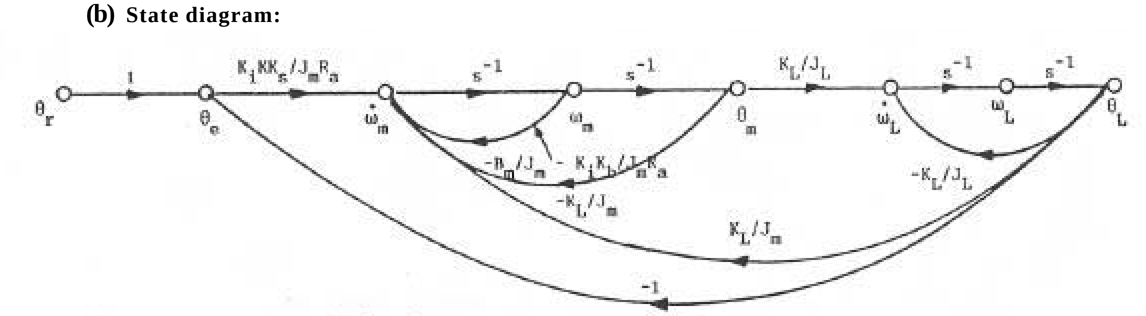
$$T_m=K_ii_a \qquad \frac{dw_m}{dt}=\frac{1}{J_m}T_m-\frac{B_m}{J_m}w-\frac{K_L}{J_m}\left(q_m-q_L\right) \qquad \frac{dw_L}{dt}=\frac{K_L}{J_L}\left(q_m-q_L\right) \qquad e_b=K_bw_m$$

$$K_b=15.5\text{ V / KRPM}=\frac{15.5}{1000\times 2p/60}=0.148\text{ V / rad / sec}$$

State equations:

$$\frac{dq_L}{dt}=w_L \qquad \frac{dw_L}{dt}=\frac{K_L}{J_L}q_m-\frac{K_L}{J_L}q_L \qquad \frac{dq_m}{dt}=w_m \qquad \frac{dw_m}{dt}=-\frac{B_m}{J_m}w_m-\frac{K_L}{J_m}q_L+\frac{1}{J_m}\frac{K_i}{R_a}\left(KK_sq_e-K_bw_m\right)$$

(b) State diagram:



(c) Forward-path transfer function:

$$G(s)=\frac{K_iK\,K_sK_L}{s\left[J_mJ_LR_as^3+\left(B_mR_a+K_iK_b\right)J_Ls^2+R_aK_L\left(J_L+J_m\right)s+K_L\left(B_mR_a+K_iK_b\right)\right]}$$

$$J_mR_aJ_L=0.03\times 1.15\times 0.05=0.001725 \qquad B_mR_aJ_L=10\times 1.15\times 0.05=0.575 \qquad K_iK_bJ_L=21\times 0.148\times 0.05=0.1554$$

$R_a K_L J_L = 1.15 \times 50000 \times 0.05 = 2875 \qquad R_a K_L J_m = 1.15 \times 50000 \times 0.03 = 1725 \qquad K_i K K_s K_L = 21 \times 1 \times 50000 \; K = 1050000 \, K$

$$K_L \left(B_m R_a + K_i K_b \right) = 50000(10 \times 1.15 + 21 \times 0.148) = 730400$$

$$G(s) = \frac{608.7 \times 10^6 K}{s\left(s^3 + 423.42 s^2 + 2.6667 \times 10^6 s + 4.2342 \times 10^8\right)}$$

(d) Closed-loop transfer function:

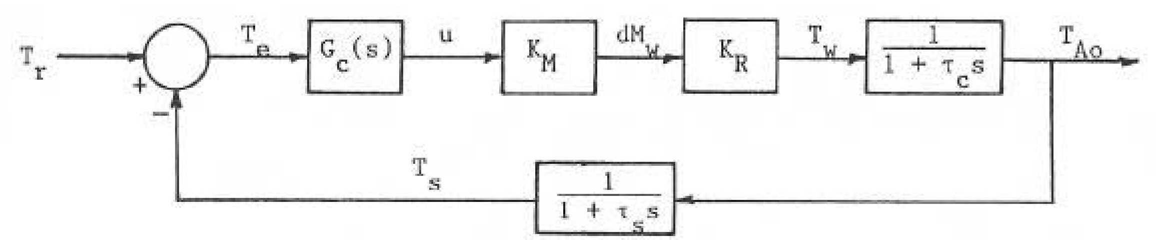
$$M(s) = \frac{\Theta_L(s)}{\Theta_r(s)} = \frac{G(s)}{1+G(s)} = \frac{K_i K K_s K_L}{J_m J_L R_a s^4 + \left(B_m R_a + K_i K_b\right) J_L s^3 + R_a K_L \left(J_L + J_m\right) s^2 + K_L \left(B_m R_a + K_i K_b\right) s + K_i K K_s K_L}$$

$$M(s) = \frac{6.087 \times 10^8 K}{s^4 + 423.42 s^3 + 2.6667 \times 10^6 s^2 + 4.2342 \times 10^8 s + 6.087 \times 10^8 K}$$

Characteristic equation roots:

$K = 1$	$K = 2738$	$K = 5476$
$s = -1.45$	$s = \pm j1000$	$s = 405 \pm j1223.4$
$s = -159.88$	$s = -211.7 \pm j1273.5$	$s = -617.22 \pm j1275$
$s = -131.05 \pm j1614.6$		

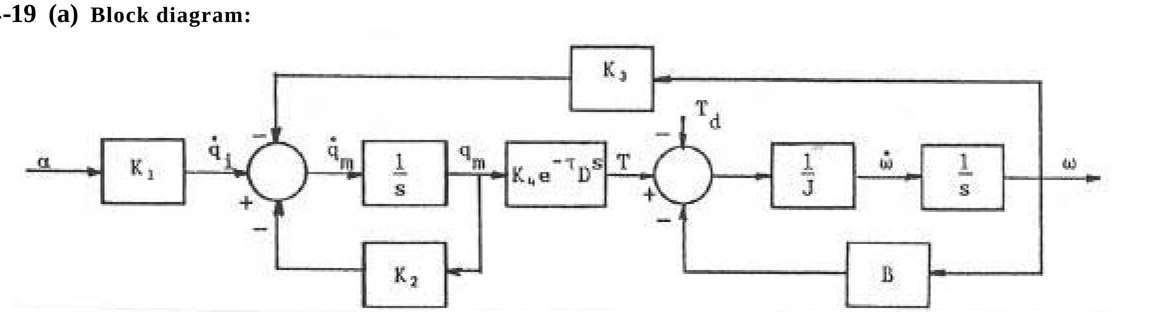
4-17 (a) Block diagram:



(b) Transfer function:

$$\frac{T_{AO}(s)}{T_r(s)} = \frac{K_M K_R}{\left(1+t_c s\right)\left(1+t_s s\right) + K_m K_R} = \frac{3.51}{20s^2 + 12s + 4.51}$$

4-19 (a) Block diagram:



(b) Transfer function:

$$\frac{\Omega(s)}{a(s)} = \frac{K_1 K_4 e^{-t_D s}}{Js^2 + \left(JK_L + B\right)s + K_2 B + K_3 K_4 e^{-t_D s}}$$

(c) Characteristic equation:

$$Js^2 + \left(JK_L + B\right)s + K_2 B + K_3 K_4 e^{-t_D s} = 0$$

(d) Transfer function:

$$\frac{\Omega(s)}{a(s)} \cong \frac{K_1 K_4 \left(2 - t_D s\right)}{\Delta(s)}$$

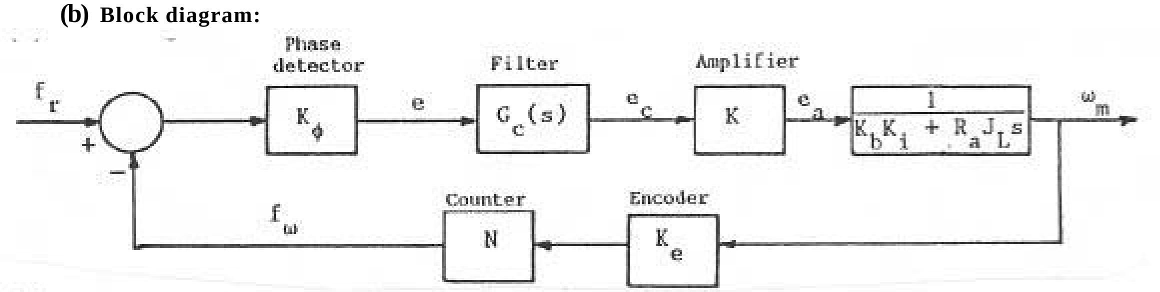
Characteristic equation:

$$\Delta(s) \cong J t_D s^3 + \left(2J + JK_2 t_D + Bt_D\right)s^2 + \left(2JK_2 + 2B - t_D K_2 B - t_D K_3 K_4\right)s + 2\left(K_2 B + K_3 K_4\right) = 0$$

4-19 (a) Transfer function:

$$G(s) = \frac{E_c(s)}{E(s)} = \frac{1 + R_2 Cs}{1 + \left(R_1 + R_2\right)Cs}$$

(b) Block diagram:



(c) Forward-path transfer function:

$$\frac{\Omega_m(s)}{E(s)} = \frac{K\left(1 + R_2 Cs\right)}{\left[1 + \left(R_1 + R_2\right)Cs\right]\left(K_b K_i + R_a J_L s\right)}$$

(d) Closed-loop transfer function:

$$\frac{\Omega_m(s)}{F_r(s)} = \frac{K_f K\left(1 + R_2 Cs\right)}{\left[1 + \left(R_1 + R_2\right)Cs\right]\left(K_b K_i + R_a J_L s\right) + K_f K K_e N\left(1 + R_2 Cs\right)}$$

(e)

$$G_c(s) = \frac{E_c(s)}{E(s)} = \frac{\left(1 + R_2 Cs\right)}{R_1 Cs}$$

Forward-path transfer function:

$$\frac{\Omega_m(s)}{E(s)} = \frac{K\left(1 + R_2 Cs\right)}{R_1 Cs\left(K_b K_i + R_a J_L s\right)}$$

Closed-loop transfer function:

$$\frac{\Omega_m(s)}{F_r(s)} = \frac{K_f K\left(1 + R_2 C s\right)}{R_1 C s\left(K_b K_i + R_a J_L s\right) + K_f K K_e N\left(1 + R_2 C s\right)}$$

$K_e = 36$ pulses/rev $= 36/2\pi$ pulses/rad $= 5.73$ pulses/rad.

(f) $f_r = 120$ pulses/sec $\quad \omega_m = 200$ RPM $= 200(2\pi/60)$ rad/sec

$f_\omega = N K_e \omega_m = 120$ pulses/sec $= N(36/2\pi)200(2\pi/60) = 120N$ pulses/sec

Thus, $N = 1$. For $\omega_m = 1800$ RPM, $120 = N(36/2\pi)1800(2\pi/60) = 1080N$. Thus, $N = 9$.

4-20 (a) Differential equations:

$$K_i i_a = J_m \frac{d^2\theta_m}{dt^2} + B_m \frac{d\theta_m}{dt} + K\left(\theta_m - \theta_L\right) + B\left(\frac{d\theta_m}{dt} - \frac{d\theta_L}{dt}\right)$$

$$K\left(\theta_m - \theta_L\right) + B\left(\frac{d\theta_m}{dt} - \frac{d\theta_L}{dt}\right) = \left(J_L \frac{d^2\theta_L}{dt^2} + B_L \frac{d\theta_L}{dt}\right) + T_L$$

(b) Take the Laplace transform of the differential equations with zero initial conditions, we get

$$K_i I_a(s) = \left(J_m s^2 + B_m s + Bs + K\right)\Theta_m(s) + \left(Bs + K\right)\Theta_L(s)$$

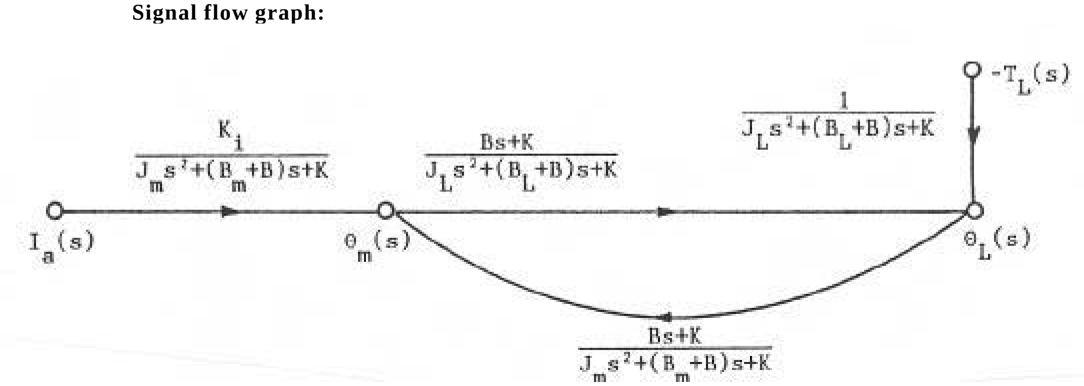
$$\left(Bs + K\right)\Theta_m(s) - \left(Bs + K\right)\Theta_L(s) = \left(J_L s^2 + B_L\right)s\Theta_L(s) + T_L(s)$$

Solving for $\Theta_m(s)$ and $\Theta_L(s)$ from the last two equations, we have

$$\Theta_m(s) = \frac{K_i}{J_m s^2 + \left(B_m + B\right)s + K} I_a(s) + \frac{Bs + K}{J_m s^2 + \left(B_m + B\right)s + K}\Theta_L(s)$$

$$\Theta_L(s) = \frac{Bs + K}{J_L s^2 + \left(B_L + B\right)s + K}\Theta_m(s) - \frac{T_L(s)}{J_L s^2 + \left(B_L + B\right)s + K}$$

Signal flow graph:



(c) Transfer matrix:

$$\begin{bmatrix}\Theta_m(s)\\ \Theta_L(s)\end{bmatrix} = \frac{1}{\Delta_o(s)}\begin{bmatrix} K_i\left[J_L s^2 + \left(B_L + B\right)s + K\right] & Bs + K \\ K_i\left(Bs + K\right) & J_m s^2 + \left(B_m + B\right)s + K\end{bmatrix}\begin{bmatrix} I_a(s) \\ -T_L(s)\end{bmatrix}$$

$$\Delta_o(s) = J_L J_m s^3 + \left[J_L\left(B_m + B\right) + J_m\left(B_L + B\right)\right] s^3 + \left[B_L B_m + \left(B_L + B_M\right) B + \left(J_m + J_L\right) K\right] s^2 + K\left(B_L + B\right) s$$

4-21 (a) Nonlinear differential equations:

$$\frac{dx(t)}{dt} = v(t) \qquad \frac{dv(t)}{dt} = -k(v) - g(x) + f(t) = -Bv(t) + f(t)$$

With $R_a = 0$, $\quad f(t) = \dfrac{e(t)}{K_b v(t)} = K_f i_f(t) = K_f i_f(t) = K_f i_a(t)$ $\qquad$ Then, $\quad i_a(t) = \dfrac{e(t0}{K_b K_f v(t)}$

$f(t) = K_i f(t) i_a(t) = \dfrac{K_i e^2(t)}{K_b^2 K_f v^2(t)}.$ $\qquad$ Thus, $\quad \dfrac{dv(t)}{dt} = -Bv(t) + \dfrac{K_i}{K_b^2 K_f v^2(t)} e^2(t)$

(b) State equations: $i_a(t)$ **as input.**

$$\frac{dx(t)}{dt} = v(t) \qquad \frac{dv(t)}{dt} = -Bv(t) + K_i K_f i_a^2(t)$$

(c) State equations: $f(t)$ **as input.**

$$f(t) = K_i K_f i_a^2(t) \qquad i_a(t) = i_f(t) = \frac{f(t)}{K_f}$$

$$\frac{dx(t)}{dt} = v(t) \qquad \frac{dv(t)}{dt} = -Bv(t) + \frac{K_i}{K_f} f^2(t)$$

4-22 (a) Force and torque equations:

Broom: vertical direction: $\quad f_v - M_b g = M_b \dfrac{d^2(L\cos\theta)}{dt}$

horizontal direction: $\quad f_x = M_b \dfrac{d^2\left[x(t) + L\sin\theta\right]}{dt^2}$

rotation about CG: $\quad J_b \dfrac{d^2\theta}{dt^2} = f_y L\sin\theta - f_x L\cos\theta$

Car: horizontal direction: $\quad u(t) = f_x + M_c \dfrac{d^2 x(t)}{dt^2} \qquad J_b = \dfrac{M_b L^2}{3}$

(b) State equations: Define the state variables as $x_1 = \theta$, $\quad x_2 = \dfrac{d\theta}{dt}$, $\quad x_3 = x$, and $\quad x_4 = \dfrac{dx}{dt}$.

Eliminating f_x and f_y from the equations above, and $\sin x_1 \cong x_1$ and $\cos x_1 \cong 1$.

$$\frac{dx_1}{dt} = x_2 \qquad \frac{dx_2}{dt} = \frac{\left(M_c + M_b\right) g x_1 - M_b L x_2^2 x_1 - u(t)}{L\left[4\left(M_b + M_c\right)/3 - M_b\right]}$$

$$\frac{dx_3}{dt} = x_4 \qquad \frac{dx_4}{dt} = \frac{u(t) + M_b L x_2^2 x_1 - 3M_b g x_1 / 4}{\left(M_b + M_c\right) - 3M_b / 4}$$

(c) Linearization:

$$\frac{\partial f_1}{\partial x_1}=0 \qquad \frac{\partial f_1}{\partial x_2}=1 \qquad \frac{\partial f_1}{\partial x_3}=0 \qquad \frac{\partial f_1}{\partial x_4}=0 \qquad \frac{\partial f_1}{\partial u}=0$$

$$\frac{\partial f_2}{\partial x_1}=\frac{\left(M_b+M_c\right)g-M_b x_2^2}{L\left(M_b+M_c-3M_b/4\right)}=0 \qquad \frac{\partial f_2}{\partial x_2}=\frac{-2M_b x_1 x_2}{L\left(M_b+M_c-3M_b/4\right)}=0 \qquad \frac{\partial f_2}{\partial x_3}=0$$

$$\frac{\partial f_2}{\partial x_4}=0 \qquad \frac{\partial f_2}{\partial u}=\frac{-1}{L\left[4\left(M_b+M_c\right)/3-M_b\right]}$$

$$\frac{\partial f_3}{\partial x_1}=0 \qquad \frac{\partial f_3}{\partial x_2}=0 \qquad \frac{\partial f_3}{\partial x_3}=0 \qquad \frac{\partial f_3}{\partial x_4}=0 \qquad \frac{\partial f_3}{\partial u}=0$$

$$\frac{\partial f_4}{\partial x_1}=\frac{M_b L x_2^2-3M_b g/4}{\left(M_b+M_c\right)-3M_b/4} \qquad \frac{\partial f_4}{\partial x_2}=\frac{2M_b L x_1 x_2}{\left(M_b+M_c\right)-3M_b/4}$$

$$\frac{\partial f_4}{\partial x_3}=0 \qquad \frac{\partial f_4}{\partial x_4}=0 \qquad \frac{\partial f_4}{\partial u}=\frac{1}{\left(M_b+M_c\right)-3M_b/4}$$

Linearized state equations:

$$\begin{bmatrix}\Delta\dot{x}_1\\ \Delta\dot{x}_2\\ \Delta\dot{x}_3\\ \Delta\dot{x}_4\end{bmatrix}=\begin{bmatrix}0 & 1 & 0 & 0\\ \dfrac{3\left(M_b+M_c\right)g}{L\left(M_b+4M_c\right)} & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ \dfrac{-3M_b g}{M_b+4M_c} & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}\Delta x_1\\ \Delta x_2\\ \Delta x_3\\ \Delta x_4\end{bmatrix}+\begin{bmatrix}0\\ \dfrac{-3}{L\left(M_b+4M_c\right)}\\ 0\\ \dfrac{4}{M_b+4M_c}\end{bmatrix}\Delta u$$

4-23 (a) Differential equations: $\left[L(y)=\dfrac{L}{y}\right]$

$$e(t)=Ri(t)+\frac{d\left[L(y)i(t)\right]}{dt}=Ri(t)+i(t)\frac{dL(y)}{dy}\frac{dy(t)}{dt}+\frac{L}{y}\frac{di(t)}{dt}=Ri(t)-\frac{L}{y^2}i(t)\frac{dy(t)}{dt}+\frac{L}{y}\frac{di(t)}{dt}$$

$My(t)=Mg-\dfrac{Ki^2(t)}{y^2(t)}$ At equilibrium, $\dfrac{di(t)}{dt}=0$, $\dfrac{dy(t)}{dt}=0$, $\dfrac{d^2y(t)}{dt^2}=0$

Thus, $i_{eq}=\dfrac{E_{eq}}{R}$ $\qquad \dfrac{dy_{eq}}{dt}=0$ $\qquad y_{eq}=\dfrac{E_{eq}}{R}\sqrt{\dfrac{K}{Mg}}$

(b) Define the state variables as $x_1=i$, $x_2=y$, and $x_3=\dfrac{dy}{dt}$.

Then, $x_{1eq}=\dfrac{E_{eq}}{R}$ $\qquad x_{2eq}=\dfrac{E_{eq}}{R}\sqrt{\dfrac{K}{Mg}}$ $\qquad x_{3eq}=0$

The differential equations are written in state equation form:

$$\frac{dx_1}{dt}=-\frac{R}{L}x_1x_2+\frac{x_1x_3}{x_2}+\frac{x_2}{L}e=f_1 \qquad \frac{dx_2}{dt}=x_3=f_2 \qquad \frac{dx_3}{dt}=g-\frac{K}{M}\frac{x_1^2}{x_2^2}=f_3$$

(c) Linearization:

$$\frac{\partial f_1}{\partial x_1} = -\frac{R}{L}x_{2eq} + \frac{x_{3eq}}{x_{2eq}} = -\frac{E_{eq}}{L}\sqrt{\frac{K}{Mg}} \qquad \frac{\partial f_1}{\partial x_2} = -\frac{R}{L}x_{1eq} - \frac{x_1 x_3}{x_2^2} + \frac{E_{eq}}{L} = 0 \qquad \frac{\partial f_1}{\partial x_3} = \frac{x_{1eq}}{x_{2eq}} = \sqrt{\frac{Mg}{K}}$$

$$\frac{\partial f_1}{\partial e} = \frac{x_{2eq}}{L} = \frac{1}{L}\sqrt{\frac{K}{Mg}}\frac{E_{eq}}{R} \qquad \frac{\partial f_2}{\partial x_1} = 0 \qquad \frac{\partial f_2}{\partial x_2} = 0 \qquad \frac{\partial f_2}{\partial x_3} = 1 \qquad \frac{\partial f_2}{\partial e} = 0$$

$$\frac{\partial f_3}{\partial x_1} = -\frac{2K}{M}\frac{x_{1eq}}{x_{2eq}^2} = -\frac{2Rg}{E_{eq}} \qquad \frac{\partial f_3}{\partial x_2} = \frac{2K}{M}\frac{x_{1eq}^2}{x_{2eq}^3} = \frac{2Rg}{E_{eq}}\sqrt{\frac{Mg}{K}} \qquad \frac{\partial f_3}{\partial e} = 0$$

The linearized state equations about the equilibrium point are written as: $\Delta\dot{\mathbf{x}} = \mathbf{A}^*\Delta\mathbf{x} + \mathbf{B}^*\Delta e$

$$\mathbf{A}^* = \begin{bmatrix} -\dfrac{E_{eq}}{L}\sqrt{\dfrac{K}{Mg}} & 0 & \sqrt{\dfrac{Mg}{K}} \\ 0 & 0 & 0 \\ -\dfrac{2Rg}{E_{eq}} & \dfrac{2Rg}{E_{eq}}\sqrt{\dfrac{Mg}{K}} & 0 \end{bmatrix} \qquad \mathbf{B}^* = \begin{bmatrix} \dfrac{E_{eq}}{RL}\sqrt{\dfrac{K}{Mg}} \\ 0 \\ 0 \end{bmatrix}$$

4-24 (a) Differential equations:

$$M_1\frac{d^2y_1(t)}{dt^2} = M_1 g - B\frac{dy_1(t)}{dt} - \frac{Ki^2(t)}{y_1^2(t)} + Ki^2(t)\frac{1}{\left[y_2(t) - y_1(t)\right]^2}$$

$$M_2\frac{d^2y_2(t)}{dt^2} = M_2 g^2 - B\frac{dy_2(t)}{dt} - \frac{Ki^2(t)}{\left[y_2(t) - y_1(t)\right]^2}$$

Define the state variables as $x_1 = y_1$, $x_2 = \dfrac{dy_1}{dt}$, $x_3 = y_2$, $x_4 = \dfrac{dy_2}{dt}$.

The state equations are:

$$\frac{dx_1}{dt} = x_2 \qquad M_1\frac{dx_2}{dt} = M_1 g - Bx_2 - \frac{Ki^2}{x_1^2} + \frac{Ki^2}{\left(x_3 - x_1\right)^2} \qquad \frac{dx_3}{dt} = x_4 \qquad M_2\frac{dx_4}{dt} = M_2 g - Bx_4 - \frac{Ki^2}{\left(x_3 - x_1\right)^2}$$

At equilibrium, $\dfrac{dx_1}{dt} = 0$, $\dfrac{dx_2}{dt} = 0$, $\dfrac{dx_3}{dt} = 0$, $\dfrac{dx_4}{dt} = 0$. Thus , $x_{2eq} = 0$ and $x_{4eq} = 0$.

$$M_1 g - \frac{KI^2}{X_1^2} + \frac{KI^2}{\left(X_3 - X_1\right)^2} = 0 \qquad M_2 g - \frac{KI^2}{\left(X_3 - X_1\right)^2} = 0$$

Solving for I, with $X_1 = 1$, we have

$$Y_2 = X_3 = 1 + \left(\frac{M_1 + M_2}{M_2}\right)^{1/2} \qquad I = \left(\frac{\left(M_1 + M_2\right)g}{K}\right)^{1/2}$$

(b) Nonlinear state equations:

$$\frac{dx_1}{dt} = x_2 \qquad \frac{dx_2}{dt} = g - \frac{B}{M_1}x_2 - \frac{K}{M_1 x_1^2}i^2 + \frac{Ki^2}{M_1\left(x_3 - x_1\right)^2} \qquad \frac{dx_3}{dt} = x_4 \qquad \frac{dx_4}{dt} = g - \frac{B}{M_2}x_4 - \frac{Ki^2}{M_2\left(x_3 - x_1\right)^2}$$

(c) Linearization:

$$\frac{\partial f_1}{\partial x_1} = 0 \qquad \frac{\partial f_1}{\partial x_2} = 0 \qquad \frac{\partial f_1}{\partial x_3} = 0 \qquad \frac{\partial f_1}{\partial x_4} = 0 \qquad \frac{\partial f_1}{\partial i} = 0$$

$$\frac{\partial f_2}{\partial x_1} = \frac{2KI^2}{M_1 x_1^3} + \frac{2KI^2}{M_1(X_3 - X_1)^3} \qquad \frac{\partial f_2}{\partial x_2} = -\frac{B}{M_1} \qquad \frac{\partial f_2}{\partial x_3} = \frac{-2KI^2}{M_1\left(X_3 - X_1\right)^3} \qquad \frac{\partial f_2}{\partial x_4} = 0$$

$$\frac{\partial f_2}{\partial i} = \frac{2KI}{M_1}\left(\frac{-1}{X_1^2} + \frac{1}{\left(X_3 - X_1\right)^2}\right) \qquad \frac{\partial f_3}{\partial x_1} = 0 \qquad \frac{\partial f_3}{\partial x_2} = 0 \qquad \frac{\partial f_3}{\partial x_3} = 0 \qquad \frac{\partial f_3}{\partial x_4} = 1 \qquad \frac{\partial f_3}{\partial i} = 0$$

$$\frac{\partial f_4}{\partial x_1} = \frac{-2KI^2}{M_2\left(X_3 - X_1\right)^3} \qquad \frac{\partial f_4}{\partial x_2} = 0 \qquad \frac{\partial f_4}{\partial x_3} = \frac{2KI^2}{M_2\left(X_3 - X_1\right)^3} \qquad \frac{\partial f_4}{\partial x_4} = -\frac{B}{M_2} \qquad \frac{\partial f_4}{\partial i} = \frac{-2KI}{M_2\left(X_3 - X_1\right)^2}$$

Linearized state equations: $M_1 = 2,\ M_2 = 1,\ g = 32.2,\ B = 0.1,\ K = 1.$

$$I = \left(\frac{32.2(1+2)}{1}\right)^{1/2} X_1 = \sqrt{96.6}X_1 = 9.8285X_1 \qquad X_1 = \frac{1}{9.8285} = 1$$

$$X_3 = \left(1 + \sqrt{1+2}\right)X_1 = 2.732X_1 = Y_2 = 2.732 \qquad X_3 - X_1 = 1.732$$

$$\mathbf{A}^* = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{2KI^2}{M_1}\left(\frac{1}{X_1^3} + \frac{1}{\left(X_3 - X_1\right)^3}\right) & \frac{-B}{M_1} & \frac{-2KI^2}{M_1\left(X_3 - X_1\right)^3} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{-2KI^2}{M_2\left(X_3 - X_1\right)^3} & 0 & \frac{2KI^2}{M_2\left(X_3 - X_1\right)^3} & \frac{-B}{M_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 115.2 & -0.05 & -18.59 & 0 \\ 0 & 0 & 0 & 1 \\ -37.18 & 0 & 37.18 & -0.1 \end{bmatrix}$$

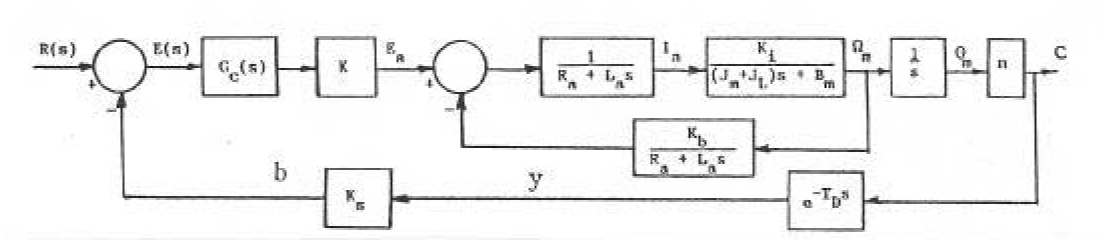
$$\mathbf{B}^* = \begin{bmatrix} 0 \\ \frac{2KI}{M_1}\left(\frac{-1}{X_1^2} + \frac{1}{\left(X_3 - X_1\right)^2}\right) \\ 0 \\ \frac{-2KI}{M_2\left(X_3 - X_1\right)^2} \end{bmatrix} = \begin{bmatrix} 0 \\ -6.552 \\ 0 \\ -6.552 \end{bmatrix}$$

4-25 (a) System equations:

$$T_m = K_i i_a = \left(J_m + J_L \right) \frac{dw_m}{dt} + B_m w_m \qquad e_a = R_a i_a + L_a \frac{di_a}{dt} + K_b w_m \qquad y = nq_m \qquad y = y\left(t - T_D \right)$$

$$T_D = \frac{d}{V} \ \text{(sec)} \qquad e = r - b \qquad b = K_s y \qquad E_a(s) = KG_c(s)E(s)$$

Block diagram:



(b) **Forward-path transfer function:**

$$\frac{Y(s)}{E(s)} = \frac{KK_i nG_c(s)e^{-T_D s}}{s\left\{\left(R_a + L_a s\right)\left[\left(J_m + J_L \right)s + B_m\right] + K_b K_i\right\}}$$

Closed-loop transfer function:

$$\frac{Y(s)}{R(s)} = \frac{KK_i nG_c(s)e^{-T_D s}}{s\left(R_a + L_a s\right)\left[\left(J_m + J_L \right)s + B_m\right] + K_b K_i s + KG_c(s)K_i ne^{-T_D s}}$$

Chapter 5 STATE VARIABLE ANALYSIS OF LINEAR DYNAMIC SYSTEMS

5-1 **(a)** **State variables:** $x_1 = y, \quad x_2 = \frac{dy}{dt}$

State equations:

$$\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 5 \end{bmatrix} r$$

Output equation:

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1$$

(b) **State variables:** $x_1 = y, \quad x_2 = \frac{dy}{dt}, \quad x_3 = \frac{d^2 y}{dt^2}$

State equations:

$$\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \\ \frac{dx_3}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2.5 & -1.5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0.5 \end{bmatrix} r$$

Output equation:

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1$$

(c) **State variables:** $x_1 = \int_0^t y(t)\,dt, \quad x_2 = \frac{dx_1}{dt}, \quad x_3 = \frac{dy}{dt}, \quad x_4 = \frac{d^2 y}{dt^2}$

State equations:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -1 & -3 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} r$$

Output equaton:

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_1$$

(d) **State variables:** $x_1 = y, \quad x_2 = \frac{dy}{dt}, \quad x_3 = \frac{d^2 y}{dt^2}, \quad x_4 = \frac{d^3 y}{dt^3}$

State equations:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -2.5 & 0 & -1.5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} r$$

Output equation:

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_1$$

5-2 We shall first show that

$$\Phi(s) = \left(s\mathbf{I} - \mathbf{A}\right)^{-1} = \frac{\mathbf{I}}{s} + \frac{\mathbf{A}}{s^2} + \frac{1}{2!}\frac{\mathbf{A}^2}{s^2} + \cdots$$

We multiply both sides of the equation by $\left(s\mathbf{I} - \mathbf{A}\right)$, and we get $\mathbf{I} = \mathbf{I}$. Taking the inverse Laplace transform on both sides of the equation gives the desired relationship for $\phi(t)$.

5-3 (a) Characteristic equation: $\Delta(s) = |s\mathbf{I} - \mathbf{A}| = s^2 + s + 2 = 0$

Eigenvalues: $s = -0.5 + j1.323, \quad -0.5 - j1.323$

State transition matrix:

$$\phi(t) = \begin{bmatrix} \cos 1.323t + 0.378\sin 1.323t & 0.756\sin 1.323t \\ -1.512\sin 1.323t & -1.069\sin\left(1.323t - 69.3^o\right) \end{bmatrix} e^{-0.5t}$$

(b) Characteristic equation: $\Delta(s) = |s\mathbf{I} - \mathbf{A}| = s^2 + 5s + 4 = 0$ **Eigenvalues:** $s = -4, \quad -1$

State transition matrix:

$$\phi(t) = \begin{bmatrix} 1.333e^{-t} - 0.333e^{-4t} & 0.333e^{-t} - 0.333e^{-4t} \\ -1.333e^{-t} - 1.333e^{-4t} & -0.333e^{-t} + 1.333e^{-4t} \end{bmatrix}$$

(c) Characteristic equation: $\Delta(s) = (s+3)^2 = 0$ **Eigenvalues:** $s = -3, \quad -3$

State transition matrix:

$$\phi(t) = \begin{bmatrix} e^{-3t} & 0 \\ 0 & e^{-3t} \end{bmatrix}$$

(d) Characteristic equation: $\Delta(s) = s^2 - 9 = 0$ **Eigenvalues:** $s = -3, \quad 3$

State transition matrix:

$$\phi(t) = \begin{bmatrix} e^{3t} & 0 \\ 0 & e^{-3t} \end{bmatrix}$$

(e) Characteristic equation: $\Delta(s) = s^2 + 4 = 0$ **Eigenvalues:** $s = j2, \quad -j2$

State transition matrix:

$$\phi(t) = \begin{bmatrix} \cos 2t & \sin 2t \\ -\sin 2t & \cos 2t \end{bmatrix}$$

(f) Characteristic equation: $\Delta(s) = s^3 + 5s^2 + 8s + 4 = 0$ **Eigenvalues:** $s = -1, \quad -2, \quad -2$

State transition matrix:

$$\phi(t) = \begin{bmatrix} e^{-t} & 0 & 0 \\ 0 & e^{-2t} & te^{-2t} \\ 0 & 0 & e^{-2t} \end{bmatrix}$$

(g) Characteristic equation: $\Delta(s) = s^3 + 15s^2 + 75s + 125 = 0$ **Eigenvalues:** $s = -5, \quad -5, \quad -5$

$$\phi(t) = \begin{bmatrix} e^{-5t} & te^{-5t} & 0 \\ 0 & e^{-5t} & te^{-5t} \\ 0 & 0 & e^{-5t} \end{bmatrix}$$

5-4 State transition equation: $\mathbf{x}(t) = \phi(t)\mathbf{x}(t) + \int_0^t \phi(t-\tau)\mathbf{Br}(\tau)d\tau$ $\phi(t)$ for each part is given in Problem 5-3.

(a)

$$\int_0^t f(t-\tau)\mathbf{B}r(\tau)d\tau = \mathcal{L}^{-1}\left[(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}R(s)\right] = \mathcal{L}^{-1}\left\{\frac{1}{\Delta(s)}\begin{bmatrix} s+1 & 1 \\ -2 & s \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}\frac{1}{s}\right\}$$

$$= \mathcal{L}^{-1}\begin{bmatrix} \dfrac{s+2}{s\left(s^2+s+2\right)} \\ \dfrac{s-2}{s\left(s^2+s+2\right)} \end{bmatrix} = \begin{bmatrix} 1+0.378\sin 1.323t-\cos 1.323t \\ -1+1.134\sin 1.323t+\cos 1.323t \end{bmatrix} \qquad t \geq 0$$

(b)

$$\int_0^t f(t-\tau)\mathbf{B}r(\tau)d\tau = \mathcal{L}^{-1}\left[(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}R(s)\right] = \mathcal{L}^{-1}\left\{\frac{1}{\Delta(s)}\begin{bmatrix} s+5 & 1 \\ -4 & s \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}\frac{1}{s}\right\}$$

$$= \mathcal{L}^{-1}\begin{bmatrix} \dfrac{s+6}{s(s+1)(s+2)} \\ \dfrac{s-4}{s(s+1)(s+4)} \end{bmatrix} = \mathcal{L}^{-1}\begin{bmatrix} \dfrac{1.5}{s}-\dfrac{1.67}{s+1}+\dfrac{0.167}{s+4} \\ \dfrac{-1}{s}+\dfrac{1.67}{s+1}-\dfrac{0.667}{s+4} \end{bmatrix} = \begin{bmatrix} 1.5-1.67e^{-t}+0.167e^{-4t} \\ -1+1.67e^{-t}-0.667e-4t \end{bmatrix} \qquad t \geq 0$$

(c)

$$\int_0^t f(t-\tau)\mathbf{B}r(\tau)d\tau = \mathcal{L}^{-1}\left[(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}R(s)\right] = \mathcal{L}^{-1}\left\{\begin{bmatrix} \dfrac{1}{s+3} & 0 \\ 0 & \dfrac{1}{s+3} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\frac{1}{s}\right\}$$

$$= \mathcal{L}^{-1}\begin{bmatrix} 0 \\ \dfrac{1}{s(s+3)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0.333\left(1-e^{-3t}\right) \end{bmatrix} \qquad t \geq 0$$

(d)

$$\int_0^t f(t-\tau)\mathbf{B}r(\tau)d\tau = \mathcal{L}^{-1}\left[(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}R(s)\right] = \mathcal{L}^{-1}\left\{\begin{bmatrix} \dfrac{1}{s-3} & 0 \\ 0 & \dfrac{1}{s+3} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\frac{1}{s}\right\}$$

$$= \mathcal{L}^{-1}\begin{bmatrix} 0 \\ \dfrac{1}{s(s+3)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0.333\left(1-e^{-3t}\right) \end{bmatrix} \qquad t \geq 0$$

(e)

$$\int_0^t f(t-t)\mathbf{B}r(t)\,dt = \mathcal{L}^{-1}\left[\left(s\mathbf{I}-\mathbf{A}\right)^{-1}\mathbf{B}R(s)\right] = \mathcal{L}^{-1}\left\{\begin{bmatrix} \dfrac{1}{s^2+4} & 2 \\ \dfrac{-2}{s^2+4} & \dfrac{s}{s^2+4}\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}\frac{1}{s}\right\}$$

$$= \mathcal{L}^{-1}\begin{bmatrix}\dfrac{2}{s} \\ \dfrac{1}{\left(s^2+4\right)}\end{bmatrix} = \begin{bmatrix}2 \\ 0.5\sin 2t\end{bmatrix} \qquad t \ge 0$$

(f)

$$\int_0^t f(t-t)\mathbf{B}r(t)\,dt = \mathcal{L}^{-1}\left[\left(s\mathbf{I}-\mathbf{A}\right)^{-1}\mathbf{B}R(s)\right] = \mathcal{L}^{-1}\left\{\begin{bmatrix} \dfrac{1}{s+1} & 0 & 0 \\ 0 & \dfrac{1}{s+2} & \dfrac{1}{(s+2)^2} \\ 0 & 0 & \dfrac{1}{s+2}\end{bmatrix}\begin{bmatrix}0\\1\\0\end{bmatrix}\frac{1}{s}\right\}$$

$$= \mathcal{L}^{-1}\begin{bmatrix}0 \\ \dfrac{1}{s(s+2)} \\ 0\end{bmatrix} = \begin{bmatrix}0 \\ 0.5\left(1-e^{-2t}\right) \\ 0\end{bmatrix} \qquad t \ge 0$$

(g)

$$\int_0^t f(t-t)\mathbf{B}r(t)\,dt = \mathcal{L}^{-1}\left[\left(s\mathbf{I}-\mathbf{A}\right)^{-1}\mathbf{B}R(s)\right] = \mathcal{L}^{-1}\left\{\begin{bmatrix} \dfrac{1}{s+5} & \dfrac{1}{(s+5)^2} & 0 \\ 0 & \dfrac{1}{s+5} & \dfrac{1}{(s+5)^2} \\ 0 & 0 & \dfrac{1}{s+5}\end{bmatrix}\begin{bmatrix}0\\0\\1\end{bmatrix}\frac{1}{s}\right\}$$

$$= \mathcal{L}^{-1}\begin{bmatrix}0 \\ \dfrac{1}{s(s+5)^2} \\ \dfrac{1}{s(s+5)}\end{bmatrix} = \mathcal{L}^{-1}\begin{bmatrix}0 \\ \dfrac{0.04}{s}-\dfrac{0.04}{s+5}-\dfrac{0.2}{(s+5)^2} \\ \dfrac{0.2}{s}-\dfrac{0.2}{s+5}\end{bmatrix} = \begin{bmatrix}0 \\ 0.04-0.04e^{-5t}-0.2te^{-5t} \\ 0.2-0.2e^{-5t}\end{bmatrix}u_s(t)$$

5-5 **(a)** Not a state transition matrix, since $f(0) \neq \mathbf{I}$ (identity matrix).

(b) Not a state transition matrix, since $f(0) \neq \mathbf{I}$ (identity matrix).

(c) $f(t)$ is a state transition matrix, since $f(0) = \mathbf{I}$ and

$$[f(t)]^{-1} = \begin{bmatrix} 1 & 0 \\ 1-e^{-t} & e^{-t} \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ 1-e^{t} & e^{t} \end{bmatrix} = f(-t)$$

(d) $f(t)$ is a state transition matrix, since $f(0) = \mathbf{I}$, and

$$[f(t)]^{-1} = \begin{bmatrix} e^{2t} & -te^{2t} & t^2e^{2t}/2 \\ 0 & e^{2t} & -te^{2t} \\ 0 & 0 & e^{2t} \end{bmatrix} = f(-t)$$

5-6 (a) (1) Eigenvalues of A: 2.325, $-0.3376 + j0.5623$, $-0.3376 - j0.5623$

(2) Transfer function relation:

$$\mathbf{X}(s) = (s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}U(s) = \frac{1}{\Delta(s)}\begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 1 & 2 & s+3 \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}U(s) = \frac{1}{\Delta(s)}\begin{bmatrix} s^2+3s+2 & s+3 & 1 \\ -1 & s(s+3) & s \\ -s & -2s-1 & s^2 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}U(s) = \frac{1}{\Delta(s)}\begin{bmatrix} 1 \\ s \\ s^2 \end{bmatrix}U(s)$$

$$\Delta(s) = s^3 + 3s^2 + 2s + 1$$

(3) Output transfer function:

$$\frac{Y(s)}{U(s)} = \mathbf{C}(s)(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}\frac{1}{\Delta(s)}\begin{bmatrix} 1 \\ s \\ s^2 \end{bmatrix} = \frac{1}{s^3+3s^2+2s+1}$$

(b) (1) Eigenvalues of A: -1, -1.

(2) Transfer function relation:

$$\mathbf{X}(s) = (s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}U(s) = \frac{1}{\Delta(s)}\begin{bmatrix} s+1 & 1 \\ 0 & s+1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}U(s) = \begin{bmatrix} \dfrac{1}{(s+1)^2} \\ \dfrac{1}{(s+1)} \end{bmatrix}U(s) \qquad \Delta(s) = s^2 + 2s + 1$$

(3) Output transfer function:

$$\frac{Y(s)}{U(s)} = \mathbf{C}(s)(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B} = \begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} \dfrac{1}{(s+1)^2} \\ \dfrac{1}{s+1} \end{bmatrix} = \frac{1}{(s+1)^2} + \frac{1}{s+1} = \frac{s+2}{(s+1)^2}$$

(c) (1) Eigenvalues of A: 0, -1, -1.

(2) Transfer function relation:

$$\mathbf{X}(s) = (s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}U(s) = \frac{1}{\Delta(s)}\begin{bmatrix} s^2+2s=1 & s+2 & 1 \\ 0 & s(s+2)) & s \\ 0 & -s & s^2 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}U(s) = \frac{1}{\Delta(s)}\begin{bmatrix} 1 \\ s \\ s^2 \end{bmatrix}U(s) \qquad \Delta(s) = s(s^2+2s+1)$$

(3) Output transfer function:

$$\frac{Y(s)}{U(s)} = \mathbf{C}(s)\left(s\mathbf{I} - \mathbf{A}\right)^{-1}\mathbf{B} = \begin{bmatrix}1 & 1 & 0\end{bmatrix}\begin{bmatrix}1 \\ s \\ s^2\end{bmatrix} = \frac{s+1}{s(s+1)^2} = \frac{1}{s(s+1)}$$

5-7 We write $\frac{dy}{dt} = \frac{dx_1}{dt} + \frac{dx_2}{dt} = x_2 + x_3 \qquad \frac{d^2y}{dt} = \frac{dx_2}{dt} + \frac{dx_3}{dt} = -x_1 - 2x_2 - 2x_3 + u$

$$\frac{d\overline{\mathbf{x}}}{dt} = \begin{bmatrix}\frac{dx_1}{dt} \\ \frac{dy}{dt} \\ \frac{d^2y}{dt^2}\end{bmatrix} = \begin{bmatrix}0 & 1 & 0 \\ 0 & 1 & 1 \\ -1 & -2 & -2\end{bmatrix}\begin{bmatrix}x_1 \\ x_2 \\ x_3\end{bmatrix} + \begin{bmatrix}0 \\ 0 \\ 1\end{bmatrix}u \qquad (1)$$

$$\overline{\mathbf{x}} = \begin{bmatrix}x_1 \\ y \\ \dot{y}\end{bmatrix} = \begin{bmatrix}1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1\end{bmatrix}\mathbf{x} \qquad \mathbf{x} = \begin{bmatrix}1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -1 & 1\end{bmatrix}\overline{\mathbf{x}} \qquad (2)$$

Substitute Eq. (2) into Eq. (1), we have

$$\frac{d\overline{\mathbf{x}}}{dt} = \mathbf{A}_1\overline{\mathbf{x}} + \mathbf{B}_1 u = \begin{bmatrix}-1 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & -2\end{bmatrix}\overline{\mathbf{x}} = \begin{bmatrix}0 \\ 0 \\ 1\end{bmatrix}u$$

5-8 (a)

$$\left|s\mathbf{I} - \mathbf{A}\right| = \begin{vmatrix}s & -2 & 0 \\ -1 & s-2 & 0 \\ 1 & 0 & s-1\end{vmatrix} = s^3 - 3s^2 + 2 = s^3 + a_2 s^2 + a_1 s + a_0 \qquad a_0 = 2, \quad a_1 = 0, \quad a_2 = -3$$

$$\mathbf{M} = \begin{bmatrix}a_1 & a_2 & 1 \\ a_2 & 1 & 0 \\ 1 & 0 & 0\end{bmatrix} = \begin{bmatrix}0 & -3 & 1 \\ -3 & 1 & 0 \\ 1 & 0 & 0\end{bmatrix} \qquad \mathbf{S} = \begin{bmatrix}\mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B}\end{bmatrix} = \begin{bmatrix}0 & 2 & 4 \\ 1 & 2 & 6 \\ 1 & 1 & -1\end{bmatrix}$$

$$\mathbf{P} = \mathbf{SM} = \begin{bmatrix}-2 & 2 & 0 \\ 0 & -1 & 1 \\ -4 & -2 & 1\end{bmatrix}$$

(b)

$$|s\mathbf{I}-\mathbf{A}| = \begin{vmatrix} s & -2 & 0 \\ -1 & s-1 & 0 \\ 1 & -1 & s-1 \end{vmatrix} = s^3 - 3s^2 + 2 = s^3 + a_2 s^2 + a_1 s + a_0 \qquad a_0 = 2, \qquad a_1 = 0, \qquad a_2 = -3$$

$$\mathbf{M} = \begin{bmatrix} a_1 & a_2 & 1 \\ a_2 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -3 & 1 \\ -3 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad \mathbf{S} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 6 \\ 1 & 3 & 8 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{P} = \mathbf{SM} = \begin{bmatrix} 0 & -1 & 1 \\ -1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

(c)

$$|s\mathbf{I}-\mathbf{A}| = \begin{vmatrix} s+2 & -1 & 0 \\ 0 & s+2 & 0 \\ 1 & 2 & s+3 \end{vmatrix} = s^3 + 7s^2 + 16s + 12 = s^3 + a_2 s^2 + a_1 s + a_0 \qquad a_0 = 12, \quad a_1 = 16, \quad a_2 = 7$$

$$\mathbf{M} = \begin{bmatrix} a_1 & a_2 & 1 \\ a_2 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 16 & 7 & 1 \\ 7 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad \mathbf{S} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 1 & -2 & 4 \\ 1 & -6 & 23 \end{bmatrix}$$

$$\mathbf{P} = \mathbf{SM} = \begin{bmatrix} 9 & 6 & 1 \\ 6 & 5 & 1 \\ -3 & 1 & 1 \end{bmatrix}$$

(d)

$$|s\mathbf{I}-\mathbf{A}| = \begin{vmatrix} s+1 & -1 & 0 \\ 0 & s=1 & -1 \\ 0 & 0 & s+1 \end{vmatrix} = s^3 + 3s^2 + 3s + 1 = s^3 + a_2 s^2 + a_1 s + a_0 \qquad a_0 = 1, \quad a_1 = 3, \quad a_2 = 3$$

$$\mathbf{M} = \begin{bmatrix} a_1 & a_2 & 1 \\ a_2 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 3 & 1 \\ 3 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad \mathbf{S} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} \end{bmatrix} = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ 1 & -1 & 1 \end{bmatrix}$$

$$\mathbf{P} = \mathbf{SM} = \begin{bmatrix} 2 & 1 & 0 \\ 2 & 3 & 1 \\ 1 & 2 & 1 \end{bmatrix}$$

(e)

$$|s\mathbf{I}-\mathbf{A}| = \begin{vmatrix} s-1 & -1 \\ 2 & s+3 \end{vmatrix} = s^2 + 2s - 1 = s^2 + a_1 s + a_0 \qquad a_0 = -1, \quad a_1 = 2$$

$$\mathbf{M} = \begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \qquad \mathbf{S} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -3 \end{bmatrix}$$

$$\mathbf{P} = \mathbf{SM} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

5-9 (a) From Problem 5-8(a),

$$\mathbf{M} = \begin{bmatrix} 0 & -3 & 1 \\ -3 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Then,

$$\mathbf{V} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \mathbf{CA}^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix} \qquad \mathbf{Q} = (\mathbf{MV})^{-1} = \begin{bmatrix} 0.5 & 1 & 3 \\ 0.5 & 1.5 & 4 \\ -0.5 & -1 & -2 \end{bmatrix}$$

(b) From Problem 5-8(b),

$$\mathbf{M} = \begin{bmatrix} 16 & 7 & 1 \\ 7 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\mathbf{V} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \mathbf{CA}^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 3 & 1 \\ 2 & 5 & 1 \end{bmatrix} \qquad \mathbf{Q} = (\mathbf{MV})^{-1} = \begin{bmatrix} 0.2308 & 0.3077 & 1.0769 \\ 0.1538 & 0.5385 & 1.3846 \\ -0.2308 & -0.3077 & -0.0769 \end{bmatrix}$$

(c) From Problem 5-8(c),

$$\mathbf{V} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \mathbf{CA}^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 4 & -4 & 0 \end{bmatrix}$$

Since $\mathbf{V}$ is singular, the OCF transformation cannot be conducted.

(d) From Problem 5-8(d),

$$\mathbf{M} = \begin{bmatrix} 3 & 3 & 1 \\ 3 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Then,

$$\mathbf{V} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \mathbf{CA}^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & -1 \\ 1 & -2 & 2 \end{bmatrix} \qquad \mathbf{Q} = (\mathbf{MV})^{-1} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 1 & -2 \\ 1 & -1 & 1 \end{bmatrix}$$

(e) From Problem 5-8(e),

$$\mathbf{M} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \qquad \text{Then,} \qquad \mathbf{V} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA}^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \qquad \mathbf{Q} = (\mathbf{MV})^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & -3 \end{bmatrix}$$

5-10 (a) Eigenvalues of $\mathbf{A}$: $1, \; 2.7321, \; -0.7321$

$$\mathbf{T} = \begin{bmatrix} \mathbf{p}_1 & \mathbf{p}_2 & \mathbf{p}_3 \end{bmatrix} = \begin{bmatrix} 0 & 0.5591 & 0.8255 \\ 0 & 0.7637 & -0.3022 \\ 1 & -0.3228 & 0.4766 \end{bmatrix}$$

where $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$ are the eigenvectors.

(b) Eigenvalues of $\mathbf{A}$: $1, \; 2.7321, \; -0.7321$

$$\mathbf{T} = \begin{bmatrix} \mathbf{p}_1 & \mathbf{p}_2 & \mathbf{p}_3 \end{bmatrix} = \begin{bmatrix} 0 & 0.5861 & 0.7546 \\ 0 & 0.8007 & -0.2762 \\ 1 & 0.1239 & 0.5952 \end{bmatrix}$$

where $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$ are the eigenvalues.

(c) Eigenvalues of $\mathbf{A}$: -3, -2, -2. A nonsingular DF transformation matrix $\mathbf{T}$ cannot be found.

(d) Eigenvalues of $\mathbf{A}$: $-1, -1, -1$

The matrix $\mathbf{A}$ is already in Jordan canonical form. Thus, the DF transformation matrix $\mathbf{T}$ is the identity matrix $\mathbf{I}$.

(e) Eigenvalues of $\mathbf{A}$: $0.4142, \; -2.4142$

$$\mathbf{T} = \begin{bmatrix} \mathbf{p}_2 & \mathbf{p}_2 \end{bmatrix} = \begin{bmatrix} 0.8629 & -0.2811 \\ -0.5054 & 0.9597 \end{bmatrix}$$

5-11 (a)

$$\mathbf{S} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \qquad \mathbf{S} \text{ is singular.}$$

(b)

$$\mathbf{S} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -2 & 2 \\ 3 & -3 & 3 \end{bmatrix} \qquad \mathbf{S} \text{ is singular.}$$

(c)

$$\mathbf{S} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} \end{bmatrix} = \begin{bmatrix} 2 & 2+2\sqrt{2} \\ \sqrt{2} & 2+\sqrt{2} \end{bmatrix} \qquad \mathbf{S} \text{ is singular.}$$

(d)

$$\mathbf{S} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} \end{bmatrix} = \begin{bmatrix} 1 & -2 & 4 \\ 0 & 0 & 0 \\ 1 & -4 & 14 \end{bmatrix} \qquad \mathbf{S} \text{ is singular.}$$

5-12 (a) Rewrite the differential equations as:

$$\frac{d^2q_m}{dt^2} = -\frac{B}{J}\frac{d^2q_m}{dt^2} - \frac{K}{J}q_m + \frac{K_i}{J}i_a \qquad \frac{di_a}{dt} = -\frac{K_b}{L_a}\frac{dq_m}{dt} - \frac{R_a}{L_a}i_a + \frac{K_aK_s}{L_a}\left(q_r - q_m\right)$$

State variables: $x_1 = q_m, \quad x_2 = \dfrac{dq_m}{dt}, \quad x_3 = i_a$

State equations:

$$\begin{bmatrix} \dfrac{dx_1}{dt} \\ \dfrac{dx_2}{dt} \\ \dfrac{dx_3}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -\dfrac{K}{J} & -\dfrac{B}{J} & \dfrac{K_i}{J} \\ -\dfrac{K_aK_s}{L_a} & -\dfrac{K_b}{L_a} & -\dfrac{R_a}{L_a} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dfrac{K_aK_s}{L_a} \end{bmatrix} q_r$$

Output equation:

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}\mathbf{x} = x_1$$

(b) **Forward-path transfer function:**

$$G(s) = \frac{\Theta_m(s)}{E(s)} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} s & -1 & 0 \\ \dfrac{K}{J} & s + \dfrac{B}{J} & -\dfrac{K_i}{J} \\ 0 & \dfrac{K_b}{L_a} & s + \dfrac{R_a}{L_a} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ \dfrac{K_a}{L_a} \end{bmatrix} = \frac{K_iK_a}{\Delta_o(s)}$$

$$\Delta_o(s) = JL_as^3 + \left(BL_a + R_aJ\right)s^2 + \left(KL_a + K_iK_b + R_aB\right)s + KR_a = 0$$

Closed-loop transfer function:

$$M(s) = \frac{\Theta_m(s)}{\Theta_r(s)} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} s & -1 & 0 \\ \dfrac{K}{J} & s + \dfrac{B}{J} & -\dfrac{K_i}{J} \\ \dfrac{K_aK_s}{L_a} & \dfrac{K_b}{L_a} & s + \dfrac{R_a}{L_a} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ \dfrac{K_aK_s}{L_a} \end{bmatrix} = \frac{K_sG(s)}{1 + K_s(s)}$$

$$= \frac{K_iK_aK_s}{JL_as^3 + \left(BL_a + R_aJ\right)s^2 + \left(KL_a + K_iK_b + R_aB\right)s + K_iK_aK_s + KR_a}$$

5-13 (a)

$$\mathbf{A}=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\qquad \mathbf{A}^2=\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}\qquad \mathbf{A}^3=\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\qquad \mathbf{A}^4=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

(1) Infinite series expansion:

$$f(t)=\mathbf{I}+\mathbf{A}t+\frac{1}{2!}\mathbf{A}^2t^2+\cdots=\begin{bmatrix}1-\frac{t^2}{2!}+\frac{t^4}{4!}-\cdots & t-\frac{t^3}{3!}+\frac{t^5}{5!}-\cdots\\ -t+\frac{t^3}{3!}-\frac{t^5}{5!}+\cdots & 1-\frac{t^2}{2!}+\frac{t^4}{4!}\cdots\end{bmatrix}=\begin{bmatrix}\cos t & \sin t\\ -\sin t & \cos t\end{bmatrix}$$

(2) Inverse Laplace transform:

$$\Phi(s)=(s\mathbf{I}-\mathbf{A})^{-1}=\begin{bmatrix}s & -1\\ 1 & s\end{bmatrix}^{-1}=\frac{1}{s^2+1}\begin{bmatrix}s & 1\\ -1 & s\end{bmatrix}\qquad f(t)=\begin{bmatrix}\cos t & \sin t\\ -\sin t & \cos t\end{bmatrix}$$

(b)

$$\mathbf{A}=\begin{bmatrix}-1 & 0\\ 0 & -2\end{bmatrix}\qquad \mathbf{A}^2=\begin{bmatrix}1 & 0\\ 0 & 4\end{bmatrix}\qquad \mathbf{A}^3=\begin{bmatrix}-1 & 0\\ 0 & -8\end{bmatrix}\qquad \mathbf{A}^4=\begin{bmatrix}-1 & 0\\ 0 & 16\end{bmatrix}$$

(1) Infinite series expansion:

$$f(t)=\mathbf{I}+\mathbf{A}t+\frac{1}{2!}\mathbf{A}^2t^2+\cdots=\begin{bmatrix}1-t+\frac{t^2}{2!}-\frac{t^3}{3!}+\frac{t^4}{4!}+\cdots & 0\\ 0 & 1-2t+\frac{4t^2}{2!}-\frac{8t^3}{3!}+\cdots\end{bmatrix}=\begin{bmatrix}e^{-t} & 0\\ 0 & e^{-2t}\end{bmatrix}$$

(2) Inverse Laplace transform:

$$\Phi(s)=(s\mathbf{I}-\mathbf{A})^{-1}=\begin{bmatrix}s+1 & 0\\ 0 & s+2\end{bmatrix}^{-1}=\begin{bmatrix}\frac{1}{s+1} & 0\\ 0 & \frac{1}{s+2}\end{bmatrix}\qquad f(t)=\begin{bmatrix}e^{-t} & 0\\ 0 & e^{-2t}\end{bmatrix}$$

(c)

$$\mathbf{A}=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\qquad \mathbf{A}^2=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\qquad \mathbf{A}^3=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\qquad \mathbf{A}^4=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

(1) Infinite series expansion:

$$f(t)=\mathbf{I}+\mathbf{A}t+\frac{1}{2!}\mathbf{A}^2t^2+\cdots=\begin{bmatrix}1+\frac{t^2}{2!}+\frac{t^4}{4!}+\cdots & t+\frac{t^3}{3!}+\frac{t^5}{5!}\cdots\\ t+\frac{t^3}{3!}+\frac{t^5}{5!}\cdots & 1+\frac{t^2}{2!}+\frac{t^4}{4!}+\cdots\end{bmatrix}=\begin{bmatrix}e^{-t}+e^{t} & -e^{-t}+e^{t}\\ -e^{-t}+e^{t} & e^{-t}+e^{t}\end{bmatrix}$$

(2) Inverse Laplace transform:

$$\Phi(s)=(s\mathbf{I}-\mathbf{A})^{-1}=\begin{bmatrix}s & -1\\ -1 & s\end{bmatrix}^{-1}=\frac{1}{s^2-1}\begin{bmatrix}s & 1\\ 1 & s\end{bmatrix}=\begin{bmatrix}\frac{0.5}{s+1}-\frac{0.5}{s-1} & \frac{-0.5}{s+1}+\frac{0.5}{s-1}\\ \frac{-0.5}{s+1}+\frac{0.5}{s-1} & \frac{0.5}{s+1}+\frac{0.5}{s-1}\end{bmatrix}$$

$$f(t) = 0.5\begin{bmatrix} e^{-t} + e^{t} & -e^{-t} + e^{t} \\ -e^{-t} + e^{t} & e^{-t} + e^{t} \end{bmatrix}$$

5-14 (a) $e = K_s\left(q_r - q_y\right) \qquad e_a = e - e_s \qquad e_s = R_s i_a \qquad e_u = Ke_a$

$$i_a = \frac{e_u - e_b}{R_a + R_s} \qquad e_b = K_b \frac{dq_y}{dt} \qquad T_m = K_i i_a = \left(J_m + J_L\right)\frac{d^2 q_y}{dt^2}$$

Solve for i_a in terms of q_y and $\frac{dq_y}{dt}$, we have

$$i_a = \frac{KK_s\left(q_r - q_y\right) - K_b \frac{dq_y}{dt}}{R_s + R_s + KR_s}$$

Differential equation:

$$\frac{d^2 q_y}{dt^2} = \frac{K_i i_a}{J_m + J_L} = \frac{K_i}{\left(J_m + J_L\right)\left(R_a + R_s + KR_s\right)}\left(-K_b \frac{dq_y}{dt} - KK_s q_y + KK_s q_y\right)$$

State variables: $x_1 = q_y, \quad x_2 = \frac{dq_y}{dt}$

State equations:

$$\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{-KK_s K_i}{\left(J_m + J_L\right)\left(R_a + R_s + KR_s\right)} & \frac{-K_b K_i}{\left(J_m + J_L\right)\left(R_a + R_s + KR_s\right)} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{-KK_s K_i}{\left(J_m + J_L\right)\left(R_a + R_s + KR_s\right)} \end{bmatrix} q_r$$

$$= \begin{bmatrix} 0 & 1 \\ -322.58 & -80.65 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 322.58 \end{bmatrix} q_r$$

We can let $v(t) = 322.58\, q_r$, then the state equations are in the form of CCF.

(b)

$$\left(s\mathbf{I} - \mathbf{A}\right)^{-1} = \begin{bmatrix} s & -1 \\ 322.58 & s + 80.65 \end{bmatrix}^{-1} = \frac{1}{s^2 + 80.65s + 322.58}\begin{bmatrix} s + 80.65 & 1 \\ -322.58 & s \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-0.06}{s + 76.42} - \frac{1.059}{s + 4.22} & \frac{-0.014}{s + 76.42} + \frac{0.014}{s + 4.22} \\ \frac{4.468}{s + 76.42} - \frac{4.468}{s + 4.22} & \frac{1.0622}{s + 76.42} - \frac{0.0587}{s + 4.22} \end{bmatrix}$$

For a unit-step function input, $u_s(t) = 1/s$.

$$\left(s\mathbf{I} - \mathbf{A}\right)^{-1}\mathbf{B}\frac{1}{s} = \begin{bmatrix} \frac{322.2}{s(s + 76.42)(s + 4.22)} \\ \frac{322.2}{s(s + 76.42)(s + 4.22)} \end{bmatrix} = \begin{bmatrix} \frac{1}{s} + \frac{0.0584}{s + 76.42} - \frac{1.058}{s + 4.22} \\ \frac{-4.479}{s + 76.42} + \frac{4.479}{s + 4.22} \end{bmatrix}$$

$$\mathbf{x}(t)=\begin{bmatrix} -0.06e^{-76.42t}-1.059e^{-4.22t} & -0.014e^{-76.42t}+0.01e^{-4.22t} \\ 4.468e^{-76.42t}-4.468e^{-4.22t} & 1.0622\,e^{-76.42t}-0.0587\,e^{-4.22t} \end{bmatrix}\mathbf{x}(0)$$

$$=\begin{bmatrix} 1+0.0584e^{-76.42t}-1.058e^{-4.22t} \\ -4.479e^{-76.42t}+4.479e^{-4.22t} \end{bmatrix} \qquad t\ge 0$$

(c) Characteristic equation: $\Delta(s)=s^2+80.65s+322.58=0$

(d) From the state equations we see that whenever there is R_a there is $(1+K)R_s$. Thus, the purpose of R_s is to increase the effective value of R_a by $(1+K)R_s$. This improves the time constant of the system.

5-15 (a) State equations:

$$\begin{bmatrix} \dfrac{dx_1}{dt} \\ \dfrac{dx_2}{dt} \end{bmatrix}=\begin{bmatrix} 0 & 1 \\ \dfrac{-KK_sK_i}{J(R+R_s+KR_s)} & \dfrac{-K_bK_i}{J(R+R_s+KR_s)} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}+\begin{bmatrix} 0 \\ \dfrac{KK_sK_i}{J(R+R_s+KR_s)} \end{bmatrix}q_r$$

$$=\begin{bmatrix} 0 & 1 \\ -818.18 & -90.91 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}+\begin{bmatrix} 0 \\ 818.18 \end{bmatrix}q_r$$

Let $v=818.18\,q_r$. The equations are in the form of CCF with v as the input.

(b) $$(s\mathbf{I}-\mathbf{A})^{-1}=\begin{bmatrix} s & -1 \\ 818.18 & s+90.91 \end{bmatrix}^{-1}=\frac{1}{(s+10.128)(s+80.782)}\begin{bmatrix} s+90.91 & 1 \\ -818.18 & s \end{bmatrix}$$

$$\mathbf{x}(t)=\begin{bmatrix} 1.143e^{-10.128t}-0.142e^{-80.78t} & 0.01415e^{-10.128t}-0.0141e^{-80.78t} \\ -11.58e^{-10.128t}+0.1433e^{-80.78t} & -0.1433e^{-10.128t}+1.143e^{-80.78t} \end{bmatrix}\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}$$

$$+\begin{bmatrix} 11.58e^{-10.128t}-11.58e^{-80.78t} \\ 1-1.1434e^{-10.128t}+0.1433e^{-80.78t} \end{bmatrix} \qquad t\ge 0$$

(c) Characteristic equation: $\Delta(s)=s^2+90.91s+818.18=0$

Eigenvalues: -10.128, -80.782

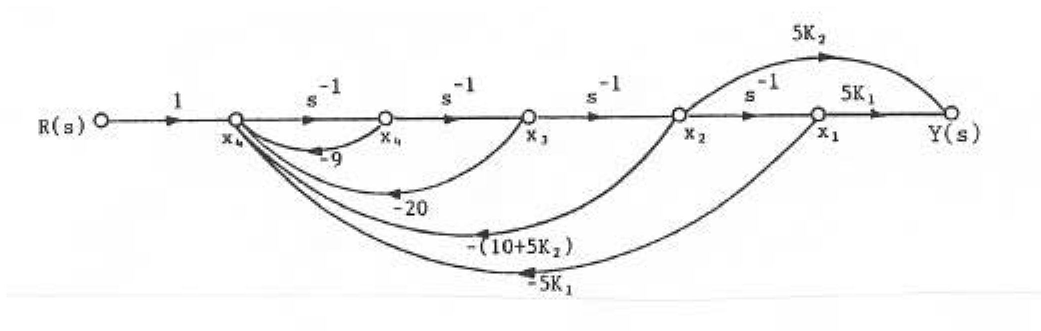
(d) Same remark as in part (d) of Problem 5-14.

5-16 (a) Forward-path transfer function: **Closed-loop transfer function:**

$$G(s)=\frac{Y(s)}{E(s)}=\frac{5(K_1+K_2s)}{s[s(s+4)(s+5)+10]}$$

$$M(s)=\frac{Y(s)}{R(s)}=\frac{G(s)}{1+G(s)}=\frac{5(K_1+K_2s)}{s^4+9s^3+20s^2+(10+5K_2)s+5K_1}$$

(b) State diagram by direct decomposition:



State equations:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -5K & -\left(10+5K_2\right) & -20 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} r$$

Output equation:

$$y = \begin{bmatrix} 5K_1 & 5K_2 & 0 \end{bmatrix} \mathbf{x}$$

(c) Final value: $r(t) = u_s(t), \quad R(s) = \dfrac{1}{s}.$

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) = \lim_{s\to 0} \frac{5\left(K_1 + K_2 s\right)}{s^4 + 9s^3 + 20s^2 + \left(10+5K_2\right)s + 5K_1} = 1$$

5-17 In CCF form,

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 & \cdots & -a_{n-1} \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$s\mathbf{I} - \mathbf{A} = \begin{bmatrix} s & -1 & 0 & 0 & \cdots & 0 \\ 0 & s & -1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & -1 \\ a_0 & a_1 & a_2 & a_3 & \cdots & s+a_n \end{bmatrix}$$

$$\left|s\mathbf{I} - \mathbf{A}\right| = s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \cdots + a_1 s + a_0$$

Since $\mathbf{B}$ has only one nonzero element which is in the last row, only the last column of $\operatorname{adj}\left(s\mathbf{I}-\mathbf{A}\right)$ is going to contribute to $\operatorname{adj}\left(s\mathbf{I}-\mathbf{A}\right)B$. The last column of $\operatorname{adj}\left(s\mathbf{I}-\mathbf{A}\right)$ is obtained from the cofactors of the last row of $\left(s\mathbf{I}-\mathbf{A}\right)$. Thus, the last column of $\operatorname{adj}\left(s\mathbf{I}-\mathbf{A}\right)B$ is $\begin{bmatrix} 1 & s & s^2 & \cdots & s^{n-1} \end{bmatrix}'$.

5-18 (a) State variables: $x_1 = y, \quad x_2 = \frac{dy}{dt}, \quad x_3 = \frac{d^2y}{dt^2}$

State equations: $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}r(t)$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -3 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

(b) State transition matrix:

$$\Phi(s) = (s\mathbf{I} - \mathbf{A})^{-1} = \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 1 & 3 & s+3 \end{bmatrix}^{-1} = \frac{1}{\Delta(s)} \begin{bmatrix} s^2+3s+3 & s+3 & 1 \\ -1 & s^2+3s & s \\ -s & -3s-1 & s^2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{s+1} + \frac{1}{(s+1)^2} + \frac{1}{(s+1)^3} & \frac{1}{(s+1)^2} + \frac{2}{(s+1)^3} & \frac{1}{(s+1)^3} \\ \frac{-1}{(s+1)^3} & \frac{1}{s+1} + \frac{1}{(s+1)^2} - \frac{2}{(s+1)^3} & \frac{s}{(s+1)^3} \\ \frac{-s}{(s+1)^3} & \frac{-3}{(s+1)^2} + \frac{2}{(s+1)^3} & \frac{s^2}{(s+1)^3} \end{bmatrix}$$

$$\Delta(s) = s^3 + 3s^2 + 3s + 1 = (s+1)^3$$

$$f(t) = \begin{bmatrix} (1+t+t^2/2)e^{-t} & (t+t^2)e^{-t} & t^2e^{-t}/2 \\ -t^2e^{-t}/2 & (1+t-t^2)e^{-t} & (t-t^2/2)e^{-t} \\ (-t+t^2/2)e^{-t} & t^2e^{-t} & (1-2t+t^2/2)e^{-t} \end{bmatrix}$$

(d) Characteristic equation: $\Delta(s) = s^3 + 3s^2 + 3s + 1 = 0$

Eigenvalues: $-1, \ -1, \ -1$

5-19 (a) State variables: $x_1 = y, \quad x_2 = \frac{dy}{dt}$

State equations:

$$\begin{bmatrix} \frac{dx_1(t)}{dt} \\ \frac{dx_2(t)}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r(t)$$

State transition matrix:

$$\Phi(s) = (s\mathbf{I} - \mathbf{A})^{-1} = \begin{bmatrix} s & -1 \\ 1 & s+2 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{s+2}{(s+1)^2} & \frac{1}{(s+1)^2} \\ \frac{-1}{(s+1)^2} & \frac{s}{(s+1)^2} \end{bmatrix} \qquad f(t) = \begin{bmatrix} (1+t)e^{-t} & te^{-t} \\ -te^{-t} & (1-t)e^{-t} \end{bmatrix}$$

Characteristic equation: $\Delta(s) = (s+1)^2 = 0$

(b) State variables: $x_1 = y, \quad x_2 = y + \dfrac{dy}{dt}$

State equations:

$$\frac{dx_1}{dt} = \frac{dy}{dt} = x_2 - y = x_2 - x_1 \qquad \frac{dx_2}{dt} = \frac{d^2y}{dt^2} + \frac{dy}{dt} = -y - \frac{dy}{dt} + r = -x_2 + r$$

$$\begin{bmatrix} \dfrac{dx_1}{dt} \\ \dfrac{dx_2}{dt} \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r$$

State transition matrix:

$$\Phi(s) = \begin{bmatrix} s+1 & -2 \\ 0 & s+1 \end{bmatrix}^{-1} = \begin{bmatrix} \dfrac{1}{s+1} & \dfrac{-2}{(s+1)^2} \\ 0 & \dfrac{1}{s+1} \end{bmatrix} \qquad f(t) = \begin{bmatrix} e^{-t} & -te^{-t} \\ 0 & e^{-t} \end{bmatrix}$$

(c) Characteristic equation: $\Delta(s) = (s+1)^2 = 0$ which is the same as in part (a).

5-20 (a) State transition matrix:

$$s\mathbf{I} - \mathbf{A} = \begin{bmatrix} s-s & w \\ -w & s-s \end{bmatrix} \qquad (s\mathbf{I} - \mathbf{A})^{-1} = \frac{1}{\Delta(s)} \begin{bmatrix} s-s & -w \\ w & s-s \end{bmatrix} \qquad \Delta(s) = s^2 - 2s + (s^2 + w^2)$$

$$f(t) = \mathcal{L}^{-1}\left[(s\mathbf{I} - \mathbf{A})^{-1}\right] = \begin{bmatrix} \cos wt & -\sin wt \\ \sin wt & \cos wt \end{bmatrix} e^{st}$$

(b) Eigenvalues of A: $s + jw, \quad s - jw$

5-21 (a)

$$\frac{Y_1(s)}{U_1(s)} = \frac{s^{-3}}{1 + s^{-1} + 2s^{-2} + 3s^{-3}} = \frac{1}{s^3 + s^2 + 2s + 3}$$

$$\frac{Y_2(s)}{U_2(s)} = \frac{s^{-3}}{1 + s^{-1} + 2s^{-2} + 3s^{-3}} = \frac{1}{s^3 + s^2 + 2s + 3} = \frac{Y_1(s)}{U_1(s)}$$

(b) State equations [Fig. 5-21(a)]: $\dot{\mathbf{x}} = \mathbf{A}_1\mathbf{x} + \mathbf{B}_1 u_1$ **Output equation:** $\mathbf{y}_1 = \mathbf{C}_1\mathbf{x}$

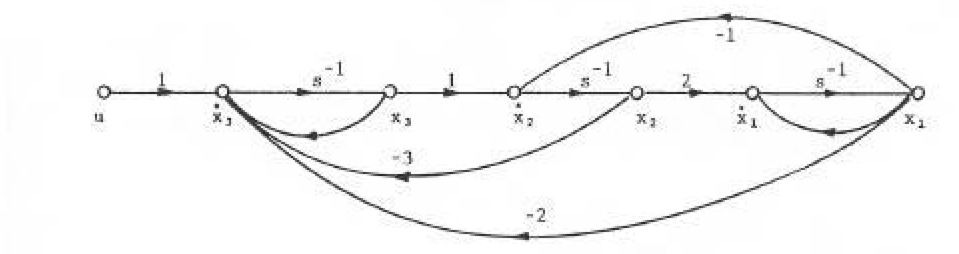
$$\mathbf{A}_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -3 & -2 & -1 \end{bmatrix} \qquad \mathbf{B}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad \mathbf{C}_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

State equations [Fig. 5-21(b)]: $\dot{\mathbf{x}} = \mathbf{A}_2\mathbf{x} + \mathbf{B}_2 u_2$ **Output equation:** $y_2 = \mathbf{C}_2\mathbf{x}$

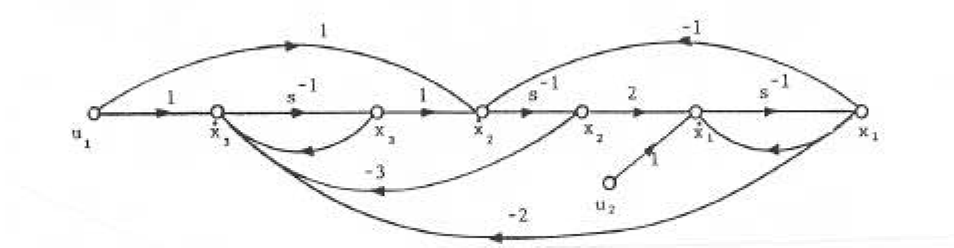
$$\mathbf{A}_2 = \begin{bmatrix} 0 & 0 & -3 \\ 1 & 0 & -2 \\ 0 & 1 & -1 \end{bmatrix} \qquad \mathbf{B}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \mathbf{C}_2 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$$

Thus, $\mathbf{A}_2 = \mathbf{A}_1'$

5-22 (a) State diagram:



(b) State diagram:

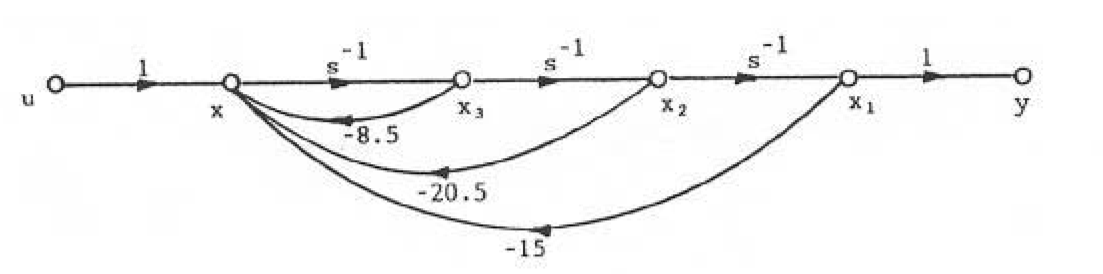


5-23 (a)

$$G(s) = \frac{Y(s)}{U(s)} = \frac{10s^{-3}}{1+8.5s^{-1}+20.5s^{-2}+15s^{-3}}\frac{X(s)}{X(s)} \qquad Y(s) = 10X(s)$$

$$X(s) = U(s) - 8.5s^{-1}X(s) - 20.5s^{-2}X(s) - 15s^{-3}X(s)$$

State diagram:



State equation: $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -15 & -20.5 & -8.5 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

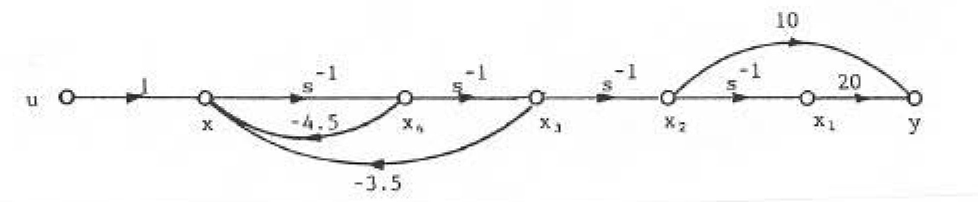
A and **B** are in CCF

(b)

$$G(s) = \frac{Y(s)}{U(s)} = \frac{10s^{-3} + 20s^{-4}}{1 + 4.5s^{-1} + 3.5s^{-2}} \frac{X(s)}{X(s)}$$

$$Y(s) = 10s^{-3}X(s) + 20s^{-4}X(s) \qquad X(s) = -4.5s^{-1}X(s) - 3.5s^{-2}X(s) + U(s)$$

State diagram:



State equations: $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -3.5 & -4.5 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

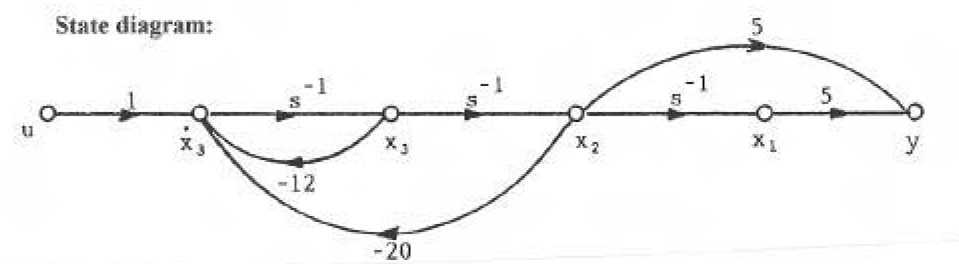
A and **B** are in CCF

(c)

$$G(s) = \frac{Y(s)}{U(s)} = \frac{5(s+1)}{s(s+2)(s+10)} = \frac{5s^{-2} + 5s^{-3}}{1 + 12s^{-1} + 20s^{-2}} \frac{X(s)}{X(s)}$$

$$Y(s) = 5s^{-2}X(s) + 5s^{-3}X(s) \qquad X(s) = U(s) - 12s^{-1}X(s) - 20s^{-2}X(s)$$

State diagram:



State equations: $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -20 & -12 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

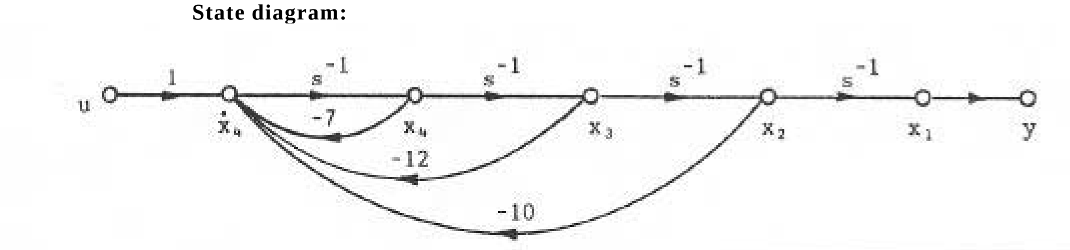
A and **B** are in CCF

(d)

$$G(s) = \frac{Y(s)}{U(s)} = \frac{1}{s(s+5)(s^2+2s+2)} = \frac{s^{-4}}{1+7s^{-1}+12s^{-2}+10s^{-3}} \frac{X(s)}{X(s)}$$

$$Y(s) = s^{-4}X(s) \qquad X(s) = U(s) - 7s^{-1}X(s) - 12s^{-2}X(s) - 10s^{-3}$$

State diagram:



State equations: $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)$

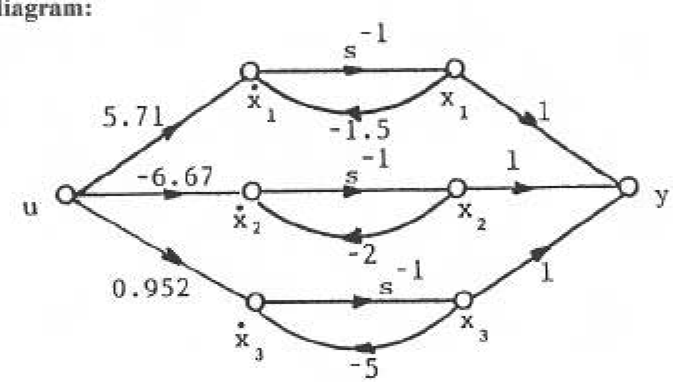
$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -10 & -12 & -7 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

A and **B** are in CCF

5-24 (a)

$$G(s) = \frac{Y(s)}{U(s)} = \frac{10}{s^3 + 8.5s^2 + 20.5s + 15} = \frac{5.71}{s+15} - \frac{6.67}{s+2} + \frac{0.952}{s+5}$$

State diagram:



State equations: $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)$

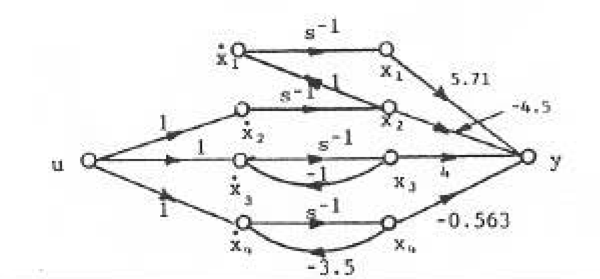
$$\mathbf{A} = \begin{bmatrix} -1.5 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -5 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 5.71 \\ -6.67 \\ 0.952 \end{bmatrix}$$

The matrix **B** is not unique. It depends on how the input and the output branches are allocated.

(b)

$$G(s) = \frac{Y(s)}{U(s)} = \frac{10(s+2)}{s^2(s=1)(s+3.5)} = \frac{-4.5}{s} + \frac{0.49}{s+3.5} + \frac{4}{s+1} + \frac{5.71}{s^2}$$

State diagram:



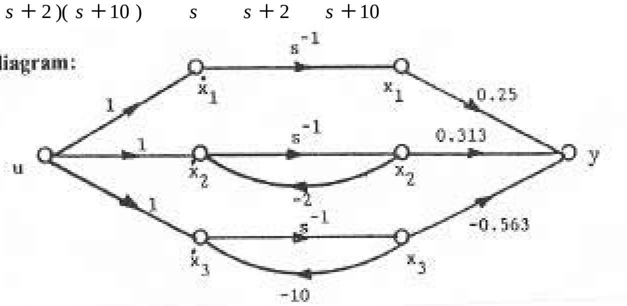
State equation: $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3.5 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

(b)

$$G(s) = \frac{Y(s)}{U(s)} = \frac{5(s+1)}{s(s+2)(s+10)} = \frac{2.5}{s} + \frac{0.313}{s+2} - \frac{0.563}{s+10}$$

State diagram:



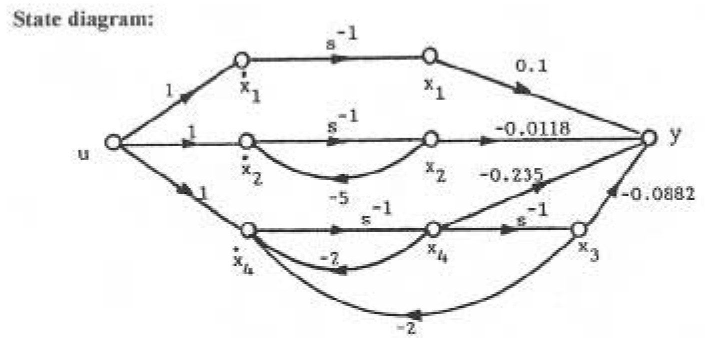
State equations: $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)$

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -10 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

(d)

$$G(s) = \frac{Y(s)}{U(s)} = \frac{1}{s(s+5)(s^2+2s+2)} = \frac{0.1}{s} - \frac{0.0118}{s+5} - \frac{0.0882s+0.235}{s^2+2s+2}$$

State diagram:



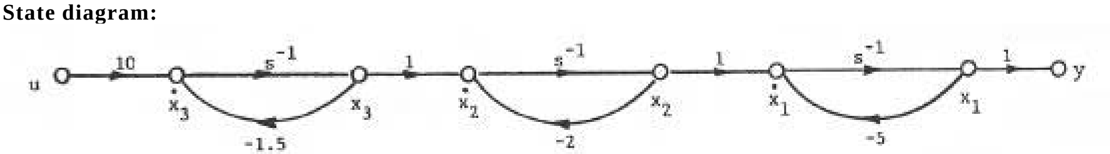
State equations: $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)$

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -5 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -2 & -2 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

5-25 (a)

$$G(s) = \frac{Y(s)}{U(s)} = \frac{10}{(s+1.5)(s+2)(s+5)}$$

State diagram:



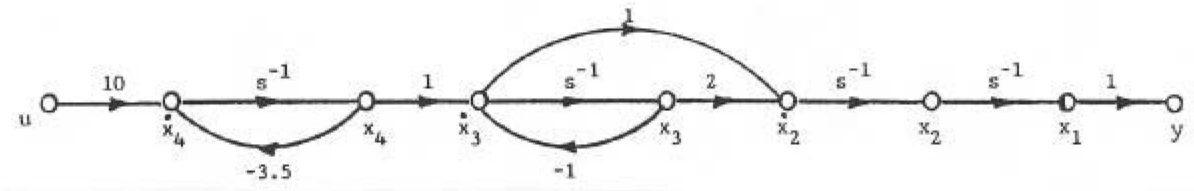
State equations: $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)$

$$\mathbf{A} = \begin{bmatrix} -5 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -1.5 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix}$$

(b)

$$G(s) = \frac{Y(s)}{U(s)} = \frac{10(s+2)}{s^2(s+1)(s+3.5)} = \left(\frac{10}{s^2}\right)\left(\frac{s+2}{s+1}\right)\left(\frac{1}{s+3.5}\right)$$

State diagram:



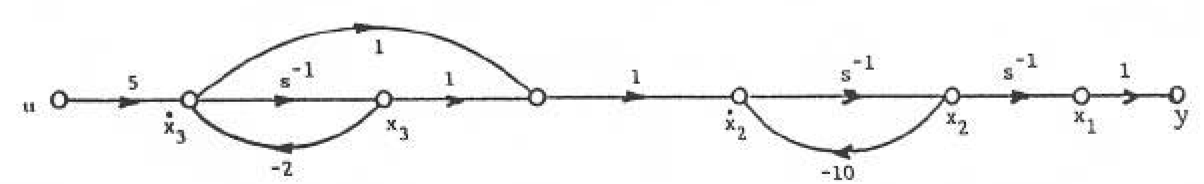
State equations: $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -3.5 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 10 \end{bmatrix}$$

(c)

$$G(s) = \frac{Y(s)}{U(s)} = \frac{59s+1)}{s(s+2)(s+10)} = \left(\frac{5}{s}\right)\left(\frac{s+1}{s+2}\right)\left(\frac{1}{s+10}\right)$$

State diagram:



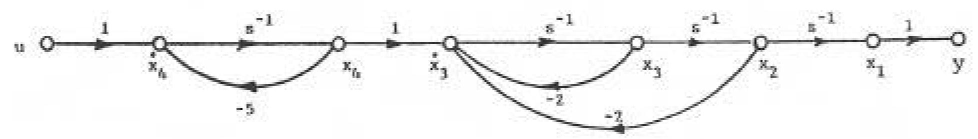
State equations: $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -10 & -1 \\ 0 & 0 & -2 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix}$$

(d)

$$G(s) = \frac{Y(s)}{U(s)} = \frac{1}{s(s+5)(s^2+2s+2)}$$

State diagram:



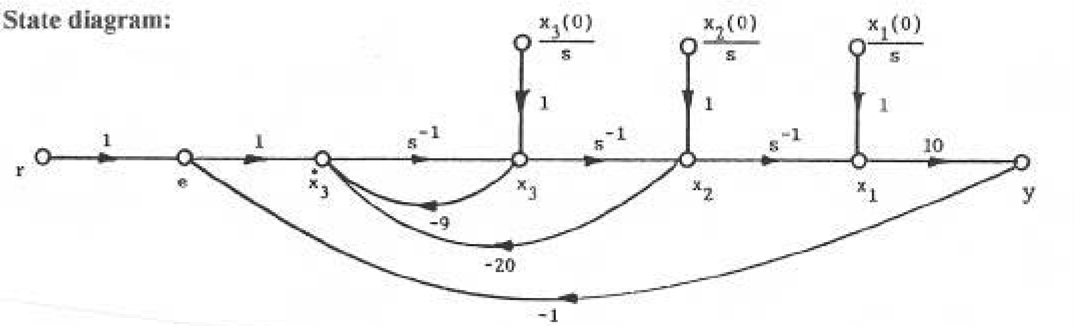
State equations: $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -2 & -2 & 1 \\ 0 & 0 & 0 & -5 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

5-26 (a)

$$G(s) = \frac{Y(s)}{E(s)} = \frac{10}{s(s+4)(s+5)} = \frac{10s^{-3}}{1+9s^{-1}+20s^{-2}} \frac{X(s)}{X(s)}$$

State diagram:



(b) Dynamic equations:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -10 & -20 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r \qquad y = \begin{bmatrix} 10 & 0 & 0 \end{bmatrix} \mathbf{x}$$

(c) State transition equation:

$$\begin{bmatrix} X_1(s) \\ X_2(s) \\ X_3(s) \end{bmatrix} = \frac{1}{\Delta(s)} \begin{bmatrix} s^{-1}\left(1+9s^{-1}+20s^{-2}\right) & s^{-2}\left(1+9s^{-1}\right) & s^{-3} \\ -10s^{-3} & s^{-1}\left(1+9s^{-1}\right) & s^{-2} \\ -10s^{-2} & -20s^{-2} & s^{-1} \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} + \frac{1}{\Delta(s)} \begin{bmatrix} s^{-3} \\ s^{-2} \\ s^{-1} \end{bmatrix} \frac{1}{s}$$

$$= \frac{1}{\Delta_c(s)} \begin{bmatrix} s^2+9s+20 & s+9 & 1 \\ -10 & s(s+9) & s \\ -10s & -10(2s+1) & s^2 \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} + \frac{1}{\Delta_c(s)} \begin{bmatrix} \frac{1}{s} \\ 1 \\ s \end{bmatrix}$$

$$\Delta(s) = 1 + 9s^{-1} + 20s^{-2} + 10s^{-3} \qquad \Delta_c(s) = s^3 + 9s^2 + 20s + 10$$

$$\mathbf{x}(t)=\left\{\begin{bmatrix}1.612 & 0.946 & 0.114\\ -1.14 & -0.669 & -0.081\\ 0.807 & 0.474 & 0.057\end{bmatrix}e^{-0.708t}+\begin{bmatrix}-0.706 & -1.117 & -0.169\\ 1.692 & 2.678 & 4.056\\ -4.056 & -6.420 & -0.972\end{bmatrix}e^{-2.397t}+\begin{bmatrix}0.0935 & 0.171 & 0.055\\ -0.551 & -1.009 & -0.325\\ 3.249 & 5.947 & 1.915\end{bmatrix}e^{-5.895t}\right\}\mathbf{x}(0)$$

$$+\begin{bmatrix}0.1-0.161e^{-0.708t}+0.0706e^{-2.397t}-0.00935e^{-5.895t}\\ 0.114e^{-0.708t}-0.169e^{-2.397t}+0.055e^{-5.895t}\\ -0.087e^{-0.708t}+0.406e^{-2.397t}-0.325e^{-5.895t}\end{bmatrix} \qquad t\ge 0$$

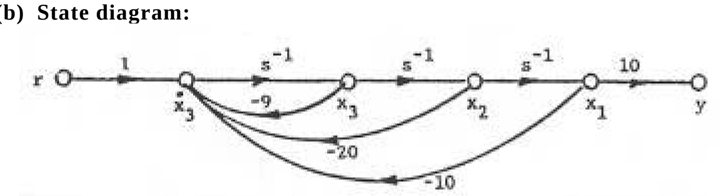
(d) Output:

$$y(t)=10x_1(t)=10\left(1.612e^{-0.708t}-0.706e^{-2.397t}+0.0935e^{-5.897t}\right)x_1(0)+10\left(0.946e^{-0.708t}-1.117e^{-2.397t}+0.1711e^{-5.895t}\right)x_2(0)$$
$$+10\left(1.141e^{-0.708t}-0.169e^{-2.397t}+0.0550e^{-5.895t}\right)x_3(0)+1-1.61e^{-0.708t}+0.706e^{-2.397t}-0.0935e^{-5.895t} \qquad t\ge 0$$

5-27 (a) Closed-loop transfer function:

$$\frac{Y(s)}{R(s)}=\frac{10}{s^3+9s^2+20s+10}$$

(b) State diagram:



(c) State equations:

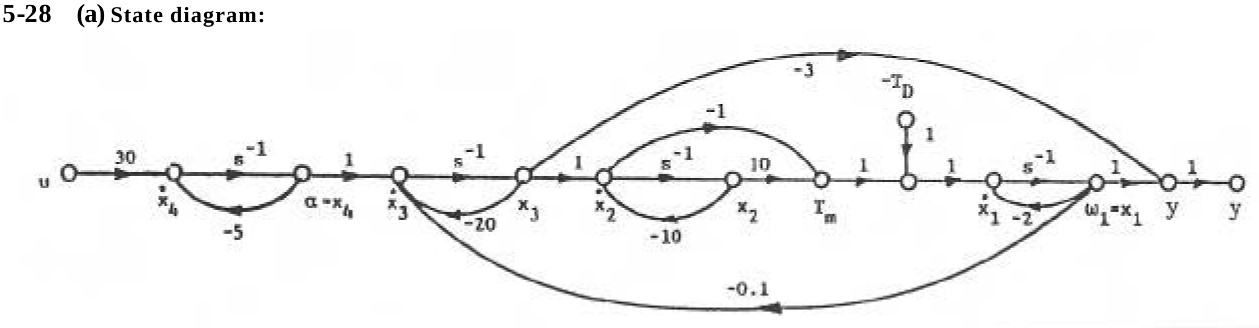
$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix}=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -10 & -20 & -9\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}+\begin{bmatrix}0\\ 0\\ 1\end{bmatrix}r$$

(d) State transition equations:

[Same answers as Problem 5-26(d)]

(e) Output: [Same answer as Problem 5-26(e)]

5-28 (a) State diagram:



(b) State equations:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} -2 & 20 & -1 & 0 \\ 0 & -10 & 1 & 0 \\ -0.1 & 0 & -20 & 1 \\ 0 & 0 & 0 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 0 & 0 \\ 0 & 0 \\ 30 & 0 \end{bmatrix} \begin{bmatrix} u \\ T_D \end{bmatrix}$$

(c) Transfer function relations:

From the system block diagram,

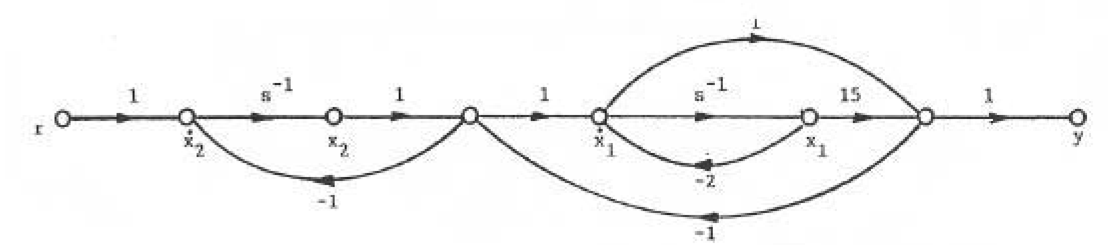
$$Y(s) = \frac{1}{\Delta(s)}\left(\frac{-1}{s+2} T_D(s) + \frac{0.3}{(s+2)(s+20)} T_D(s) + \frac{30e^{-0.2s}U(s)}{(s+2)(s+5)(s+20)} + \frac{90U(s)}{(s+5)(s+20)} \right)$$

$$\Delta(s) = 1 + \frac{0.1e^{-0.2s}}{(s+2)(s+20)} = \frac{(s+2)(s+20) + 0.1e^{-0.2s}}{(s+2)(s+20)}$$

$$Y(s) = \frac{-(s+19.7)}{(s+2)(s+20)+0.1e^{-0.2s}} T_D(s) + \frac{30e^{-0.2s} + 90(s+2)U(s)}{(s+5)\left[(s+2)(s+20)+0.1e^{-0.2s}\right]}$$

$$\Omega(s) = \frac{-(s+20)}{(s+2)(s+20)+0.1e^{-0.2s}} T_D(s) + \frac{30e^{-0.2s}U(s)}{(s+5)\left[(s+2)(s+20)+0.1e^{-0.2s}\right]}$$

5-29 (a) There should not be any incoming branches to a state variable node other than the s^{-1} branch. Thus, we should create a new node as shown in the following state diagram.



(b) State equations: Notice that there is a loop with gain -1 after all the s^{-1} branches are deleted, so $\Delta = 2$.

$$\frac{dx_1}{dt} = \frac{17}{2}x_1 + \frac{1}{2}x_2 \qquad \frac{dx_2}{dt} = \frac{15}{2}x_1 - \frac{1}{2}x_2 + \frac{1}{2}r$$

Output equation: $y = 6.5x_1 + 0.5x_2$

5-30 (a) Transfer function:

$$\frac{Y(s)}{R(s)} = \frac{Ks^2 + 5s + 1}{(s+1)(s^2 + 11s + 2)}$$

(b) Characteristic equation:

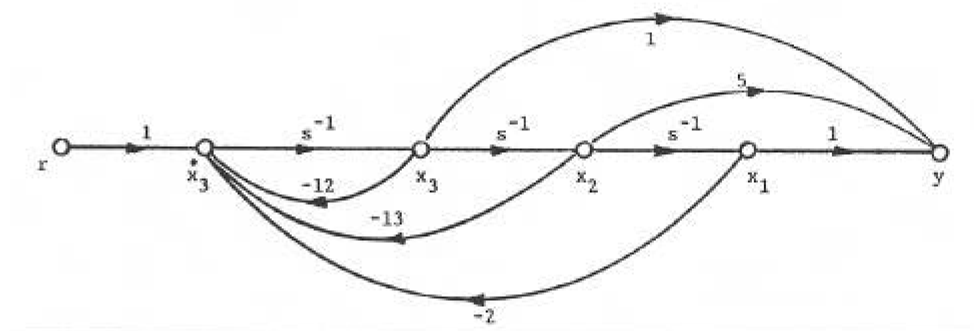
$$(s+1)(s^2 + 11s + 2) = 0$$

Roots of characteristic equation: -1, -0.185, -10.82. These are not functions of K.

(c) When $K = 1$:

$$\frac{Y(s)}{R(s)} = \frac{s^2 + 5s + 1}{s^3 + 12s^2 + 13s + 2}$$

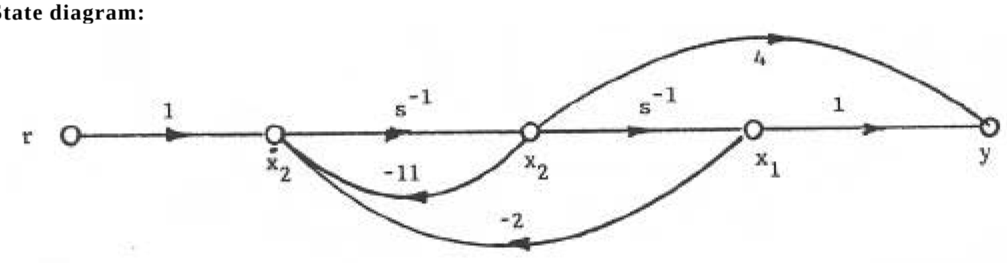
State diagram:



(d) **When** $K = 4$**:**

$$\frac{Y(s)}{R(s)} = \frac{4s^2 + 5s + 1}{(s+1)\left(s^2 + 11s + 2\right)} = \frac{(s+1)(4s+1)}{(s+1)\left(s^2 + 11s + 2\right)} = \frac{4s+1}{s^2 + 11s + 2}$$

State diagram:



(e)

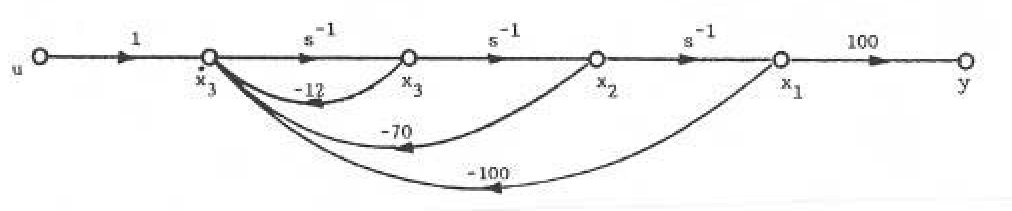
$$\frac{Y(s)}{R(s)} = \frac{Ks^2 + 5s = 1}{(s+1)(s+0.185)(s+10.82)}$$

When $K = 4$, 2.1914, 0.4536, pole-zero cancellation occurs.

5-31 **(a)**

$$G_p(s) = \frac{Y(s)}{U(s)} = \frac{1}{(1+0.5s)\left(1+0.2s+0.02s^2\right)} = \frac{100}{s^3 + 12s^2 + 70s + 100}$$

State diagram by direct decomposition:



State equations:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -100 & -70 & -12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

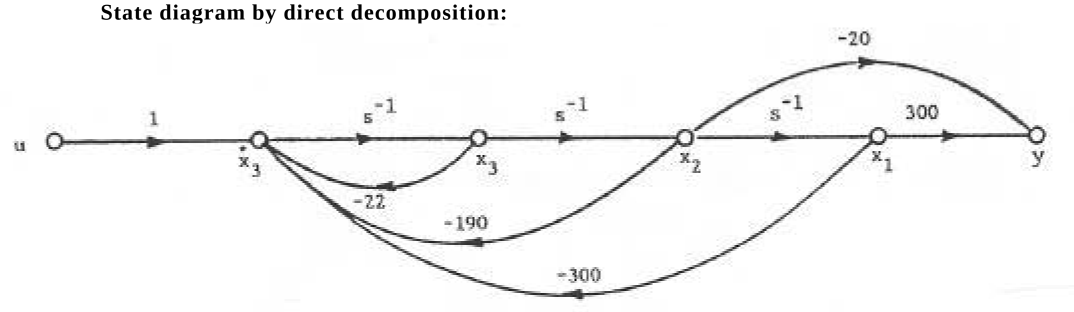
(b) **Characteristic equation of closed-loop system:** **Roots of characteristic equation:**

$s^3 + 12s^2 + 70s + 200 = 0$ $\qquad -5.88, \quad -3.06 + j4.965, \quad -3.06 - j4.965$

5-32 (a)

$$G_p(s) = \frac{Y(s)}{U(s)} \cong \frac{1 - 0.066s}{(1+0.5s)(1+0.133s+0.0067s^2)} = \frac{-20(s-15)}{s^3 + 22s^2 + 190s + 300}$$

State diagram by direct decomposition:



State equations:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -300 & -190 & -22 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Characteristic equation of closed-loop system: $\qquad$ **Roots of characteristic equation:**

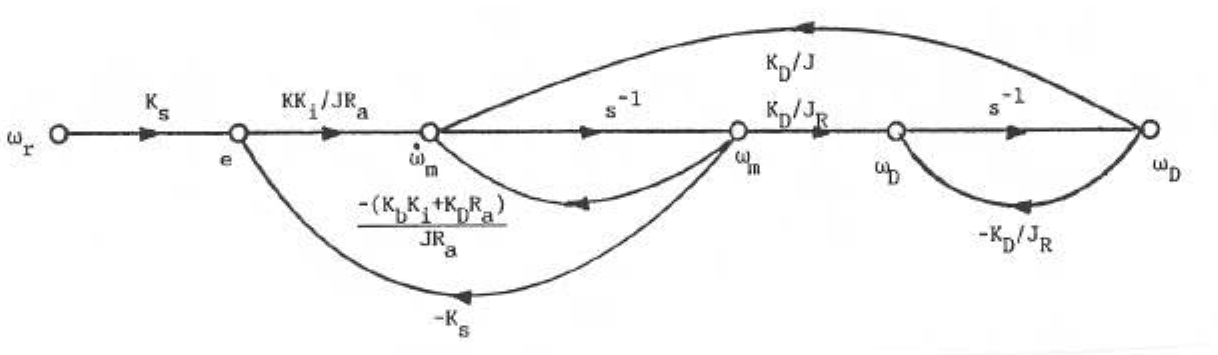
$s^3 + 22s^2 + 170s + 600 = 0$ $\qquad -12, \ -5 + j5, \ -5 - j5$

5-33 (a) State variables: $x_1 = \omega_m$ and $x_2 = \omega_D$

State equations:

$$\frac{d\omega_m}{dt} = -\frac{K_b K_i + K_b R_a}{JR_a} + \frac{K_D}{J}\omega_D + \frac{KK_i}{JR_a}e \qquad \frac{d\omega_D}{dt} = \frac{K_D}{J_R}\omega_m - \frac{K_D}{J_R}\omega_D$$

(b) State diagram:



(c) **Open-loop transfer function:**

$$\frac{\Omega_m(s)}{E(s)} = \frac{KK_i\left(J_R s + K_D\right)}{JJ_RR_a s^2 + \left(K_bJ_RK_i + K_DR_aJ_R + K_DJR_a\right)s + K_DK_bK_i}$$

Closed-loop transfer function:

$$\frac{\Omega_m(s)}{\Omega_r(s)} = \frac{K_sKK_i\left(J_R s + K_D\right)}{JJ_RR_a s^2 + \left(K_bJ_RK_i + K_DR_aJ_R + K_DJR_a + K_sKK_iJ_R\right)s + K_DK_bK_i + K_sKK_iK_D}$$

(d) **Characteristic equation of closed-loop system:**

$$\Delta(s) = JJ_RR_a s^2 + \left(K_DJ_RK_i + K_DR_aJ_R + K_DJR_a + K_sKK_iJ_R\right)s + K_DK_bK_i + K_sKK_iK_D = 0$$

$$\Delta(s) = s^2 + 1037\ s + 20131\ .2 = 0$$

Characteristic equation roots: $-19.8, \quad -1017.2$

5-34 **(a)** **State equations:** $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}r(t)$

$$\mathbf{A} = \begin{bmatrix} -b & d \\ c & -a \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 2 & -1 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \mathbf{S} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$$

Since $\mathbf{S}$ is nonsingular, the system is controllable.

(b)

$$\mathbf{S} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} \end{bmatrix} = \begin{bmatrix} 0 & d \\ 1 & -a \end{bmatrix}$$

The system is controllable for $d \neq 0$.

5-35 **(a)**

$$\mathbf{S} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$$

$\mathbf{S}$ is singular. The system is uncontrollable.

(b)

$$\mathbf{S} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & 4 \\ 1 & -3 & 9 \end{bmatrix}$$

$\mathbf{S}$ is nonsingular. The system is controllable.

5-36 (a) State equations: $\dot{\mathbf{x}}(t) = \mathbf{Ax}(t) + \mathbf{B}u(t)$

$$\mathbf{A} = \begin{bmatrix} -2 & 3 \\ 1 & 0 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \mathbf{S} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$\mathbf{S}$ is singular. The system is uncontrollable.

Output equation: $y = \begin{bmatrix} 1 & 0 \end{bmatrix}\mathbf{x} = \mathbf{Cx}$ $\qquad \mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}$

$$\mathbf{V} = \begin{bmatrix} \mathbf{C}' & \mathbf{A}'\mathbf{C}' \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix}$$

$\mathbf{V}$ is nonsingular. The system is observable.

(b) Transfer function:

$$\frac{Y(s)}{U(s)} = \frac{s+3}{s^2+2s-3} = \frac{1}{s-1}$$

Since there is pole-zero cancellation in the input-output transfer function, the system is either uncontrollable or unobservable or both. In this case, the state variables are already defined, and the system is uncontrollable as found out in part (a).

5-37 (a) $a = 1,\ 2,$ or 4. These values of α will cause pole-zero cancellation in the transfer function.

(b) The transfer function is expanded by partial fraction expansion,

$$\frac{Y(s)}{R(s)} = \frac{a-1}{3(s+1)} - \frac{a-2}{2(s+2)} + \frac{a-4}{6(s+4)}$$

By parallel decomposition, the state equations are: $\dot{\mathbf{x}}(t) = \mathbf{Ax}(t) + \mathbf{B}r(t)$, output equation: $y(t) = \mathbf{Cx}(t)$.

$$\mathbf{A} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -4 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} a-1 \\ a-2 \\ a-4 \end{bmatrix} \qquad \mathbf{D} = \begin{bmatrix} \frac{1}{3} & -\frac{1}{2} & \frac{1}{6} \end{bmatrix}$$

The system is uncontrollable for α = 1, or α = 2, or α = 4.

(c) Define the state variables so that

$$\mathbf{A} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -4 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} \frac{1}{3} \\ \frac{-1}{2} \\ \frac{-1}{6} \end{bmatrix} \qquad \mathbf{D} = \begin{bmatrix} a-1 & a-2 & a-4 \end{bmatrix}$$

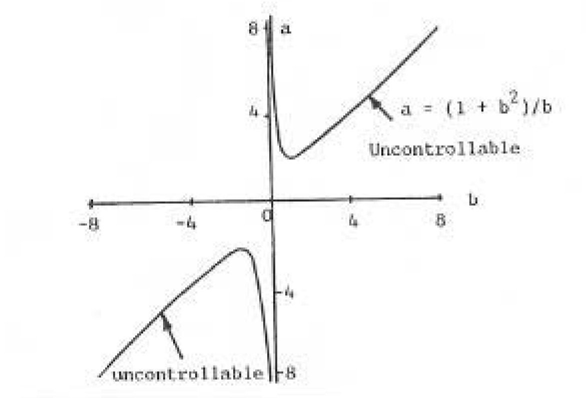
The system is unobservable for α = 1, or α = 2, or α = 4.

5-38

$$\mathbf{S} = [\mathbf{B} \quad \mathbf{AB}] = \begin{bmatrix} 1 & b \\ b & ab-1 \end{bmatrix} \qquad |\mathbf{S}| = ab - 1 - b^2 \neq 0$$

The boundary of the region of controllability is described by $ab - 1 - b^2 = 0$.

Regions of controllability:



5-39

$$\mathbf{S} = [\mathbf{B} \quad \mathbf{AB}] = \begin{bmatrix} b_1 & b_1 + b_2 \\ b_2 & b_2 \end{bmatrix} \qquad |\mathbf{S}| = 0 \text{ when } b_1 b_2 - b_1 b_2 - b_2^2 = 0, \text{or } b_2 = 0$$

The system is completely controllable when $b_2 \neq 0$.

$$\mathbf{V} = \begin{bmatrix} \mathbf{C}' & \mathbf{A}'\mathbf{C}' \end{bmatrix} = \begin{bmatrix} d_1 & d_2 \\ d_2 & d_1 + d_2 \end{bmatrix} \qquad |\mathbf{V}| = 0 \text{ when } d_1 \neq 0.$$

The system is completely observable when $d_2 \neq 0$.

5-40 (a) State equations:

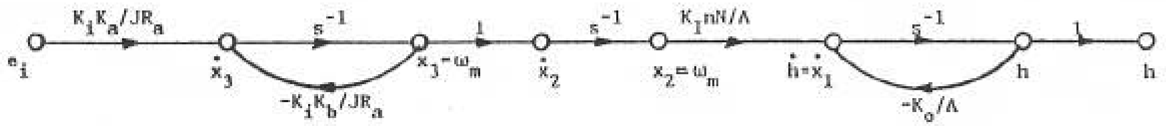
$$\frac{dh}{dt} = \frac{1}{A}(q_i - q_o) = \frac{K_I nN}{A} q_m - \frac{K_o}{A} h \qquad \frac{dq_m}{dt} = w_m \qquad \frac{dw_m}{dt} = -\frac{K_i K_b}{JR_a} w_m + \frac{K_i K_a}{JR_a} e_i \qquad J = J_m + n^2 J_L$$

State variable: $x_1 = h, \quad x_2 = q_m, \quad x_3 = \frac{dq_m}{dt} = w_m$

State equations: $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}e_i$

$$\mathbf{A} = \begin{bmatrix} \frac{-K_o}{A} & \frac{K_I nN}{A} & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -\frac{K_i K_b}{JR_a} \end{bmatrix} = \begin{bmatrix} -1 & 0.016 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -11.767 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ \frac{K_i K_a}{JR_a} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 8333.33 \end{bmatrix}$$

State diagram:



(b) Characteristic equation of A:

$$|s\mathbf{I}-\mathbf{A}| = \begin{vmatrix} s+\dfrac{K_o}{A} & \dfrac{-K_I nN}{A} & 0 \\ 0 & s & -1 \\ 0 & 0 & s+\dfrac{K_i K_b}{JR_a} \end{vmatrix} = s\left(s+\frac{K_o}{A}\right)\left(s+\frac{K_i K_b}{JR_a}\right) = s(s+1)(s+11.767)$$

Eigenvalues of A: $0, -1, -11.767.$

(c) Controllability:

$$\mathbf{S} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 133.33 \\ 0 & 8333.33 & -98058 \\ 8333.33 & -98058 & 1153848 \end{bmatrix}$$

$|\mathbf{S}| \neq 0.$ The system is controllable.

(d) Observability:

(1) $\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$:

$$\mathbf{V} = \begin{bmatrix} \mathbf{C}' & \mathbf{A}'\mathbf{C}' & (\mathbf{A}')^2\mathbf{C}' \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0.016 & -0.016 \\ 0 & 0 & 0.016 \end{bmatrix}$$

V is nonsingular. The system is observable.

(2) $\mathbf{C} = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$:

$$\mathbf{V} = \begin{bmatrix} \mathbf{C}' & \mathbf{A}'\mathbf{C}' & (\mathbf{A}')^2\mathbf{C}' \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & -11.767 \end{bmatrix}$$

V is singular. The system is unobservable.

(3) $\mathbf{C} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$:

$$\mathbf{V} = \begin{bmatrix} \mathbf{C}' & \mathbf{A}'\mathbf{C}' & (\mathbf{A}')^2\mathbf{C}' \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & -11.767 & 138.46 \end{bmatrix}$$

V is singular. The system is unobservable.

5-41 (a) Characteristic equation: $\Delta(s) = |s\mathbf{I}-\mathbf{A}^*| = s^4 - 25.92 s^2 = 0$

Roots of characteristic equation: $-5.0912, \quad 5.0912, \; 0, \; 0$

(b) Controllability:

$$\mathbf{S} = \begin{bmatrix} \mathbf{B}^* & \mathbf{A}^*\mathbf{B}^* & \mathbf{A}^{*2}\mathbf{B}^* & \mathbf{A}^{*3}\mathbf{B}^* \end{bmatrix} = \begin{bmatrix} 0 & -0.0732 & 0 & -1.8973 \\ -0.0732 & 0 & -1.8973 & 0 \\ 0 & 0.0976 & 0 & 0.1728 \\ 0.0976 & 0 & 0.1728 & 0 \end{bmatrix}$$

$\mathbf{S}$ is nonsingular. Thus, $\begin{bmatrix}\mathbf{A}^*, & \mathbf{B}^*\end{bmatrix}$ is controllable.

(c) Observability:

(1) $\mathbf{C}^* = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$

$$\mathbf{V} = \begin{bmatrix} \mathbf{C}^{*\prime} & \mathbf{A}^{*\prime}\mathbf{C}^{*\prime} & (\mathbf{A}^{*\prime})^2\mathbf{C}^{*\prime} & (\mathbf{A}^{*\prime})^3\mathbf{C}^{*\prime} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 25.92 & 0 \\ 0 & 1 & 0 & 25.92 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\mathbf{S}$ is singular. The system is unobservable.

(2) $\mathbf{C}^* = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}$

$$\mathbf{V} = \begin{bmatrix} \mathbf{C}^{*\prime} & \mathbf{A}^{*\prime}\mathbf{C}^{*\prime} & (\mathbf{A}^{*\prime})^2\mathbf{C}^{*\prime} & (\mathbf{A}^{*\prime})^3\mathbf{C}^{*\prime} \end{bmatrix} = \begin{bmatrix} 0 & 25.92 & 0 & 671.85 \\ 1 & 0 & 25.92 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\mathbf{S}$ is singular. The system is unobservable.

(3) $\mathbf{C}^* = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}$

$$\mathbf{V} = \begin{bmatrix} \mathbf{C}^{*\prime} & \mathbf{A}^{*\prime}\mathbf{C}^{*\prime} & (\mathbf{A}^{*\prime})^2\mathbf{C}^{*\prime} & (\mathbf{A}^{*\prime})^3\mathbf{C}^{*\prime} \end{bmatrix} = \begin{bmatrix} 0 & 0 & -2.36 & 0 \\ 0 & 0 & 0 & -2.36 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$\mathbf{S}$ is nonsingular. The system is observable.

(4) $\mathbf{C}^* = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}$

$$\mathbf{V} = \begin{bmatrix} \mathbf{C}^{*'} & \mathbf{A}^{*'}\mathbf{C}^{*'} & (\mathbf{A}^{*'})^2\mathbf{C}^{*'} & (\mathbf{A}^{*'})^3\mathbf{C}^{*'} \end{bmatrix} = \begin{bmatrix} 0 & -2.36 & 0 & -61.17 \\ 0 & 0 & -2.36 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$\mathbf{S}$ is singular. The system is unobservable.

5-42 The controllability matrix is

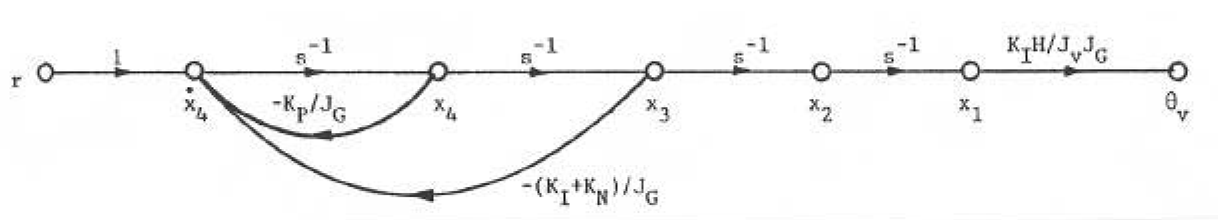
$$\mathbf{S} = \begin{bmatrix} 0 & -1 & 0 & -16 & 0 & -384 \\ -1 & 0 & -16 & 0 & -384 & 0 \\ 0 & 0 & 0 & 16 & 0 & 512 \\ 0 & 0 & 16 & 0 & 512 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Rank of $\mathbf{S}$ is 6. The system is controllable.

5-43 (a) Transfer function:

$$\frac{\Theta_v(s)}{R(s)} = \frac{K_I H}{J_v s^2 \left(J_G s^2 + K_P s + K_I + K_N \right)}$$

State diagram by direct decomposition:



State equations: $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}r(t)$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \dfrac{-\left(K_I + K_N\right)}{J_G} & \dfrac{-K_P}{J_G} \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

(b) Characteristic equation: $J_v s^2 \left(J_G s^2 + K_P s + K_I + K_N \right) = 0$

5-44 (a) **State equations:** $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u_1(t)$

$$\mathbf{A} = \begin{bmatrix} -3 & 1 \\ 0 & -2 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \mathbf{S} = [\mathbf{B} \quad \mathbf{AB}] = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}$$

$\mathbf{S}$ is nonsingular. $[\mathbf{A}, \quad \mathbf{B}]$ is controllable.

Output equation: $y_2 = \mathbf{C}\mathbf{x}$ $\qquad \mathbf{C} = [-1 \quad 1]$

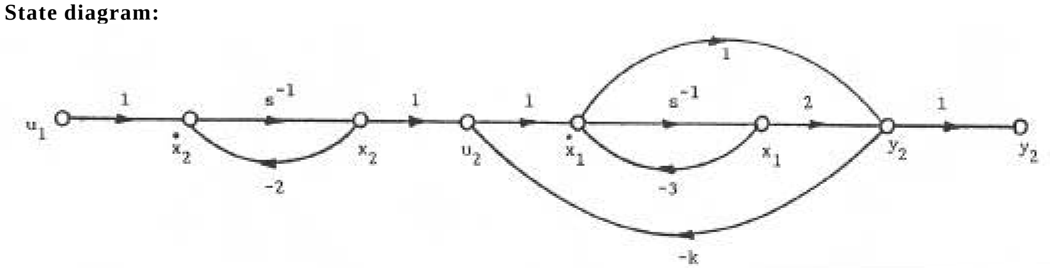
$$\mathbf{V} = [\mathbf{C}' \quad \mathbf{A}'\mathbf{C}'] = \begin{bmatrix} -1 & 3 \\ 1 & -3 \end{bmatrix}$$ $\mathbf{V}$ is singular. The system is unobservable.

(b) With feedback, $u_2 = -kc_2$, the state equation is: $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u_1(t)$.

$$\mathbf{A} = \begin{bmatrix} \frac{-3-2k}{1+g} & \frac{1}{1+k} \\ 0 & -2 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \mathbf{S} = \begin{bmatrix} 0 & \frac{1}{1+k} \\ 1 & -2 \end{bmatrix}$$

$\mathbf{S}$ is nonsingular for all finite values of k. The system is controllable.

State diagram:



Output equation: $y_2 = \mathbf{C}\mathbf{x}$ $\qquad \mathbf{C} = \begin{bmatrix} \frac{-1}{1+k} & \frac{1}{1+k} \end{bmatrix}$

$$\mathbf{V} = [\mathbf{D}' \quad \mathbf{A}'\mathbf{D}'] = \begin{bmatrix} \frac{-1}{1+K} & \frac{3+2k}{(1+k)^2} \\ \frac{1}{1+k} & -\frac{3+2k}{(1+k)^2} \end{bmatrix}$$

$\mathbf{V}$ is singular for any k. The system with feedback is unobservable.

5-45 (a)

$$\mathbf{S} = [\mathbf{B} \quad \mathbf{AB}] = \begin{bmatrix} 1 & 2 \\ 2 & -7 \end{bmatrix}$$ $\mathbf{S}$ is nonsingular. System is controllable.

$$\mathbf{V} = [\mathbf{C}' \quad \mathbf{A}'\mathbf{C}'] = \begin{bmatrix} 1 & -1 \\ 1 & -2 \end{bmatrix}$$ $\mathbf{V}$ is nonsingular. System is observable.

(b) $u = -[k_1 \quad k_2]\mathbf{x}$

$$\mathbf{A}_c = \mathbf{A} - \mathbf{BK} = \begin{bmatrix} 0 & 1 \\ -1 & -3 \end{bmatrix} - \begin{bmatrix} k_1 & k_2 \\ 2k_1 & 2k_2 \end{bmatrix} = \begin{bmatrix} -k_1 & 1-k_2 \\ -1-2k_1 & -3-2k_2 \end{bmatrix}$$

$$\mathbf{S} = \left[\mathbf{B} \quad \mathbf{A}_c\mathbf{B}\right] = \begin{bmatrix} 1 & -k_1 - 2k_2 + 2 \\ 2 & -7 - 2k_1 - 4k_2 \end{bmatrix} \qquad \left|\mathbf{S}\right| = -11 - 2k_2 \neq 0$$

For controllabillity, $k_2 \neq -\dfrac{11}{2}$

$$\mathbf{V} = \left[\mathbf{C}' \quad \mathbf{A}_c'\mathbf{C}'\right] = \begin{bmatrix} -1 & -1 - 3k_1 \\ 1 & -2 - 3k_2 \end{bmatrix}$$

For observability, $\left|\mathbf{V}\right| = -1 + 3k_1 - 3k_2 \neq 0$

Chapter 6 STABILITY OF LINEAR CONTROL SYSTEMS

6-1 (a) Poles are at $s = 0, -1.5 + j1.6583, -1.5 - j1.6583$ One poles at $s = 0$. **Marginally stable**.

(b) Poles are at $s = -5, -j\sqrt{2}, j\sqrt{2}$ Two poles on $j\omega$ axis. **Marginally stable**.

(c) Poles are at $s = -0.8688, 0.4344 + j2.3593, 0.4344 - j2.3593$ Two poles in RHP. **Unstable**.

(d) Poles are at $s = -5, -1 + j, -1 - j$ All poles in the LHP. **Stable**.

(e) Poles are at $s = -1.3387, 1.6634 + j2.164, 1.6634 - j2.164$ Two poles in RHP. **Unstable**.

(f) Poles are at $s = -22.8487 \pm j22.6376, 21.3487 \pm j22.6023$ Two poles in RHP. **Unstable**.

6-2 (a) $s^3 + 25s^2 + 10s + 450 = 0$ **Roots:** $-25.31, 0.1537 + j4.214, 0.1537 - 4.214$

Routh Tabulation:

$$\begin{array}{lll} s^3 & 1 & 10 \\ s^2 & 25 & 450 \\ s^1 & \dfrac{250 - 450}{25} = -8 & 0 \\ s^0 & 450 & \end{array}$$

Two sign changes in the first column. Two roots in RHP.

(b) $s^3 + 25s^2 + 10s + 50 = 0$ **Roots:** $-24.6769, -0.1616 + j1.4142, -0.1616 - j1.4142$

Routh Tabulation:

$$\begin{array}{lll} s^3 & 1 & 10 \\ s^2 & 25 & 50 \\ s^1 & \dfrac{250 - 50}{25} = 8 & 0 \\ s^0 & 50 & \end{array}$$

No sign changes in the first column. No roots in RHP.

(c) $s^3 + 25s^2 + 250s + 10 = 0$ **Roots:** $-0.0402, -12.48 + j9.6566, -j9.6566$

Routh Tabulation:

$$\begin{array}{lll} s^3 & 1 & 250 \\ s^2 & 25 & 10 \\ s^1 & \dfrac{6250 - 10}{25} = 249.6 & 0 \\ s^0 & 10 & \end{array}$$

No sign changes in the first column. No roots in RHP.

(d) $2s^4 + 10s^3 + 5.5s^2 + 5.5s + 10 = 0$ **Roots:** $-4.466, -1.116, 0.2888 + j0.9611, 0.2888 - j0.9611$

Routh Tabulation:

s^4	2	5.5	10
s^3	10	5.5	
s^2	$\dfrac{55-11}{10}=4.4$	10	
s^1	$\dfrac{24.2-100}{4.4}=-75.8$		
s^0	10		

Two sign changes in the first column. Two roots in RHP.

(e) $s^6+2s^5+8s^4+15s^3+20s^2+16s+16=0$ **Roots:** $-1.222\pm j0.8169,\ 0.0447\pm j1.153,\ 0.1776\pm j2.352$

Routh Tabulation:

s^6	1	8	20	16
s^5	2	15	16	
s^4	$\dfrac{16-15}{2}=0.5$	$\dfrac{40-16}{2}=12$		
s^3	-33	-48		
s^2	$\dfrac{-396+24}{-33}=11.27$	16		
s^1	$\dfrac{-541.1+528}{11.27}=-1.16$	0		
s^0	0			

Four sign changes in the first column. Four roots in RHP.

(f) $s^4+2s^3+10s^2+20s+5=0$ **Roots:** $-0.29,\ -1.788,\ 0.039+j3.105,\ 0.039-j3.105$

Routh Tabulation:

s^4	1	10	5	
s^3	2	20		
s^2	$\dfrac{20-20}{2}=0$	5		
s^2	ϵ	5		Replace 0 in last row by ϵ
s^1	$\dfrac{20\epsilon-10}{\epsilon}\cong-\dfrac{10}{\epsilon}$			**Two sign changes in first column. Two roots in RHP.**
s^0	5			

6-3 (a) $s^4+25s^3+15s^2+20s+K=0$

Routh Tabulation:

s^4	1	15	K	
s^3	25	20		
s^2	$\dfrac{375-20}{25}=14.2$	K		
s^1	$\dfrac{284-25K}{14.2}=20-1.76K$			$20-1.76K>0$ or $K<11.36$
s^0	K			$K>0$

Thus, the system is stable for $0 < K < 11.36$. When $K = 11.36$, the system is marginally stable. The auxiliary equation equation is $A(s)=14.2s^2+11.36=0$. The solution of $A(s) = 0$ is $s^2=-0.8$. The frequency of oscillation is 0.894 rad/sec.

(b) $s^4+Ks^3+2s^2+(K+1)s+10=0$

Routh Tabulation:

s^4	1	2	10	
s^3	K	$K+1$		$K>0$
s^2	$\dfrac{2K-K-1}{K}=\dfrac{K-1}{K}$	10		$K>1$
s^1	$\dfrac{-9K^2-1}{K-1}$			$-9K^2-1>0$
s^0	10			

The conditions for stability are: $K > 0$, $K > 1$, and $-9K^2-1>0$. Since K^2 is always positive, the last condition cannot be met by any real value of K. Thus, the system is unstable for all values of K.

(c) $s^3+(K+2)s^2+2Ks+10=0$

Routh Tabulation:

s^3	1	$2K$	
s^2	$K+2$	10	$K>-2$
s^1	$\dfrac{2K^2+4K-10}{K+2}$		$K^2+2K-5>0$
s^0	10		

The conditions for stability are: $K > -2$ and $K^2+2K-5>0$ or $(K+3.4495)(K-1.4495) > 0$, or $K > 1.4495$. Thus, the condition for stability is $K > 1.4495$. When $K = 1.4495$ the system is marginally stable. The auxiliary equation is $A(s)=3.4495s^2+10=0$. The solution is $s^2=-2.899$. The frequency of oscillation is 1.7026 rad/sec.

(d) $s^3+20s^2+5s+10K=0$

Routh Tabulation:

s^3	1	5	
s^2	20	$10K$	
s^1	$\dfrac{100-10K}{20}=5-0.5K$		$5-0.5K>0$ or $K<10$
s^0	$10K$		$K>0$

The conditions for stability are: $K > 0$ and $K < 10$. Thus, $0 < K < 10$. When $K = 10$, the system is marginally stable. The auxiliary equation is $A(s)=20s^2+100=0$. The solution of the auxiliary equation is $s^2=-5$. The frequency of oscillation is 2.236 rad/sec.

(e) $s^4+Ks^3+5s^2+10s+10K=0$

Routh Tabulation:

s^4	1	5	$10K$	
s^3	K	10		$K>0$
s^2	$\dfrac{5K-10}{K}$	$10K$		$5K-10>0$ or $K>2$
s^1	$\dfrac{\dfrac{50K-100}{K}-10K^2}{\dfrac{5K-10}{K}}=\dfrac{50K-100-10K^3}{5K-10}$			$5K-10-K^3>0$
s^0	$10K$			$K>0$

The conditions for stability are: $K > 0$, $K > 2$, and $5K-10-K^3>0$. The last condition is written as $(K+2.9055)(K^2-2.9055K+3.4419)<0$. The second-order term is positive for all values of K.

Thus, the conditions for stability are: $K > 2$ and $K < -2.9055$. Since these are contradictory, the system is unstable for all values of K.

(f) $s^4+12.5s^3+s^2+5s+K=0$

Routh Tabulation:

s^4	1	1	K	
s^3	12.5	5		
s^2	$\dfrac{12.5-5}{12.5}=0.6$	K		
s^1	$\dfrac{3-12.5K}{0.6}=5-20.83K$			$5-20.83K>0$ or $K<0.24$
s^0	K			$K>0$

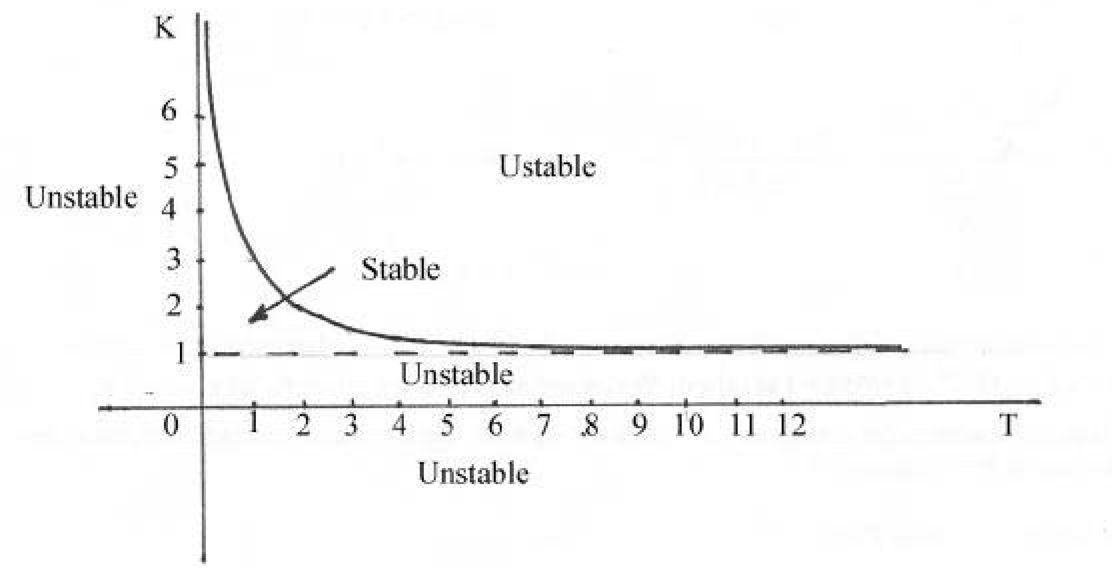
The condition for stability is $0 < K < 0.24$. When $K = 0.24$ the system is marginally stable. The auxiliary equation is $A(s)=0.6s^2+0.24=0$. The solution of the auxiliary equation is $s^2=-0.4$. The frequency of oscillation is 0.632 rad/sec.

6-4 The characteristic equation is $Ts^3+(2T+1)s^2+(2+K)s+5K=0$

Routh Tabulation:

s^3	T	$K+2$	$T>0$
s^2	$2T+1$	$5K$	$T>-1/2$
s^1	$\dfrac{(2T+1)(K+2)-5KT}{2T+1}$		$K(1-3T)+4T+2>0$
s^0	$5K$		$K>0$

The conditions for stability are: $T > 0$, $K > 0$, and $K < \dfrac{4T+2}{3T-1}$. The regions of stability in the T-versus-K parameter plane is shown below.



6-5 (a) Characteristic equation: $s^5 + 600s^4 + 50000s^3 + Ks^2 + 24Ks + 80K = 0$

Routh Tabulation:

s^5	1	50000	$24K$	
s^4	600	K	$80K$	
s^3	$\dfrac{3\times10^7 - K}{600}$	$\dfrac{14320K}{600}$		$K < 3\times10^7$
s^2	$\dfrac{21408000K - K^2}{3\times10^7 - K}$	$80K$		$K < 21408000$
s^1	$\dfrac{-7.2\times10^{16} + 3.113256\times10^{11}K - 14400K^2}{600(21408000 - K)}$			$K^2 - 2.162\times10^7 K + 5\times10^{12} < 0$
s^0	$80K$			$K > 0$

Conditions for stability:

From the s^3 row: $K < 3 \times 10^7$

From the s^2 row: $K < 2.1408 \times 10^7$

From the s^1 row: $K^2 - 2.162 \times 10^7 K + 5 \times 10^{12} < 0$ or $(K - 2.34 \times 10^5)(K - 2.1386 \times 10^7) < 0$

Thus, $2.34 \times 10^5 < K < 2.1386 \times 10^7$

From the s^0 row: $K > 0$

Thus, the final condition for stability is: $2.34 \times 10^5 < K < 2.1386 \times 10^7$

When $K = 2.34 \times 10^5$ $\omega = 10.6$ rad/sec.

When $K = 2.1386 \times 10^7$ $\omega = 188.59$ rad/sec.

(b) Characteristic equation: $s^3 + (K+2)s^2 + 30Ks + 200K = 0$

Routh tabulation:

s^3	1	$30K$	
s^2	$K+2$	$200K$	$K > -2$
s^1	$\dfrac{30K^2 - 140K}{K+2}$		$K > 4.6667$
s^0	$200K$		$K > 0$

Stability Condition: $K > 4.6667$

When $K = 4.6667$, the auxiliary equation is $A(s) = 6.6667s^2 + 933.333 = 0$. The solution is $s^2 = -140$.
The frequency of oscillation is 11.832 rad/sec.

(c) Characteristic equation: $s^3 + 30s^2 + 200s + K = 0$

Routh tabulation:

s^3	1	200	
s^2	30	K	
s^1	$\dfrac{6000 - K}{30}$		$K < 6000$
s^0	K		$K > 0$

Stabililty Condition: $0 < K < 6000$

When $K = 6000$, the auxiliary equation is $A(s) = 30s^2 + 6000 = 0$. The solution is $s^2 = -200$.
The frequency of oscillation is 14.142 rad/sec.

(d) Characteristic equation: $s^3 + 2s^2 + (K+3)s + K + 1 = 0$

Routh tabulation:

s^3	1	$K+3$	
s^2	2	$K+1$	
s^1	$\dfrac{K+5}{30}$		$K > -5$
s^0	$K+1$		$K > -1$

Stability condition: $K > -1$. When $K = -1$ the zero element occurs in the first element of the

s^0 row. Thus, there is no auxiliary equation. When $K = -1$, the system is marginally stable, and one of the three characteristic equation roots is at $s = 0$. There is no oscillation. The system response would increase monotonically.

6-6 State equation: **Open-loop system:** $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)$

$$\mathbf{A} = \begin{bmatrix} 1 & -2 \\ 10 & 0 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Closed-loop system: $\dot{\mathbf{x}}(t) = (\mathbf{A} - \mathbf{BK})\mathbf{x}(t)$

$$\mathbf{A} - \mathbf{BK} = \begin{bmatrix} 1 & -2 \\ 10 - k_1 & -k_2 \end{bmatrix}$$

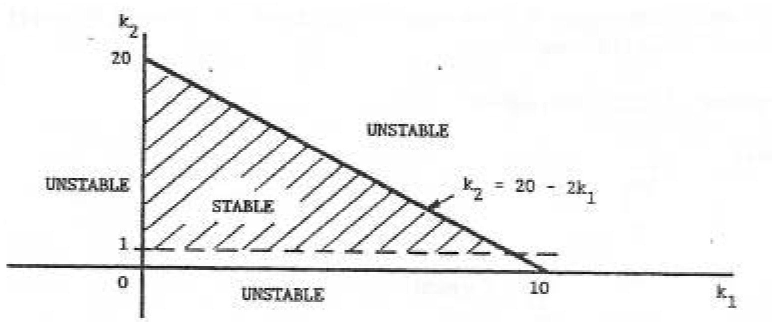
Characteristic equation of the closed-loop system:

$$\left| s\mathbf{I} - \mathbf{A} + \mathbf{BK} \right| = \begin{vmatrix} s-1 & 2 \\ -10 + k_1 & s + k_2 \end{vmatrix} = s^2 + \left(k_2 - 1\right)s + 20 - 2k_1 - k_2 = 0$$

Stability requirements:

$$k_2 - 1 > 0 \quad \text{or} \quad k_2 > 1$$
$$20 - 2k_1 - k_2 > 0 \quad \text{or} \quad k_2 < 20 - 2k_1$$

Parameter plane:



6-7 Characteristic equation of closed-loop system:

$$\left| s\mathbf{I} - \mathbf{A} + \mathbf{BK} \right| = \begin{vmatrix} s & -1 & 0 \\ 0 & s & -1 \\ k_1 & k_2 + 4 & s + k_3 + 3 \end{vmatrix} = s^3 + \left(k_3 + 3\right)s^2 + \left(k_2 + 4\right)s + k_1 = 0$$

Routh Tabulation:

s^3	1	$k_2 + 4$	
s^2	$k_3 + 3$	k_1	$k_3 + 3 > 0$ or $k_3 > -3$
s^1	$\dfrac{\left(k_3 + 3\right)\left(k_2 + 4\right) - k_1}{k_3 + 3}$		$\left(k_3 + 3\right)\left(k_2 + 4\right) - k_1 > 0$
s^0	k_1	$k_1 > 0$	

Stability Requirements:

$$k_3 > -3, \quad k_1 > 0, \quad \left(k_3 + 3\right)\left(k_2 + 4\right) - k_1 > 0$$

6-8 (a) Since **A** is a diagonal matrix with distinct eigenvalues, the states are decoupled from each other. The second row of **B** is zero; thus, the second state variable, x_2 is uncontrollable. Since the uncontrollable state has the eigenvalue at -3 which is stable, and the unstable state x_3 with the eigenvalue at -2 is controllable, the system is stabilizable.

(b) Since the uncontrollable state x_1 has an unstable eigenvalue at 1, the system is no stabilizable.

6-9 The closed-loop transfer function of the sysetm is

$$\frac{Y(s)}{R(s)} = \frac{1000}{s^3 + 15.6s^2 + \left(56 + 100K_t\right)s + 1000}$$

The characteristic equation is: $s^3 + 15.6s^2 + (56 + 100K_t)s + 1000 = 0$

Routh Tabulation:

$$\begin{array}{llll}
s^3 & 1 & 56 + 100K_t & \\
s^2 & 15.6 & 1000 & \\
s^1 & \dfrac{873.6 + 1560K_t - 1000}{15.6} & & 1560K_t - 126.4 > 0 \\
s^0 & 1000 & &
\end{array}$$

Stability Requirements: $K_t > 0.081$

6-10 The closed-loop transfer function is

$$\frac{Y(s)}{R(s)} = \frac{K(s+2)(s+\alpha)}{s^3 + Ks^2 + (2K + \alpha K - 1)s + 2\alpha K}$$

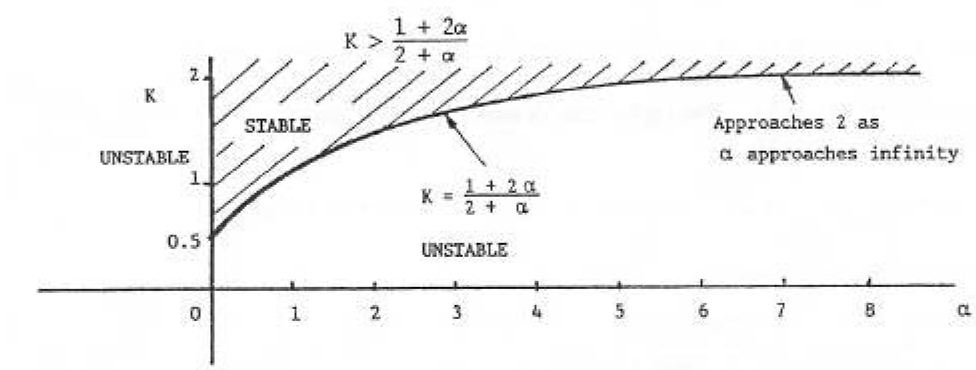
The characteristic equation: $s^3 + Ks^2 + (2K + \alpha K - 1)s + 2\alpha K = 0$

Routh Tabulation:

$$\begin{array}{llll}
s^3 & 1 & 2K + \alpha K - 1 & \\
s^2 & K & 2\alpha K & K > 0 \\
s^1 & \dfrac{(2+\alpha)K^2 - K - 2\alpha K}{K} & & (2+\alpha)K - 1 - 2\alpha > 0 \\
s^0 & 2\alpha K & & \alpha > 0
\end{array}$$

Stability Requirements: $\alpha > 0, \quad K > 0, \quad K > \dfrac{1 + 2\alpha}{2 + \alpha}$.

***K*-versus-α Parameter Plane:**



6-11 (a) Only the attitude sensor loop is in operation: $K_t = 0$. The system transfer function is:

$$\frac{\Theta(s)}{\Theta_r(s)} = \frac{G_p(s)}{1 + K_s G_p(s)} = \frac{K}{s^2 - a + KK_s}$$

If $KK_s > a$, the characteristic equation roots are on the imaginary axis, and the missible will oscillate.

If $KK_s \le a$, the characteristic equaton roots are at the origin or in the right-half plane, and the system is unstable. The missile will tumble end over end.

(b) Both loops are in operation: The system transfer function is

$$\frac{\Theta(s)}{\Theta_r(s)} = \frac{G_p(s)}{1 + K_t s G_p(s) + K_s G_p(s)} = \frac{K}{s^2 + KK_t s + KK_s - a}$$

For stability: $KK_t > 0$, $KK_s - a > 0$.

When $K_t = 0$ and $KK_s > a$, the characteristic equation roots are on the imaginary axis, and the missile will oscillate back and forth.

For any $KK_s - a$ if $KK_t < 0$, the characteristic equation roots are in the right-half plane, and the system is unstable. The missile will tumble end over end.

If $KK_t > 0$, and $KK_t < a$, the characteristic equation roots are in the right-half plane, and the system is unstable. The missile will tumble end over end.

6-12 Let $s_1 = s + a$, then when $s = -a$, $s_1 = 0$. This transforms the $s = -a$ axis in the *s*-plane onto the imaginary axis of the s_1-plane.

(a) $F(s) = s^2 + 5s + 3 = 0$ Le t $s = s_1 - 1$ We get $(s_1 - 1)^2 + 5(s_1 - 1) + 3 = 0$

Or $s_1^2 + 3s_1 - 1 = 0$

Routh Tabulation:

s_1^2	1	-1
s_1^1	3	
s_1^0	-1	

Since there is one sign change in the first column of the Routh tabulation, there is one root in the region to the right of $s = -1$ in the *s*-plane. The roots are at -3.3028 and 0.3028.

(b) $F(s) = s^3 + 3s^2 + 3s + 1 = 0$ Let $s = s_1 - 1$ We get $(s_1 - 1)^3 + 3(s_1 - 1)^2 + 3(s_1 - 1) + 1 = 0$

Or $s_1^3 = 0$. The three roots in the s_1-plane are all at $s_1 = 0$. Thus, $F(s)$ has three roots at $s = -1$.

(c) $F(s) = s^3 + 4s^2 + 3s + 10 = 0$ Let $s = s_1 - 1$ We get $(s_1 - 1)^3 + 4(s_1 - 1)^2 + 3(s_1 - 1) + 10 = 0$

Or $s_1^3 + s_1^2 - 2s_1 + 10 = 0$

Routh Tabulation:

s_1^3	1	-2
s_1^2	1	10
s_1^1	-12	
s_1^0	10	

Since there are two sign changes in the first column of the Routh tabulation, $F(s)$ has two roots in the region to the right of $s = -1$ in the s-plane. The roots are at -3.8897, $-0.0552 + j1.605$, and $-0.0552 - j1.6025$.

(d) $F(s) = s^3 + 4s^2 + 4s + 4 = 0$ Let $s = s_1 - 1$ We get $(s_1 - 1)^3 + 4(s_1 - 1)^2 + 4(s_1 - 1) + 4 = 0$

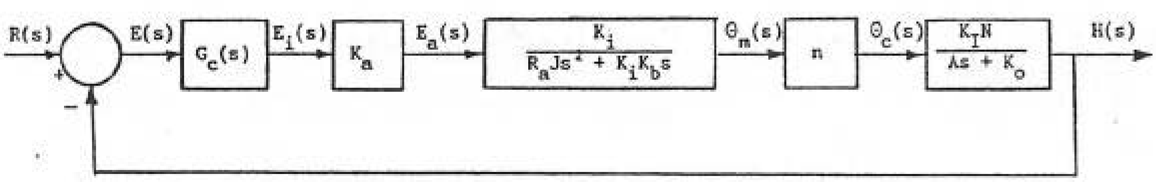
Or $s_1^3 + s_1^2 - s_1 + 3 = 0$

Routh Tabulation:

s_1^3	1	-1
s_1^2	1	3
s_1^1	-4	
s_1^0	3	

Since there are two sign changes in the first column of the Routh tabulation, $F(s)$ has two roots in the region to the right of $s = -1$ in the s-plane. The roots are at -3.1304, $-0.4348 + j1.0434$, and $-0.4348 - j1.04348$.

6-13 (a) Block diagram:



(b) Open-loop transfer function:

$$G(s) = \frac{H(s)}{E(s)} = \frac{K_a K_i n K_I N}{s(R_a Js + K_i K_b)(As + K_o)} = \frac{16.667N}{s(s+1)(s+11.767)}$$

Closed-loop transfer function:

$$\frac{H(s)}{R(s)} = \frac{G(s)}{1 + G(s)} = \frac{16.667N}{s^3 + 12.767s^2 + 11.767s + 16.667N}$$

(c) Characteristic equation:

$$s^3 + 12.767 s^2 + 11.767 s + 16.667 N = 0$$

Routh Tabulation:

s^3	1	11.767	
s^2	12.767	$16.667 N$	
s^1	$\dfrac{150.22 - 16.667 N}{12.767}$		$150.22 - 16.667 N > 0$ or $N < 9$
s^0	$16.667N$		$N > 0$

Stability condition: $0 < N < 9$.

6-14 (a) **The closed-loop transfer function:**

$$\frac{H(s)}{R(s)} = \frac{250 N}{s(0.06 s + 0.706)(As + 50) + 250 N}$$

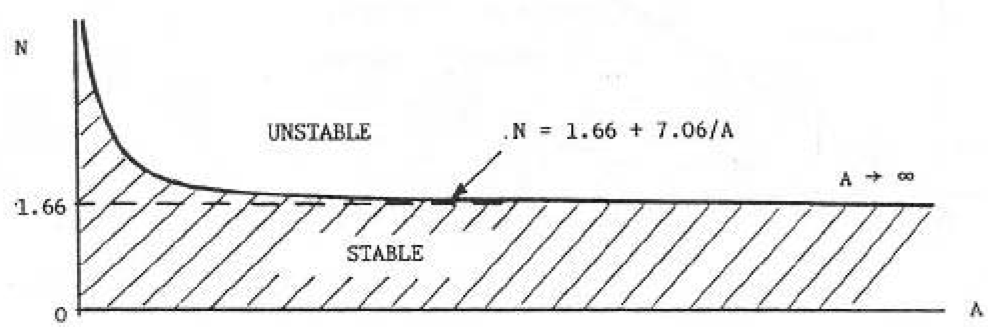
The characteristic equation:

$$0.06 As^3 + (0.706 A + 3) s^2 + 35.3 s + 250 N = 0$$

Routh Tabulation:

s^3	$0.06 A$	35.3	$A > 0$
s^2	$0.706 A + 3$	$250 N$	$0.706 A + 3 > 0$
s^1	$\dfrac{24.92 A + 105.9 - 15 NA}{07.06 A + 3}$		$24.92 A + 105.9 - 15 NA > 0$
s^0	$250N$		$N > 0$

From the s^1 row, $N < 1.66 + 7.06/A$ When $A \to \infty$ $N_{max} \to 1.66$ Thus, $N_{max} = 1.$

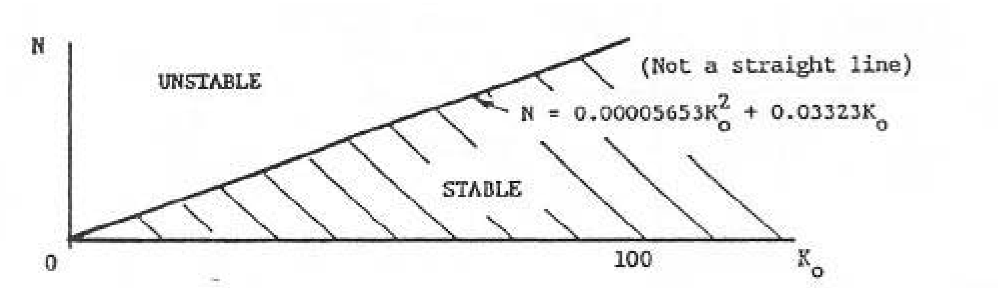


(b) For $A = 50$, the characteristic equation is

$$3s^3 + (35.3 + 0.06K_o) s^2 + 0.706 K_o s + 250 N = 0$$

Routh tabulation

s^3	3	0706 K_o	
s^2	$35.3 + 0.06K_o$	250 N	$K_o > -588.33$
s^1	$\dfrac{0.0424K_o^2 + 24.92K_o - 750N}{35.3 + 0.06K}$		$N < 0.00005653K_o^2 + 0.03323K_o$
s^0	250 N		$N > 0$

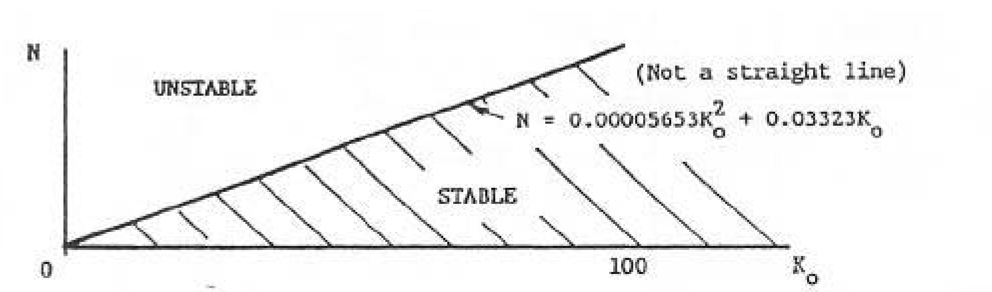


(c) $N = 10$, $A = 50$. The characteristic equation is

$$s^3 + (35.3 + 0.06K_o)s^2 + 0.706K_o s + 50K_I = 0$$

Routh Tabulution:

s^3	1	$0.706K_o$	
s^2	$35.3 + 0.06K_o$	$50K_I$	$K_o > -588.33$
s^1	$\dfrac{0.04236K_o^2 + 24.92K_o - 50K_I}{35.3 + 0.06K_o}$		$K_1 < 0.0008472K_o^2 + 0.498K_o$
s^0	$50K_I$		$K_1 > 0$

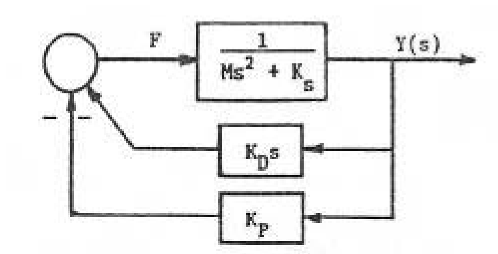


6-15 (a) Block diagram:

(b) Characteristic equation:

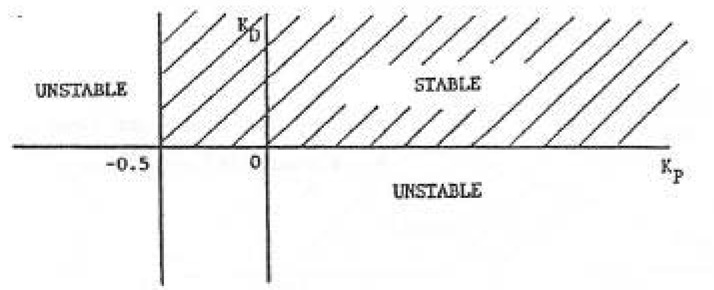
$$Ms^2 + K_D s + K_s + K_P = 0$$

$$500s^2 + K_D s + 500 + K_P = 0$$

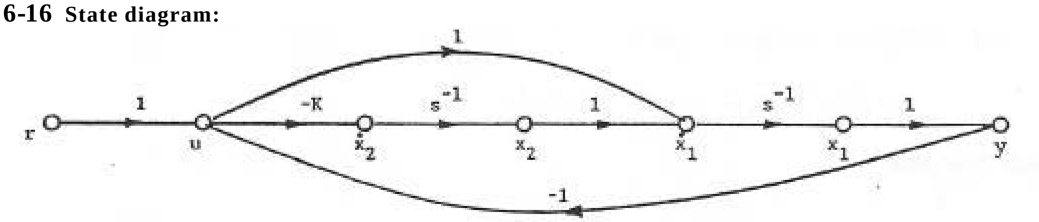


(c) For stability, $K_D > 0$, $0.5 + K_P > 0$. Thus, $K_P > -0.5$

Stability Region:



6-16 **State diagram:**



$$\Delta = 1 + s + Ks^2$$

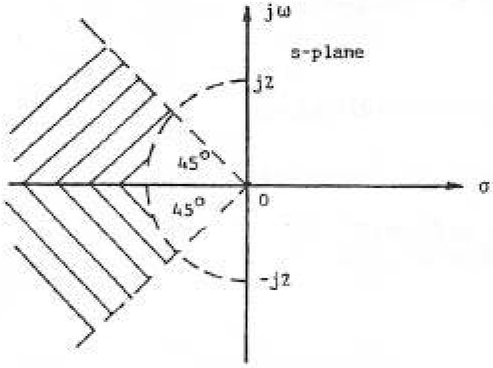
Characteristic equation:

$$s^2 + s + K = 0$$

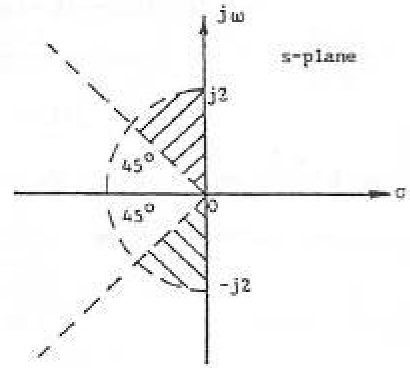
Stability requirement: $K > 0$

Chapter 7 TIME-DOMAIN ANALYSIS OF CONTROL SYSTEMS

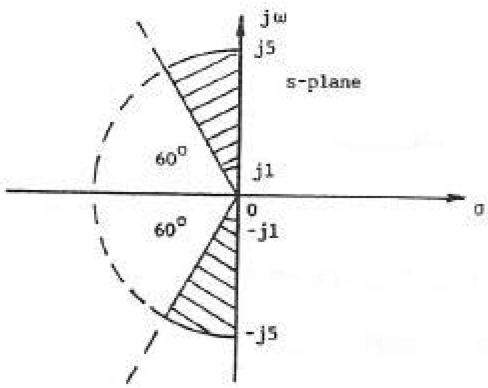
7-1 (a) $z \geq 0.707 \quad w_n \geq 2 \text{ rad / sec}$



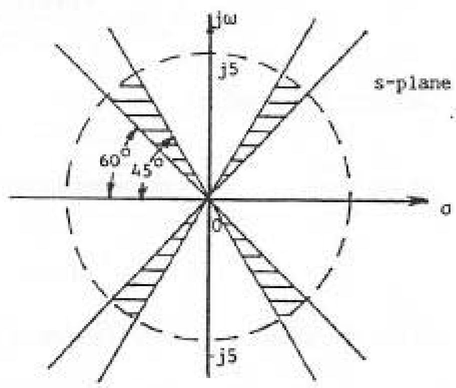
(b) $0 \leq z \leq 0.707 \quad w_n \leq 2 \text{ rad / sec}$



(c) $z \leq 0.5 \quad 1 \leq w_n \leq 5 \text{ rad / sec}$



(d) $0.5 \leq z \leq 0.707 \quad w_n \leq 0.5 \text{ rad / sec}$



7-2 (a) Type 0 **(b)** Type 0 **(c)** Type 1 **(d)** Type 2 **(e)** Type 3 **(f)** Type 3

7-3 (a) $K_p = \lim_{s \to 0} G(s) = 1000 \qquad K_v = \lim_{s \to 0} sG(s) = 0 \qquad K_a = \lim_{s \to 0} s^2 G(s) = 0$

(b) $K_p = \lim_{s \to 0} G(s) = \infty \qquad K_v = \lim_{s \to 0} sG(s) = 1 \qquad K_a = \lim_{s \to 0} s^2 G(s) = 0$

(c) $K_p = \lim_{s \to 0} G(s) = \infty \qquad K_v = \lim_{s \to 0} sG(s) = K \qquad K_a = \lim_{s \to 0} s^2 G(s) = 0$

(d) $K_p = \lim_{s \to 0} G(s) = \infty \qquad K_v = \lim_{s \to 0} sG(s) = \infty \qquad K_a = \lim_{s \to 0} s^2 G(s) = 1$

(e) $K_p = \lim_{s \to 0} G(s) = \infty \qquad K_v = \lim_{s \to 0} sG(s) = 1 \qquad K_a = \lim_{s \to 0} s^2 G(s) = 0$

(f) $K_p = \lim_{s \to 0} G(s) = \infty \qquad K_v = \lim_{s \to 0} sG(s) = \infty \qquad K_a = \lim_{s \to 0} s^2 G(s) = K$

7-4 (a)

Input	Error Constants	Steady-state Error
$u_s(t)$	$K_p = 1000$	$1/1001$
$tu_s(t)$	$K_v = 0$	$\propto$
$t^2u_s(t)/2$	$K_a = 0$	$\propto$

(b)

Input	Error Constants	Steady-state Error
$u_s(t)$	$K_p = \infty$	0
$tu_s(t)$	$K_v = 1$	1
$t^2u_s(t)/2$	$K_a = 0$	$\propto$

(c)

Input	Error Constants	Steady-state Error
$u_s(t)$	$K_p = \infty$	0
$tu_s(t)$	$K_v = K$	$1/K$
$t^2u_s(t)/2$	$K_a = 0$	$\propto$

The above results are valid if the value of K corresponds to a stable closed-loop system.

(d) The closed-loop system is unstable. It is meaningless to conduct a steady-state error analysis.

(e)

Input	Error Constants	Steady-state Error
$u_s(t)$	$K_p = \infty$	0
$tu_s(t)$	$K_v = 1$	1
$t^2u_s(t)/2$	$K_a = 0$	$\propto$

(f)

Input	Error Constants	Steady-state Error
$u_s(t)$	$K_p = \infty$	0
$tu_s(t)$	$K_v = \infty$	0
$t^2u_s(t)/2$	$K_a = K$	$1/K$

The closed-loop system is stable for all positive values of K. Thus the above results are valid.

7-5 (a) $K_H = H(0) = 1$

$$M(s) = \frac{G(s)}{1+G(s)H(s)} = \frac{s+1}{s^3+2s^2+3s+3}$$

$$a_0 = 3, \quad a_1 = 3, \quad a_2 = 2, \quad b_0 = 1, \quad b_1 = 1.$$

Unit-step Input:

$$e_{ss} = \frac{1}{K_H}\left(1 - \frac{b_0 K_H}{a_0}\right) = \frac{2}{3}$$

Unit-ramp input:

$$a_0 - b_0 K_H = 3 - 1 = 2 \neq 0. \quad \text{Thus} \quad e_{ss} = \infty.$$

Unit-parabolic Input:

$$a_0 - b_0 K_H = 2 \neq 0 \quad \text{and} \quad a_1 - b_1 K_H = 1 \neq 0. \quad \text{Thus} \quad e_{ss} = \infty.$$

(b) $K_H = H(0) = 5$

$$M(s) = \frac{G(s)}{1+G(s)H(s)} = \frac{1}{s^2+5s+5} \qquad a_0 = 5, \quad a_1 = 5, \quad b_0 = 1, \quad b_1 = 0.$$

Unit-step Input:

$$e_{ss} = \frac{1}{K_H}\left(1 - \frac{b_0 K_H}{a_0}\right) = \frac{1}{5}\left(1 - \frac{5}{5}\right) = 0$$

Unit-ramp Input:

$$i = 0: \quad a_0 - b_0 K_H = 0 \qquad i = 1: \quad a_1 - b_1 K_H = 5 \neq 0$$

$$e_{ss} = \frac{a_1 - b_1 K_H}{a_0 K_H} = \frac{5}{25} = \frac{1}{5}$$

Unit-parabolic Input:

$$e_{ss} = \infty$$

(c) $K_H = H(0) = 1/5$

$$M(s) = \frac{G(s)}{1 + G(s)H(s)} = \frac{s + 5}{s^4 + 15s^3 + 50s^2 + s + 1} \qquad \text{The system is stable.}$$

$$a_0 = 1, \quad a_1 = 1, \quad a_2 = 50, \quad a_3 = 15, \quad b_0 = 5, \quad b_1 = 1$$

Unit-step Input:

$$e_{ss} = \frac{1}{K_H}\left(1 - \frac{b_0 K_H}{a_0}\right) = 5\left(1 - \frac{5/5}{1}\right) = 0$$

Unit-ramp Input:

$$i = 0: \quad a_0 - b_0 K_H = 0 \qquad i = 1: \quad a_1 - b_1 K_H = 4/5 \neq 0$$

$$e_{ss} = \frac{a_1 - b_1 K_H}{a_0 K_H} = \frac{1 - 1/5}{1/5} = 4$$

Unit-parabolic Input:

$$e_{ss} = \infty$$

(d) $K_H = H(0) = 10$

$$M(s) = \frac{G(s)}{1 + G(s)H(s)} = \frac{1}{s^3 + 12s^2 + 5s + 10} \qquad \text{The system is stable.}$$

$$a_0 = 10, \quad a_1 = 5, \quad a_2 = 12, \quad b_0 = 1, \quad b_1 = 0, \quad b_2 = 0$$

Unit-step Input:

$$e_{ss} = \frac{1}{K_H}\left(1 - \frac{b_0 K_H}{a_0}\right) = \frac{1}{10}\left(1 - \frac{10}{10}\right) = 0$$

Unit-ramp Input:

$$i = 0: \quad a_0 - b_0 K_H = 0 \qquad i = 1: \quad a_1 - b_1 K_H = 5 \neq 0$$

$$e_{ss} = \frac{a_1 - b_1 K_H}{a_0 K_H} = \frac{5}{100} = 0.05$$

Unit-parabolic Input:

$$e_{ss} = \infty$$

7-6 (a) $M(s) = \dfrac{s + 4}{s^4 + 16s^3 + 48s^2 + 4s + 4} \qquad K_H = 1$ The system is stable.

$$a_0 = 4, \quad a_1 = 4, \quad a_2 = 48, \quad a_3 = 16, \quad b_0 = 4, \quad b_1 = 1, \quad b_2 = 0, \quad b_3 = 0$$

Unit-step Input:

$$e_{ss} = \frac{1}{K_H}\left(1 - \frac{b_0 K_H}{a_0}\right) = \left(1 - \frac{4}{4}\right) = 0$$

Unit-ramp input:

$$i = 0: \quad a_0 - b_0 K_H = 0 \qquad i = 1: \quad a_1 - b_1 K_H = 4 - 1 = 3 \neq 0$$

$$e_{ss} = \frac{a_1 - b_1 K_H}{a_0 K_H} = \frac{4-1}{4} = \frac{3}{4}$$

Unit-parabolic Input:

$$e_{ss} = \infty$$

(b) $M(s) = \dfrac{K(s+3)}{s^3 + 3s^2 + (K+2)s + 3K}$ $\quad K_H = 1 \quad$ The system is stable for $K > 0$.

$$a_0 = 3K, \quad a_1 = K+2, \quad a_2 = 3, \quad b_0 = 3K, \quad b_1 = K$$

Unit-step Input:

$$e_{ss} = \frac{1}{K_H}\left(1 - \frac{b_0 K_H}{a_0}\right) = \left(1 - \frac{3K}{3K}\right) = 0$$

Unit-ramp Input:

$$i = 0: \quad a_0 - b_0 K_H = 0 \qquad i = 1: \quad a_1 - b_1 K_H = K + 2 - K = 2 \neq 0$$

$$e_{ss} = \frac{a_1 - b_1 K_H}{a_0 K_H} = \frac{K + 2 - K}{3K} = \frac{2}{3K}$$

Unit-parabolic Input:

$$e_{ss} = \infty$$

The above results are valid for $K > 0$.

(c) $M(s) = \dfrac{s+5}{s^4 + 15s^3 + 50s^2 + 10s}$ $\quad H(s) = \dfrac{10s}{s+5}$ $\quad K_H = \lim_{s \to 0} \dfrac{H(s)}{s} = 2$

$$a_0 = 0, \quad a_1 = 10, \quad a_2 = 50, \quad a_3 = 15, \quad b_0 = 5, \quad b_1 = 1$$

Unit-step Input:

$$e_{ss} = \frac{1}{K_H}\left(\frac{a_2 - b_1 K_H}{a_1}\right) = \frac{1}{2}\left(\frac{50 - 1 \times 2}{10}\right) = 2.4$$

Unit-ramp Input:

$$e_{ss} = \infty$$

Unit-parabolic Input:

$$e_{ss} = \infty$$

(d) $M(s) = \dfrac{K(s+5)}{s^4 + 17s^3 + 60s^2 + 5Ks + 5K}$ $\quad K_H = 1 \quad$ The system is stable for $0 < K < 204$.

$$a_0 = 5K, \quad a_1 = 5K, \quad a_2 = 60, \quad a_3 = 17, \quad b_0 = 5K, \quad b_1 = K$$

Unit-step Input:

$$e_{ss} = \frac{1}{K_H}\left(1 - \frac{b_0 K_H}{a_0}\right) = \left(1 - \frac{5K}{5K}\right) = 0$$

Unit-ramp Input:

$$i = 0: \quad a_0 - b_0 K_H = 0 \qquad i = 1: \quad a_1 - b_1 K_H = 5K - K = 4K \neq 0$$

$$e_{ss} = \frac{a_1 - b_1 K_H}{a_0 K_H} = \frac{5K - K}{5K} = \frac{4}{5}$$

Unit-parabolic Input:

$$e_{ss} = \infty$$

The results are valid for $0 < K < 204$.

7-7

$$G(s)=\frac{Y(s)}{E(s)}=\frac{KG_p(s)\big/20s}{1+K_tG_p(s)}=\frac{100K}{20s(1+0.2s+100K_t)}$$ Type-1 system.

Error constants: $K_p=\infty,\quad K_v=\dfrac{5K}{1+100K_t},\quad K_a=0$

(a) $r(t)=u_s(t)$: $\quad e_{ss}=\dfrac{1}{1+K_p}=0$

(b) $r(t)=tu_s(t)$: $\quad e_{ss}=\dfrac{1}{K_v}=\dfrac{1+100K_t}{5K}$

(c) $r(t)=t^2u_s(t)/2$: $\quad e_{ss}=\dfrac{1}{K_a}=\infty$

7-8

$$G_p(s)=\frac{100}{(1+0.1s)(1+0.5s)}\qquad G(s)=\frac{Y(s)}{E(s)}=\frac{KG_p(s)}{20s\left[1+K_tG_p(s)\right]}$$

$$G(s)=\frac{100K}{20s\left[(1+0.1s)(1+0.5s)+100K_t\right]}$$

Error constants: $K_p=\infty,\quad K_v=\dfrac{5K}{1+100K_t},\quad K_a=0$

(a) $r(t)=u_s(t)$: $\quad e_{ss}=\dfrac{1}{1+K_p}=0$

(b) $r(t)=tu_s(t)$: $\quad e_{ss}=\dfrac{1}{K_v}=\dfrac{1+100K_t}{5K}$

(c) $r(t)=t^2u_s(t)/2$: $\quad e_{ss}=\dfrac{1}{K_a}=\infty$

Since the system is of the third order, the values of K and K_t must be constrained so that the system is stable. The characteristic equation is

$$s^3+12s^2+\left(20+2000K_t\right)s+100K=0$$

Routh Tabulation:

s^3	1	$20+2000K_t$
s^2	12	$100K$
s^1	$\dfrac{240+24000K_t-100K}{12}$	
s^0	$100K$	

Stability Conditions: $K>0\qquad 12\left(1+100K_t\right)-5K>0$ or $\dfrac{1+100K_t}{5K}>\dfrac{1}{12}$

Thus, the minimum steady-state error that can be obtained with a unit-ramp input is 1/12.

7-9 (a) From Figure 3P-19,

$$\frac{\Theta_o(s)}{\Theta_r(s)} = \frac{1+\dfrac{K_1K_2}{R_a+L_as}+\dfrac{K_iK_b+KK_1K_iK_t}{(R_a+L_as)(B_t+J_ts)}}{1+\dfrac{K_1K_2}{R_a+L_as}+\dfrac{K_iK_b+KK_1K_iK_t}{(R_a+L_as)(B_t+J_ts)}+\dfrac{KK_sK_1K_iN}{s(R_a+L_as)(B_t+J_ts)}}$$

$$\frac{\Theta_o(s)}{\Theta_r(s)} = \frac{s\left[(R_a+L_as)(B_t+J_ts)+K_1K_2(B_t+J_ts)+K_iK_b+KK_1K_iK_t\right]}{L_aJ_ts^3+(L_aB_t+R_aJ_t+K_1K_2J_t)s^2+(R_aB_t+K_iK_b+KK_iK_1K_t+K_1K_2B_t)s+KK_sK_1K_iN}$$

$$q_r(t)=u_s(t), \quad \Theta_r(s)=\frac{1}{s} \qquad \lim_{s\to 0} s\Theta_e(s)=0$$

Provided that all the poles of $s\Theta_e(s)$ are all in the left-half s-plane.

(b) For a unit-ramp input, $\Theta_r(s)=1/s^2$.

$$e_{ss}=\lim_{t\to\infty} q_e(t)=\lim_{s\to 0} s\Theta_e(s)=\frac{R_aB_t+K_1K_2B_t+K_iK_b+KK_1K_iK_t}{KK_sK_1K_iN}$$

if the limit is valid.

7-10 (a) Forward-path transfer function: [$n(t)$ = 0]:

$$G(s)=\frac{Y(s)}{E(s)}=\frac{\dfrac{K(1+0.02s)}{s^2(s+25)}}{1+\dfrac{KK_ts}{s^2(s+25)}}=\frac{K(1+0.02s)}{s\left(s^2+25s+KK_t\right)}$$

Type-1 system.

Error Constants: $K_p=\infty, \quad K_v=\dfrac{1}{K_t}, \quad K_a=0$

For a unit-ramp input, $r(t)=tu_s(t), \quad R(s)=\dfrac{1}{s^2}, \quad e_{ss}=\lim_{t\to\infty} e(t)=\lim_{s\to 0} sE(s)=\dfrac{1}{K_v}=K_t$

Routh Tabulation:

$$\begin{array}{lll}
s^3 & 1 & KK_t+0.02K \\
s^2 & 25 & K \\
s^1 & \dfrac{25K(K_t+0.02)-K}{25} & \\
s^0 & K &
\end{array}$$

Stability Conditions: $K>0 \qquad 25(K_t+0.02)-K>0$ or $K_t>0.02$

(b) With $r(t)$ = 0, $n(t)=u_s(t), \quad N(s)=1/s$.

System Transfer Function with $N(s)$ as Input:

$$\frac{Y(s)}{N(s)}=\frac{\dfrac{K}{s^2(s+25)}}{1+\dfrac{K(1+0.02s)}{s^2(s+25)}+\dfrac{KK_ts}{s^2(s+25)}}=\frac{K}{s^3+25s^2+K(K_t+0.02)s+K}$$

Steady-State Output due to $n(t)$:

$$y_{ss} = \lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) = 1 \qquad \text{if the limit is valid.}$$

7-11 (a) $n(t) = 0, \quad r(t) = tu_s(t).$

Forward-path Transfer function:

$$G(s) = \left.\frac{Y(s)}{E(s)}\right|_{n=0} = \frac{K(s+\alpha)(s+3)}{s\left(s^2-1\right)} \qquad \text{Type-1 system.}$$

Ramp-error constant: $K_v = \lim_{s\to 0} sG(s) = -3K\alpha$

Steady-state error: $e_{ss} = \frac{1}{K_v} = -\frac{1}{3K_v}$

Characteristic equation: $s^3 + Ks^2 + [K(3+\alpha)-1]s + 3\alpha K = 0$

Routh Tabulation:

s^3	1	$3K + \alpha K - 1$
s^2	K	$3\alpha K$
s^1	$\frac{K(3K+\alpha K-1) - 3\alpha K}{K}$	
s^0	$3\alpha K$	

Stability Conditions:

$3K + \alpha K - 1 - 3\alpha > 0 \quad$ or $\quad K > \frac{1+3K}{3+\alpha}$

$\alpha K > 0$

(b) When $r(t) = 0$, $n(t) = u_s(t)$, $\quad N(s) = 1/s$.

Transfer Function between $n(t)$ and $y(t)$:

$$\left.\frac{Y(s)}{N(s)}\right|_{r=0} = \frac{\dfrac{K(s+3)}{s^2-1}}{1+\dfrac{K(s+\alpha)(s+3)}{s\left(s^2-1\right)}} = \frac{Ks(s+3)}{s^3 + Ks^2 + [K(s+\alpha)-1]s + 3\alpha K}$$

Steady-State Output due to $n(t)$:

$$y_{ss} = \lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) = 0 \qquad \text{if the limit is valid.}$$

7-12

Percent maximum overshoot $= 0.25 = e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}}$

Thus

$$\pi\zeta\sqrt{1-\zeta^2} = -\ln 0.25 = 1.386 \qquad \pi^2\zeta^2 = 1.922\left(1-\zeta^2\right)$$

Solving for ζ from the last equation, we have $\zeta = 0.404$.

Peak Time $t_{max} = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = 0.01$ sec. Thus, $\omega_n = \frac{\pi}{0.01\sqrt{1-(0.404)^2}} = 343.4$ rad / sec

Transfer Function of the Second-order Prototype System:

$$\frac{Y(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{117916}{s^2 + 277.3s + 117916}$$

7-13 Closed-Loop Transfer Function: **Characteristic equation:**

$$\frac{Y(s)}{R(s)} = \frac{25K}{s^2 + (5 + 500K_t)s + 25K} \qquad s^2 + (5 + 500K_t)s + 25K = 0$$

For a second-order prototype system, when the maximum overshoot is 4.3%, $\zeta = 0.707$.

$$\omega_n = \sqrt{25K}, \qquad 2\zeta\omega_n = 5 + 500K_t = 1.414\sqrt{25K}$$

Rise Time: [Eq. (7-104)]

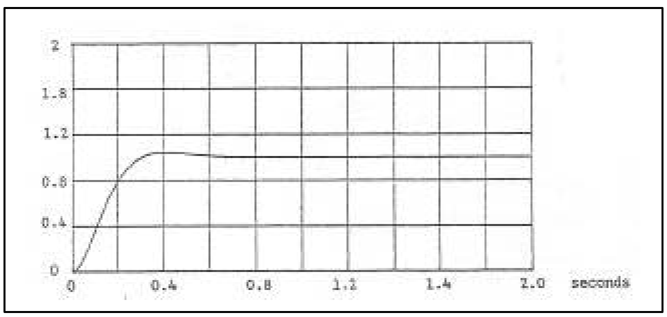
$$t_r = \frac{1 - 0.4167\zeta + 2.917\zeta^2}{\omega_n} = \frac{2.164}{\omega_n} = 0.2 \text{ sec} \qquad \text{Thus } \omega_n = 10.82 \text{ rad / sec}$$

Thus, $K = \frac{\omega_n^2}{25} = \frac{(10.82)^2}{25} = 4.68 \qquad 5 + 500K_t = 1.414\omega_n = 15.3 \qquad$ Thus $K_t = \frac{10.3}{500} = 0.0206$

With $K = 4.68$ and $K_t = 0.0206$, the system transfer function is

$$\frac{Y(s)}{R(s)} = \frac{117}{s^2 + 15.3s + 117}$$

Unit-step Response:



$y = 0.1$ at $t = 0.047$ sec.
$y = 0.9$ at $t = 0.244$ sec.
$t_r = 0.244 - 0.047 = 0.197$ sec.

$y_{\max} = 0.0432$ (4.32% max. overshoot)

7-14 Closed-loop Transfer Function: **Characteristic Equation:**

$$\frac{Y(s)}{R(s)} = \frac{25K}{s^2 + (5 + 500K_t)s + 25K} \qquad s^2 + (5 + 500K_t)s + 25K = 0$$

When Maximum overshoot = 10%, $\frac{\pi\zeta}{\sqrt{1-\zeta^2}} = -\ln 0.1 = 2.3 \qquad \pi^2\zeta^2 = 5.3(1-\zeta^2)$

Solving for ζ , we get $\zeta = 0.59$.

The Natural undamped frequency is $\omega_n = \sqrt{25K}$ Thus, $5 + 500K_t = 2\zeta\omega_n = 1.18\omega_n$

Rise Time: [Eq. (7-114)]

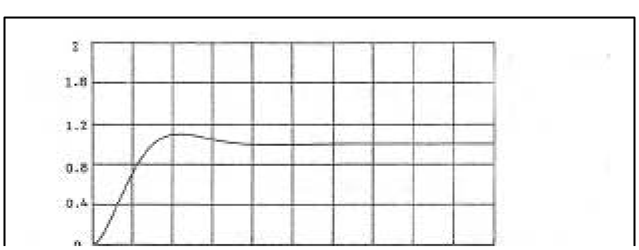
$$t_r = \frac{1 - 0.4167\zeta + 2.917\zeta^2}{\omega_n} = 0.1 = \frac{1.7696}{\omega_n} \text{ sec.} \qquad \text{Thus } \omega_n = 17.7 \text{ rad / sec}$$

$$K = \frac{\omega_n^2}{25} = 12.58 \qquad \text{Thus} \quad K_t = \frac{15.88}{500} = 0.0318$$

With $K = 12.58$ and $K_t = 0.0318$, the system transfer function is

$$\frac{Y(s)}{R(s)} = \frac{313}{s^2 + 20.88s + 314.5}$$

Unit-step Response:



$y = 0.1$ when $t = 0.028$ sec.
$y = 0.9$ when $t = 0.131$ sec.

$$t_r = 0.131 - 0.028 = 0.103 \quad \text{sec.}$$

$$y_{\max} = 1.1 \quad (10\% \quad \text{max. overshoot})$$

7-15 **Closed-Loop Transfer Function:**

$$\frac{Y(s)}{R(s)} = \frac{25K}{s^2 + (5 + 500K_t)s + 25K}$$

Characteristic Equation:

$$s^2 + \left(5 + 500K_t\right)s + 25K = 0$$

When Maximum overshoot = 20%, $\dfrac{\pi\zeta}{\sqrt{1-\zeta^2}} = -\ln 0.2 = 1.61 \qquad \pi^2\zeta^2 = 2.59\left(1-\zeta^2\right)$

Solving for ζ, we get $\zeta = 0.456$.

The Natural undamped frequency $\omega_n = \sqrt{25K} \qquad 5 + 500K_t = 2\zeta\omega_n = 0.912\,\omega_n$

Rise Time: [Eq. (7-114)]

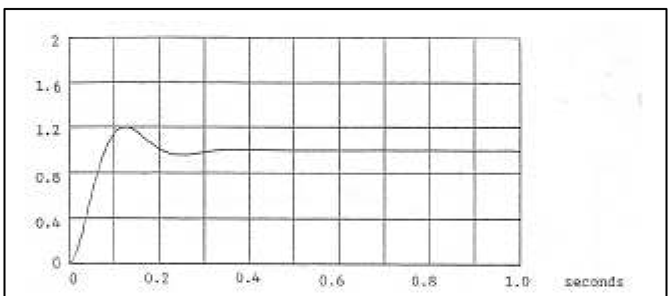
$$t_r = \frac{1 - 0.4167\zeta + 2.917\zeta^2}{\omega_n} = 0.05 = \frac{1.4165}{\omega_n} \quad \text{sec.} \qquad \text{Thus,} \quad \omega_n = \frac{1.4165}{0.05} = 28.33$$

$$K = \frac{\omega_n^2}{25} = 32.1 \qquad 5 + 500K_t = 0.912\,\omega_n = 25.84 \qquad \text{Thus,} \quad K_t = 0.0417$$

With K = 32.1 and $K_t = 0.0417$, the system transfer function is

$$\frac{Y(s)}{R(s)} = \frac{802.59}{s^2 + 25.84s + 802.59}$$

Unit-step Response:



$y = 0.1$ when $t = 0.0178$ sec.
$y = 0.9$ when $t = 0.072$ sec.

$$t_r = 0.072 - 0.0178 = 0.0542 \quad \text{sec.}$$

$$y_{\max} = 1.2 \quad (20\% \quad \text{max. overshoot})$$

7-16 **Closed-Loop Transfer Function:**

$$\frac{Y(s)}{R(s)} = \frac{25K}{s^2 + \left(5 + 500K_t\right)s + 25K}$$

Characteristic Equation:

$$s^2 + \left(5 + 500K_t\right)s + 25K = 0$$

From Eq. (7-102), Delay time $t_d \cong \dfrac{1.1 + 0.125\zeta + 0.469\zeta^2}{\omega_n} = 0.1$ sec.

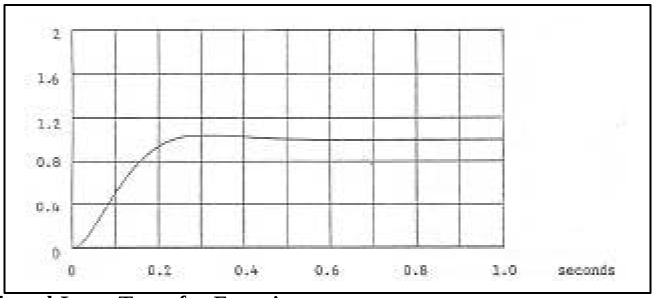
When Maximum overshoot = 4.3%, $\zeta = 0.707$. $\qquad t_d = \dfrac{1.423}{\omega_n} = 0.1$ sec. $\qquad$ Thus $\omega_n = 14.23$ rad/sec.

$$K = \left(\frac{\omega_n}{5}\right)^2 = \left(\frac{14.23}{5}\right)^2 = 8.1 \qquad 5 + 500K_t = 2\zeta\omega_n = 1.414\omega_n = 20.12 \qquad \text{Thus} \quad K_t = \frac{15.12}{500} = 0.0302$$

With K = 20.12 and $K_t = 0.0302$, the system transfer function is

$$\frac{Y(s)}{R(s)} = \frac{202.5}{s^2 + 20.1s + 202.5}$$

Unit-Step Response:



When $y = 0.5$, $t = 0.1005$ sec.
Thus, $t_d = 0.1005$ sec.

$y_{max} = 1.043$ (4.3% max. overshoot)

7-17 Closed-Loop Transfer Function:

$$\frac{Y(s)}{R(s)} = \frac{25K}{s^2 + (5 + 500K_t)s + 25K}$$

Characteristic Equation:

$$s^2 + (5 + 500K_t)s + 25K = 0$$

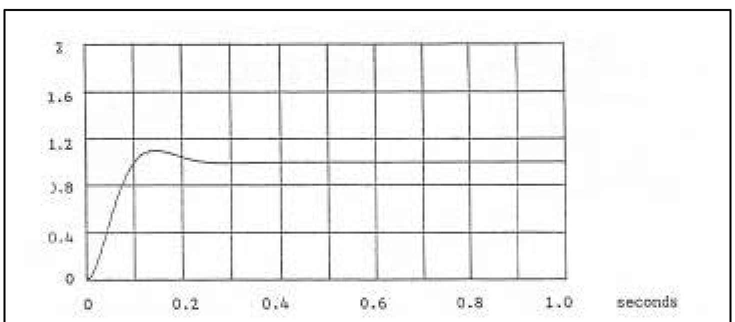
From Eq. (7-102), Delay time $t_d \cong \dfrac{1.1 + 0.125\zeta + 0.469\zeta^2}{\omega_n} = 0.05 = \dfrac{1.337}{\omega_n}$ Thus, $\omega_n = \dfrac{1.337}{0.05} = 26.74$

$$K = \left(\frac{\omega_n}{5}\right)^2 = \left(\frac{26.74}{5}\right)^2 = 28.6 \qquad 5 + 500K_t = 2\zeta\omega_n = 2 \times 0.59 \times 26.74 = 31.55 \qquad \text{Thus} \quad K_t = 0.0531$$

With $K = 28.6$ and $K_t = 0.0531$, the system transfer function is

$$\frac{Y(s)}{R(s)} = \frac{715}{s^2 + 31.55s + 715}$$

Unit-Step Response:



$y = 0.5$ when $t = 0.0505$ sec.
Thus, $t_d = 0.0505$ sec.

$y_{max} = 1.1007$ (10.07% max. overshoot)

7-18 Closed-Loop Transfer Function:

$$\frac{Y(s)}{R(s)} = \frac{25K}{s^2 + (5 + 500K_t)s + 25K}$$

Characteristic Equation:

$$s^2 + (5 + 500K_t)s + 25K = 0$$

For Maximum overshoot = 0.2, $\zeta = 0.456$.

From Eq. (7-102), Delay time $t_d = \dfrac{1.1 + 0.125\zeta + 0.469\zeta^2}{\omega_n} = \dfrac{1.2545}{\omega_n} = 0.01$ sec.

Natural Undamped Frequency $\omega_n = \dfrac{1.2545}{0.01} = 125.45$ rad/sec. Thus, $K = \left(\dfrac{\omega_n}{5}\right)^2 = \dfrac{15737.7}{25} = 629.5$

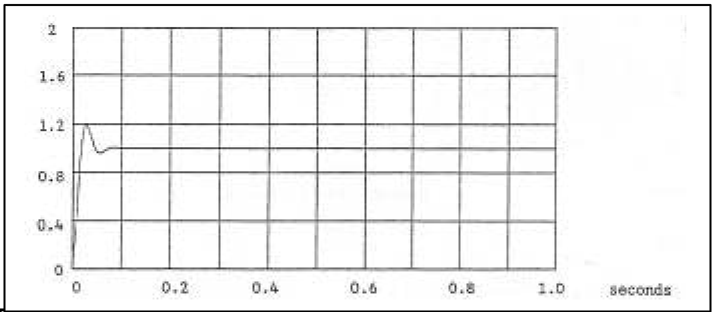
$5 + 500K_t = 2\zeta\omega_n = 2 \times 0.456 \times 125.45 = 114.41$ Thus, $K_t = 0.2188$

With $K = 629.5$ and $K_t = 0.2188$, the system transfer function is

$$\frac{Y(s)}{R(s)} = \frac{15737.7}{s^2 + 114.41s + 15737.7}$$

Unit-step Response:

$y = 0.5$ when $t = 0.0101$ sec.



Thus, $t_d = 0.0101$ sec.

$y_{\max} = 1.2$ (20% max. overshoot)

7-19 Closed-Loop Transfer Function:

$$\frac{Y(s)}{R(s)} = \frac{25K}{s^2 + (5+500K_t)s + 25K}$$

Characteristic Equation:

$$s^2 + (5+5000K_t)s + 25K = 0$$

$\zeta = 0.6 \qquad 2\zeta\omega_n = 5 + 500K_t = 1.2\omega_n$

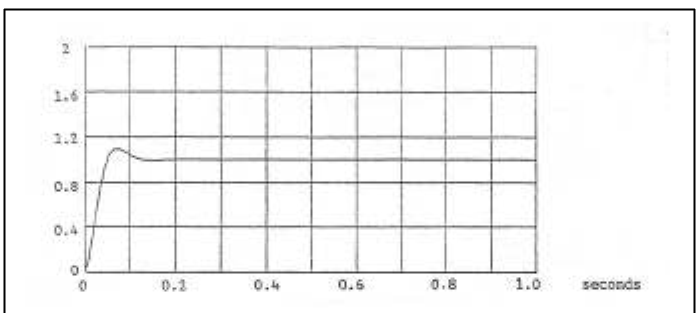
From Eq. (7-109), settling time $t_s \cong \dfrac{3.2}{\zeta\omega_n} = \dfrac{3.2}{0.6\omega_n} = 0.1$ sec. Thus, $\omega_n = \dfrac{3.2}{0.06} = 53.33$ rad / sec

$$K_t = \frac{1.2\omega_n - 5}{500} = 0.118 \qquad K = \frac{\omega_n^2}{25} = 113.76$$

System Transfer Function:

$$\frac{Y(s)}{R(s)} = \frac{2844}{s^2 + 64s + 2844}$$

Unit-step Response:



$y(t)$ reaches 1.00 and never exceeds this value at t = 0.098 sec.

Thus, $t_s = 0.098$ sec.

7-20 (a) Closed-Loop Transfer Function:

$$\frac{Y(s)}{R(s)} = \frac{25K}{s^2 + (5+500K_t)s + 25K}$$

Characteristic Equation:

$$s^2 + (5+500K_t)s + 25K = 0$$

For maximum overshoot = 0.1, $\zeta = 0.59$. $\quad 5 + 500K_t = 2\zeta\omega_n = 2 \times 0.59\omega_n = 1.18\omega_n$

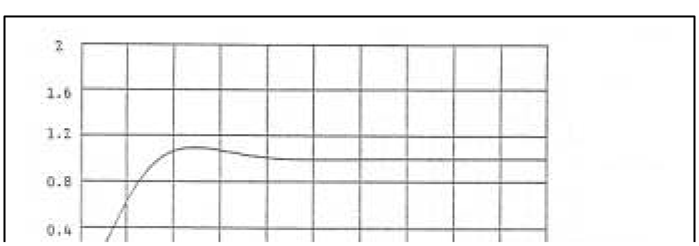
Settling time: $t_s = \dfrac{3.2}{\zeta\omega_n} = \dfrac{3.2}{0.59\omega_n} = 0.05$ sec. $\qquad \omega_n = \dfrac{3.2}{0.05 \times 0.59} = 108.47$

$$K_t = \frac{1.18\omega_n - 5}{500} = 0.246 \qquad K = \frac{\omega_n^2}{25} = 470.63$$

System Transfer Function:

$$\frac{Y(s)}{R(s)} = \frac{11765.74}{s^2 + 128s + 11765.74}$$

Unit-Step Response:



$y(t)$ reaches 1.05 and never exceeds this value at t = 0.048 sec.

Thus, $t_s = 0.048$ sec.

(b) For maximum overshoot = 0.2, $\zeta = 0.456$. $5 + 500K_t = 2\zeta\omega_n = 0.912\omega_n$

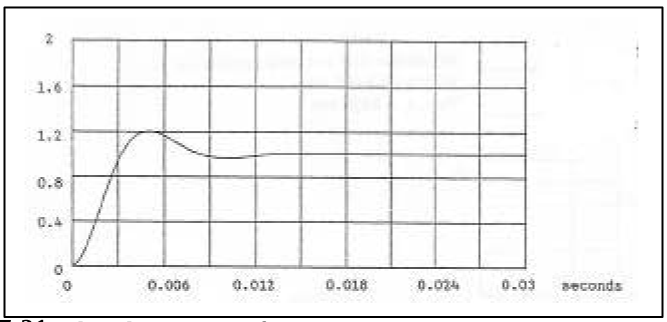
Settling time $t_s = \dfrac{3.2}{\zeta\omega_n} = \dfrac{3.2}{0.456\omega_n} = 0.01$ sec. $\omega_n = \dfrac{3.2}{0.456 \times 0.01} = 701.75$ rad / sec

$$K_t = \frac{0.912\omega_n - 5}{500} = 1.27$$

System Transfer Function:

$$\frac{Y(s)}{R(s)} = \frac{492453}{s^2 + 640s + 492453}$$

Unit-Step Response:



$y(t)$ reaches 1.05 and never exceeds this value at t = 0.0074 sec. Thus, $t_s = 0.0074$ sec. This is less than the calculated value of 0.01 sec.

7-21 Closed-Loop Transfer Function:

$$\frac{Y(s)}{R(s)} = \frac{25K}{s^2 + (5 + 500K_t)s + 25K}$$

Characteristic Equation:

$$s^2 + (5 + 500K_t)s + 25K = 0$$

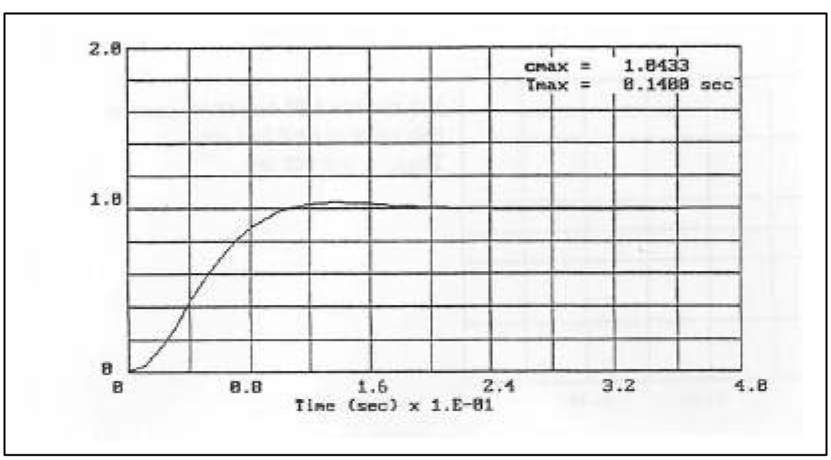
Damping ratio $\zeta = 0.707$. Settling time $t_s = \dfrac{4.5\zeta}{\omega_n} = \dfrac{3.1815}{\omega_n} = 0.1$ sec. Thus, $\omega_n = 31.815$ rad/sec.

$5 + 500K_t = 2\zeta\omega_n = 44.986$ Thus , $K_t = 0.08$ $K = \dfrac{\omega_n^2}{2\zeta} = 40.488$

System Transfer Function:

$$\frac{Y(s)}{R(s)} = \frac{1012.2}{s^2 + 44.986s + 1012.2}$$

Unit-Step Response: The unit-step response reaches 0.95 at t = 0.092 sec. which is the measured t_s.



7-22 (a) When $\zeta = 0.5$, the rise time is

$$t_r \cong \frac{1 - 0.4167\zeta + 2.917\zeta^2}{\omega_n} = \frac{1.521}{\omega_n} = 1 \text{ sec.} \quad \text{Thus } \omega_n = 1.521 \text{ rad/sec.}$$

The second-order term of the characteristic equation is written

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = s^2 + 1.521s + 2.313 = 0$$

The characteristic equation of the system is $\quad s^3 + (a+30)s^2 + 30as + K = 0$

Dividing the characteristic equation by $s^2 + 1.521s + 2.313$, we have

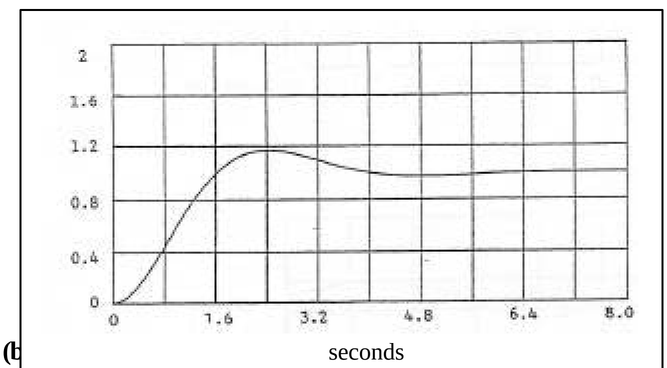
$$\begin{array}{r|l}
 & s + (28.48 + a) \\
\hline
s^2 + 1.521s + 2.313 & s^3 + (a+30)s^2 + 30as + K \\
 & \underline{s^3 + 1.521s^2 + 2.313s} \\
 & (28.48 + a)s^2 + (30a - 2.323)s + K \\
 & \underline{(28.48 + a)s^2 + (1.521a + 43.32)s + 65.874 + 2.313a} \\
 & (28.48a - 45.63)s + K - 0.744 - 2.313a
\end{array}$$

For zero remainders, $\quad 28.48a = 45.63 \quad$ Thus, $\quad a = 1.6 \quad K = 65.874 + 2.313a = 69.58$

Forward-Path Transfer Function:

$$G(s) = \frac{69.58}{s(s+1.6)(s+30)}$$

Unit-Step Response:



$y = 0.1$ when $t = 0.355$ sec.
$y = 0.9$ when $t = 1.43$ sec.

Rise Time:
$t_r = 1.43 - 0.355 = 1.075$ sec.

(b)

(i) For a unit-step input, $e_{ss} = 0$.

(ii) For a unit-ramp input, $\quad K_v = \lim_{s \to 0} sG(s) = \frac{K}{30a} = \frac{60.58}{30 \times 1.6} = 1.45 \qquad e_{ss} = \frac{1}{K_v} = 0.69$

7-23 (a) Characteristic Equation:

$$s^3 + 3s^2 + (2+K)s - K = 0$$

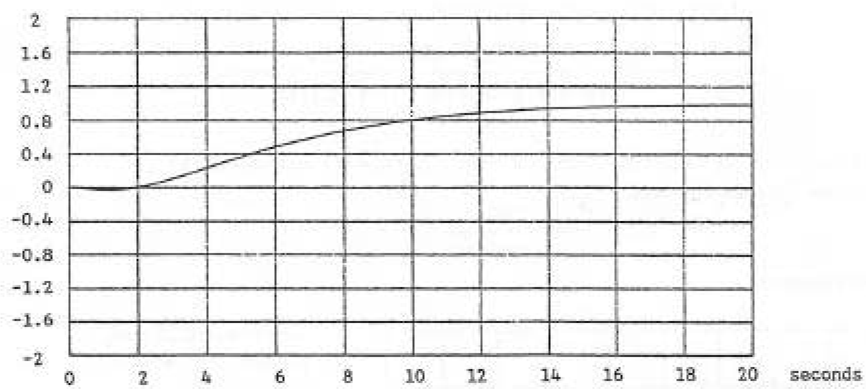
Apply the Routh-Hurwitz criterion to find the range of K for stability.

Routh Tabulation:

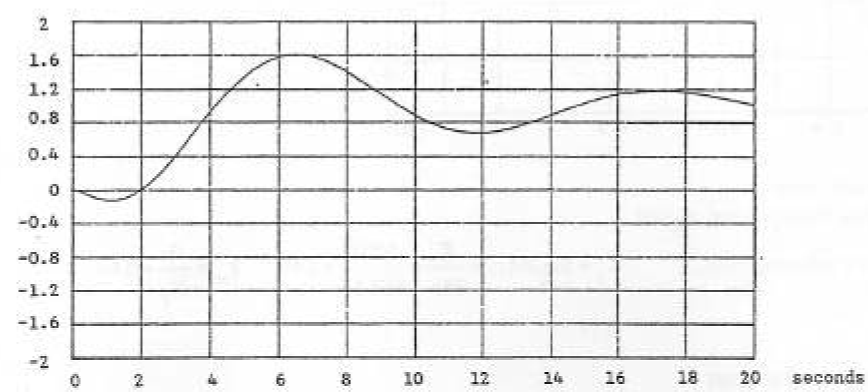
s^3	1	$2+K$
s^2	3	$-K$
s^1	$\dfrac{6+4K}{3}$	
s^0	$-K$	

Stability Condition: $-1.5 < K < 0$ This simplifies the search for K for two equal roots. When $K = -0.27806$, the characteristic equation roots are: -0.347, -0.347, and -2.3054.

(b) Unit-Step Response: ($K = -0.27806$)



(c) Unit-Step Response ($K = -1$)



7-24 (a) The state equations of the closed-loop system are:

$$\frac{dx_1}{dt} = -x_1 + 5x_2 \qquad \frac{dx_2}{dt} = -6x_1 - k_1 x_1 - k_2 x_2 + r$$

The characteristic equation of the closed-loop system is

$$\Delta = \begin{vmatrix} s+1 & -5 \\ 6+k_1 & s+k_2 \end{vmatrix} = s^2 + \left(1+k_2\right)s + \left(30+5k_1+k_2\right) = 0$$

For $\omega_n = 10$ rad / sec, $\quad 30 + 5k_1 + k_2 = \omega_n^2 = 100$. Thus $5k_1 + k_2 = 70$

(b) For $\zeta = 0.707$, $2\zeta\omega_n = 1 + k_2$. Thus $\omega_n = 1 + \dfrac{k_2}{1.414}$.

$$\omega_n^2 = \frac{\left(1+k_2\right)^2}{2} = 30 + 5k_1 + k_2 \qquad \text{Thus} \quad k_2^2 = 59 + 10k_1$$

(c) For $\omega_n = 10$ rad / sec and $\zeta = 0.707$,

$$5k_1 + k_2 = 100 \qquad \text{and} \quad 1 + k_2 = 2\zeta\omega_n = 14.14 \qquad \text{Thus} \quad k_2 = 13.14$$

Solving for k_1, we have $k_1 = 11.37$.

(d) The closed-loop transfer function is

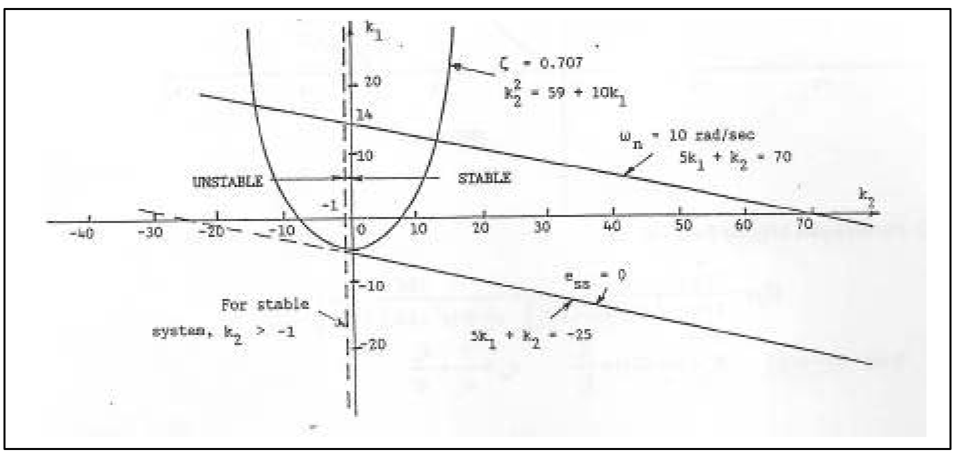
$$\frac{Y(s)}{R(s)} = \frac{5}{s^2 + \left(k_2 + 1\right)s + \left(30 + 5k_1 + k_2\right)} = \frac{5}{s^2 + 14.14s + 100}$$

For a unit-step input, $\quad \lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) = \dfrac{5}{100} = 0.05$

(e) For zero steady-state error due to a unit-step input,

$$30 + 5k_1 + k_2 = 5 \qquad \text{Thus} \qquad 5k_1 + k_2 = -25$$

Parameter Plane k_1 versus k_2:



7-25 (a) Closed-Loop Transfer Function

$$\frac{Y(s)}{R(s)} = \frac{100\left(K_P + K_D s\right)}{s^2 + 100K_D s + 100K_P}$$

The system is stable for $K_P > 0$ and $K_D > 0$.

(b) Characteristic Equation:

$$s^2 + 100K_D s + 100K_P = 0$$

(b) For $\zeta = 1$, $\quad 2\zeta\omega_n = 100K_D$.

$$\omega_n = 10\sqrt{K_P} \qquad \text{Thus} \qquad 2\omega_n = 100K_D = 20\sqrt{K_P} \qquad K_D = 0.2\sqrt{K_P}$$

(c) See parameter plane in part **(g)**.

(d) See parameter plane in part **(g)**.

(e) Parabolic error constant $K_a = 1000 \ \sec^{-2}$

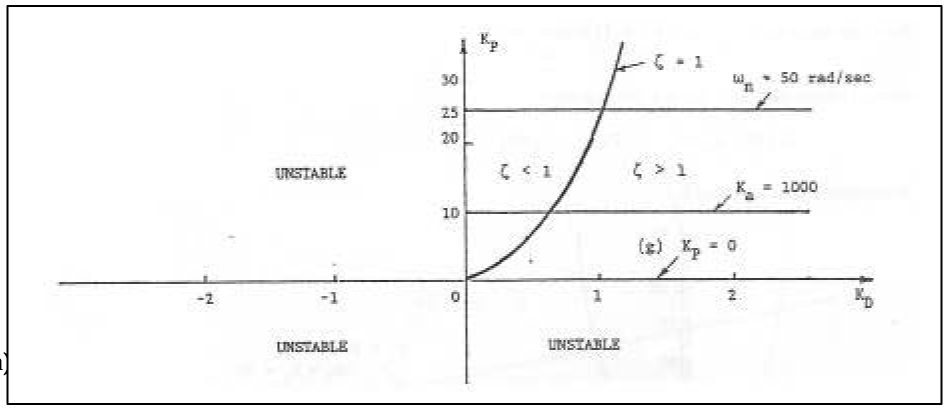
$$K_a = \lim_{s \to 0} s^2 G(s) = \lim_{s \to 0} 100\left(K_P + K_D s\right) = 100 K_P = 1000 \qquad \text{Thus} \quad K_P = 10$$

(f) Natural undamped frequency $\omega_n = 50$ rad/sec.

$$\omega_n = 10\sqrt{K_P} = 50 \qquad \text{Thus} \qquad K_P = 25$$

(g) When $K_P = 0$,

$$G(s) = \frac{100 K_D s}{s^2} = \frac{100 K_D}{s} \qquad \text{(pole-zero cancellation)}$$



7-26 (a)

$$G(s) = \frac{Y(s)}{E(s)} = \frac{KK_i}{s\left[Js(1+Ts) + K_i K_t\right]} = \frac{10K}{s\left(0.001 s^2 + 0.01 s + 10K_t\right)}$$

When $r(t) = tu_s(t)$, $\quad K_v = \lim_{s \to 0} sG(s) = \dfrac{K}{K_t} \qquad e_{ss} = \dfrac{1}{K_v} = \dfrac{K_t}{K}$

(b) When $r(t) = 0$

$$\frac{Y(s)}{T_d(s)} = \frac{1+Ts}{s\left[Js(1+Ts) + K_i K_t\right] + KK_i} = \frac{1+0.1s}{s\left(0.001 s^2 + 0.01 s + 10K_t\right) + 10K}$$

For $T_d(s) = \dfrac{1}{s} \qquad \lim_{t \to \infty} y(t) = \lim_{s \to 0} sY(s) = \dfrac{1}{10K}$ $\qquad$ if the system is stable.

(c) The characteristic equation of the closed-loop system is

$$0.001 s^3 + 0.01 s^2 + 0.1 s + 10K = 0$$

The system is unstable for $K > 0.1$. So we can set K to just less than 0.1. Then, the minimum value of the steady-state value of $y(t)$ is

$$\left.\frac{1}{10K}\right|_{K = 0.1^-} = 1^+$$

However, with this value of K, the system response will be very oscillatory. The maximum overshoot will be nearly 100%.

(d) For $K = 0.1$, the characteristic equation is

$$0.001 s^3 + 0.01 s^2 + 10 K_t s + 1 = 0 \quad \text{or} \quad s^3 + 10 s^2 + 10^4 K_t s + 1000 = 0$$

For the two complex roots to have real parts of $-2/5$. we let the characteristic equation be written as

$$(s+a)\left(s^2+5s+b\right)=0 \qquad \text{or} \qquad s^3+(s+5)s^2+(5a+b)s+ab=0$$

Then, $\quad a+5=10 \quad a=5 \quad ab=1000 \quad b=200 \quad 5a+b=10^4 K_t \quad K_t=0.0225$

The three roots are: $s=-a=-5 \quad s=-a=-5 \quad s=-2.5\pm j13.92$

7-27 (a) $K_t=10000$ oz - in / rad

The Forward-Path Transfer Function:

$$G(s)=\frac{9\times10^{12}K}{s\left(s^4+5000s^3+1.067\times10^7s^2+50.5\times10^9s+5.724\times10^{12}\right)}$$

$$=\frac{9\times10^{12}K}{s(s+116)(s+4883)(s+41.68+j3178.3)(s+41.68-j3178.3)}$$

Routh Tabulation:

s^5	1	1.067×10^7	5.724×10^{12}
s^4	5000	50.5×10^9	$9\times10^{12}K$
s^3	5.7×10^5	$5.72\times10^{12}-1.8\times10^9K$	0
s^2	$2.895\times10^8+1.579\times10^7K$	$9\times10^{12}K$	
s^1	$\dfrac{16.6\times10^{13}+8.473\times10^{12}K-2.8422\times10^9K^2}{29+1.579K}$		
s^0	$9\times10^{12}K$		

From the s^1 row, the condition of stability is $165710+8473K-2.8422K^2>0$

or $\quad K^2-2981.14K-58303.427<0 \quad$ or $\quad (K+19.43)(K-3000.57)<0$

Stability Condition: $\quad 0<K<3000.56$

The critical value of K for stability is 3000.56. With this value of K, the roots of the characteristic equation are: $\quad -4916.9, \quad -41.57+j3113.3, \quad -41.57+j3113.3, \quad -j752.68$, and $j752.68$

(b) $K_L=1000$ oz-in/rad. The forward-path transfer function is

$$G(s)=\frac{9\times10^{11}K}{s\left(s^4+5000s^3+1.582\times10^6s^2+5.05\times10^9s+5.724\times10^{11}\right)}$$

$$=\frac{9\times10^{11}K}{s(1+116.06)(s+4882.8)(s+56.248+j1005)(s+56.248-j1005)}$$

(c) Characteristic Equation of the Closed-Loop System:

$$s^5+5000s^4+1.582\times10^6s^3+5.05\times10^9s^2+5.724\times10^{11}s+9\times10^{11}K=0$$

Routh Tabulation:

s^5	1	1.582×10^6	5.724×10^{11}
s^4	5000	5.05×10^9	$9\times10^{11}K$
s^3	5.72×10^5	$5.724\times10^{11}-1.8\times10^8K$	0
s^2	$4.6503\times10^7+1.5734\times10^6K$	$9\times10^{11}K$	

$$s^1 \quad \frac{26.618\times 10^{18} + 377.43\times 10^{15}K - 2.832\times 10^{14}K^2}{4.6503\times 10^{7} + 1.5734\times 10^{6}K}$$

$$s^0 \quad 9\times 10^{11}K$$

From the s^1 row, the condition of stability is $26.618\times 10^4 + 3774.3K - 2.832K^2 > 0$

Or, $K^2 - 1332.73K - 93990 < 0$ or $(K - 1400)(K + 67.14) < 0$

Stability Condition: $0 < K < 1400$

The critical value of K for stability is 1400. With this value of K, the characteristic equation root are:

-4885.1, $-57.465 + j676$, $-57.465 - j676$, $j748.44$, and $-j748.44$

(c) $K_L = \infty$.

Forward-Path Transfer Function:

$$G(s) = \frac{nK_sK_iK}{s\left[L_aJ_Ts^2 + \left(R_aJ_T + R_mL_a\right)s + R_aB_m + K_iK_b\right]} \qquad J_T = J_m + n^2J_L$$

$$= \frac{891100K}{s\left(s^2 + 5000s + 566700\right)} = \frac{891100K}{s(s+116)(s+4884)}$$

Characteristic Equation of the Closed-Loop System:

$$s^3 + 5000s^2 + 566700s + 891100K = 0$$

Routh Tabulation:

s^3	1	566700
s^2	5000	$891100K$
s^1	$566700 - 178.22K$	
s^0	$8991100K$	

From the s^1 row, the condition of K for stability is $566700 - 178.22K > 0$.

Stability Condition: $0 < K < 3179.78$

The critical value of K for stability is 3179.78. With $K = 3179.78$, the characteristic equation roots are

-5000, $j752.79$, and $-j752.79$.

When the motor shaft is flexible, K_L is finite, two of the open-loop poles are complex. As the shaft becomes stiffer, K_L increases, and the imaginary parts of the open-loop poles also increase. When $K_L = \infty$, the shaft is rigid, the poles of the forward-path transfer function are all real. Similar effects are observed for the roots of the characteristic equation with respect to the value of K_L.

7-28 (a)

$$G_c(s) = 1 \qquad G(s) = \frac{100(s+2)}{s^2 - 1} \qquad K_p = \lim_{s\to 0} G(s) = -200$$

When $d(t) = 0$, the steady-state error due to a unit-step input is

$$e_{ss} = \frac{1}{1+K_p} = \frac{1}{1-200} = -\frac{1}{199} = -0.005025$$

(b)

$$G_c(s) = \frac{s+a}{s} \qquad G(s) = \frac{100(s+2)(s+a)}{s\left(s^2-1\right)} \qquad K_p = \infty \qquad e_{ss} = 0$$

(c)

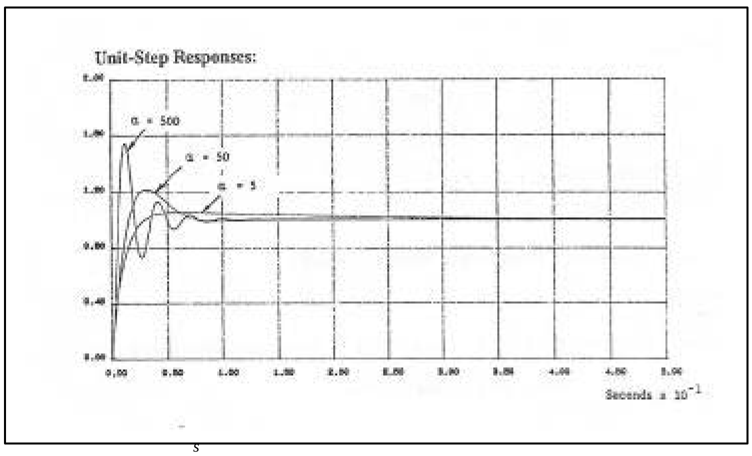
$a = 5$ maximum overshoot = 5.6%

$a = 50$ maximum overshoot = 22%

$a = 500$ maximum overshoot = 54.6%

As the value of a increases, the maximum overshoot increases because the damping effect of the zero at $s = -a$ becomes less effective.

Unit-Step Responses:



(d) $r(t) = 0$ and G_c s

s

System Transfer Function: (*r* = 0)

$$\left.\frac{Y(s)}{D(s)}\right|_{r=0} = \frac{100(s+2)}{s^3 + 100s^2 + (199 + 100a)s + 200a}$$

Output Due to Unit-Step Input:

$$Y(s) = \frac{100(s+2)}{s\left[s^3 + 100s^2 + (199 + 100a)s + 200a\right]}$$

$$y_{ss} = \lim_{t \to \infty} y(t) = \lim_{s \to 0} sY(s) = \frac{200}{200a} = \frac{1}{a}$$

(e) $r(t) = 0, \quad d(t) = u_s(t)$

$$G_c(s) = \frac{s+a}{s}$$

System Transfer Function [*r*(*t*) = 0]

$$\left.\frac{Y(s)}{D(s)}\right|_{r=0} = \frac{100s(s+20}{s^3 + 100s^2 + (199 + 100a)s + 200a} \qquad D(s) = \frac{1}{s}$$

$$y_{ss} = \lim_{t \to \infty} y(t) = \lim_{s \to 0} sY(s) = 0$$

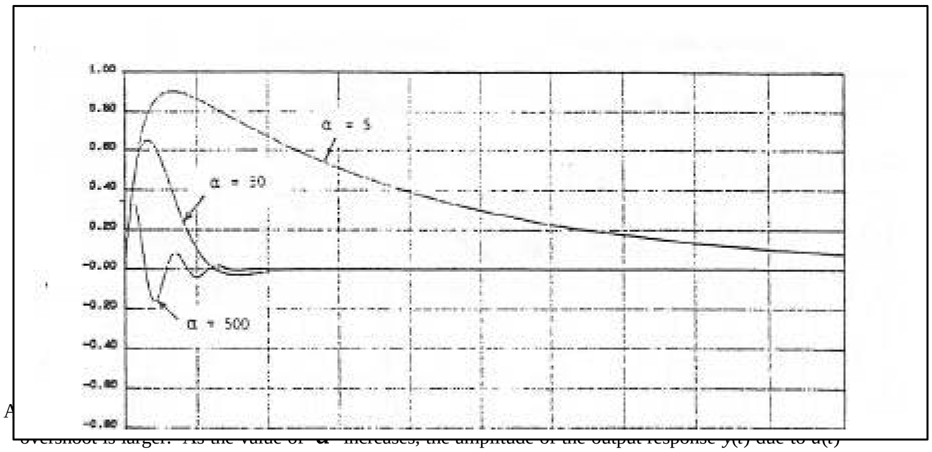
(f)

$$\alpha = 5 \qquad \left.\frac{Y(s)}{D(s)}\right|_{r=0} = \frac{100\, s(s+2)}{s^3 + 100 s^2 + 699 s + 1000}$$

$$\alpha = 50 \qquad \left.\frac{Y(s)}{D(s)}\right|_{r=0} = \frac{100\, s(s+2)}{s^3 + 100 s^2 + 5199 s + 10000}$$

$$\alpha = 5000 \qquad \left.\frac{Y(s)}{D(s)}\right|_{r=0} = \frac{100\, s(s+2)}{s^3 + 100 s^2 + 50199 s + 100000}$$

Unit-Step Responses:



(g) A

overshoot is larger. As the value of α increases, the amplitude of the output response $y(t)$ due to $d(t)$ becomes smaller and more oscillatory.

7-29 (a) Forward-Path Transfer function: **Characteristic Equation:**

$$G(s) = \frac{H(s)}{E(s)} = \frac{10N}{s(s+1)(s+10)} \cong \frac{N}{s(s+1)} \qquad s^2 + s + N = 0$$

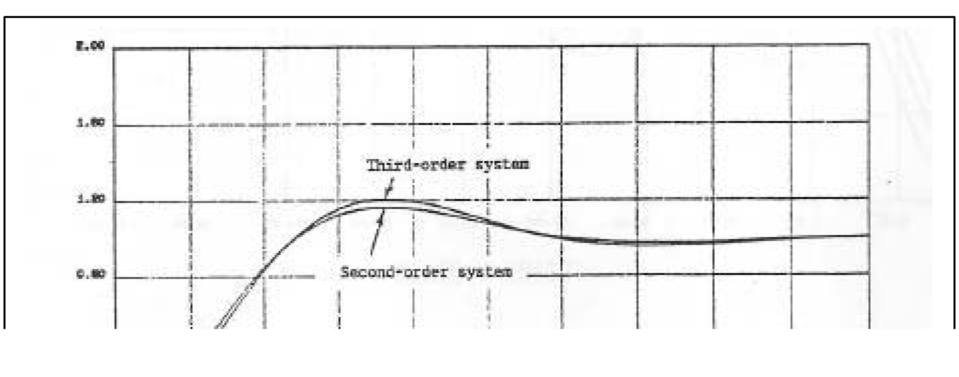
N=1: Characteristic Equation: $s^2 + s + 1 = 0$ $\qquad \zeta = 0.5 \qquad \omega_n = 1$ rad/sec.

Maximum overshoot $= e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}} = 0.163$ (16.3%) Peak time $t_{max} = \dfrac{\pi}{\omega_n\sqrt{1-\zeta^2}} = 3.628$ sec.

N=10: Characteristic Equation: $s^2 + s = 10 = 0$ $\qquad \zeta = 0.158 \qquad \omega_n = 10$ rad/sec.

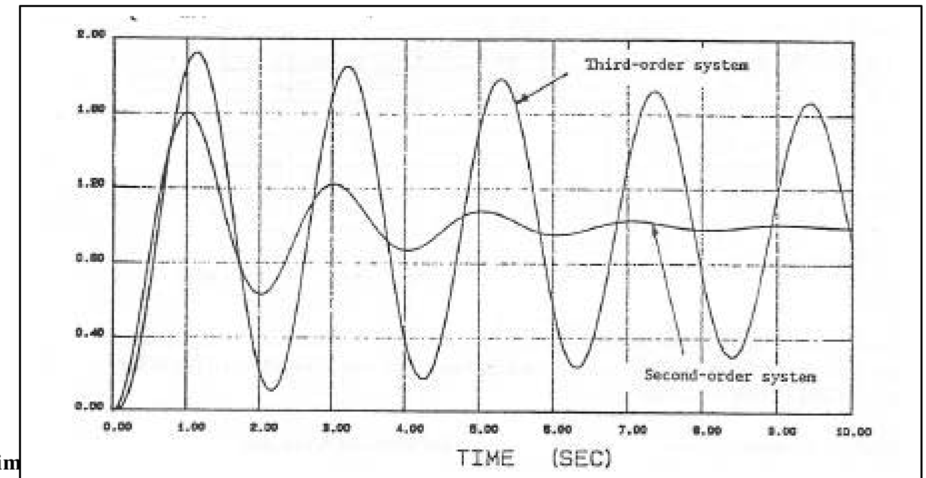
Maximum overshoot $= e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}} = 0.605$ (60.5%) Peak time $t_{max} = \dfrac{\pi}{\omega_n\sqrt{1-\zeta^2}} = 1.006$ sec.

(b) Unit-Step Response: $N = 1$



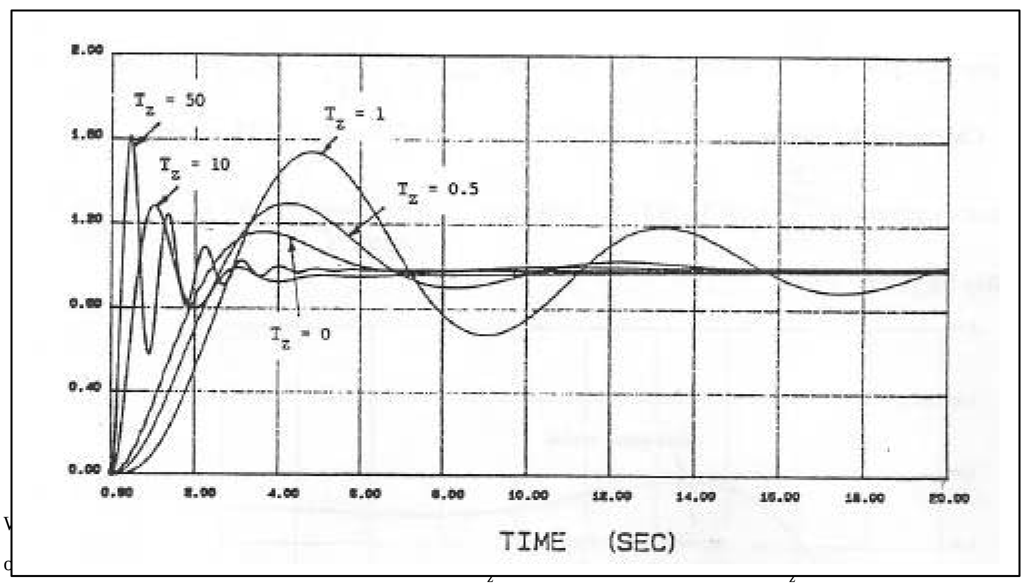
	Second-order System	Third-order System
Maximum overshoot	0.163	0.206
Peak time	3.628 sec.	3.628 sec.

Unit-Step Response: $N = 10$



Maxim		
Peak time	1.006 sec.	1.13 sec.

7-30 Unit-Step Responses:

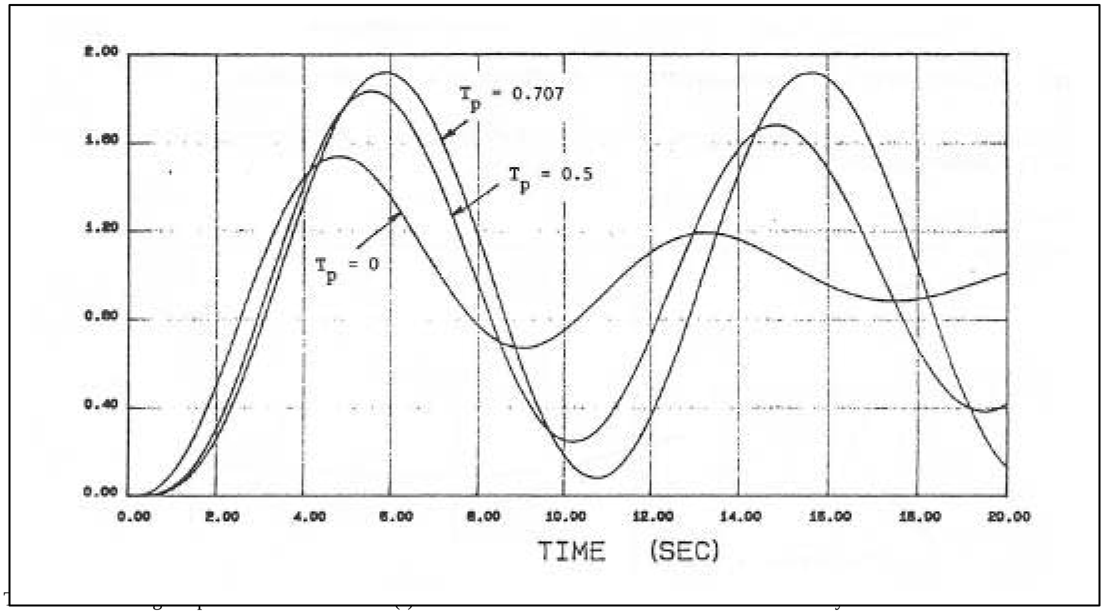


V

d z z

derivative control or a high-pass filter.

7-31 **Unit-Step Responses**



T_p

less stable. When $T_p > 0.707$, the closed-loop system is stable.

7-32 **(a)** **N=1 Closed-loop Transfer Function:**

$$M_H(s) = \frac{Y(s)}{R(s)} = \frac{10}{s^3 + 11s^2 + 10s + 10} = \frac{1}{1 + s + 1.1s^2 + 0.1s^3}$$

Second-Order Approximating system:

$$M_L(s) = \frac{1}{1 + d_1 s + d_2 s^2}$$

$$\frac{M_H(s)}{M_L(s)} = \frac{1 + d_1 s + d_2 s^2}{1 + s + 1.1s^2 + 0.1s^3} = \frac{1 + m_1 s + m_2 s^2}{1 + l_1 s + l_2 s^2 + l_3 s^3}$$

$d_1 = m_1 \qquad d_2 = m_2 \qquad l_1 = 1 \qquad l_2 = 1.1 \qquad l_3 = 0.1$

$e_2 = f_2 = 2m_2 - m_1^2 = 2d_2 - d_1^2 \qquad e_4 = f_4 = 2m_4 - 2m_1 m_3 + m_2^2 = m_2^2 = d_2^2$

$f_2 = 2l_2 - l_1^2 = 2 \times 1.1 - 1 = 1.2 \qquad f_4 = 2l_4 - 2l_1 l_3 + l_2^2 = -2 \times 1 \times 0.1 + (1.1)^2 = 1.01$

$f_6 = 2l_6 - 2l_1 l_5 + 2l_2 l_4 - l_3^2 = -l_3^2 = -0.01$

Thus, $f_2 = 1.2 \qquad e_2 = 1.2 = 2d_2 - d_1^2 \qquad f_4 = 1.01 \qquad e_4 = 1.01 = d_2^2 \qquad$ Thus $d_2 = 1.005$

$f_6 = -0.01 \qquad d_1^2 = 2d_2 - 1.2 = 0.81 \qquad$ Thus $d_1 = 0.9$

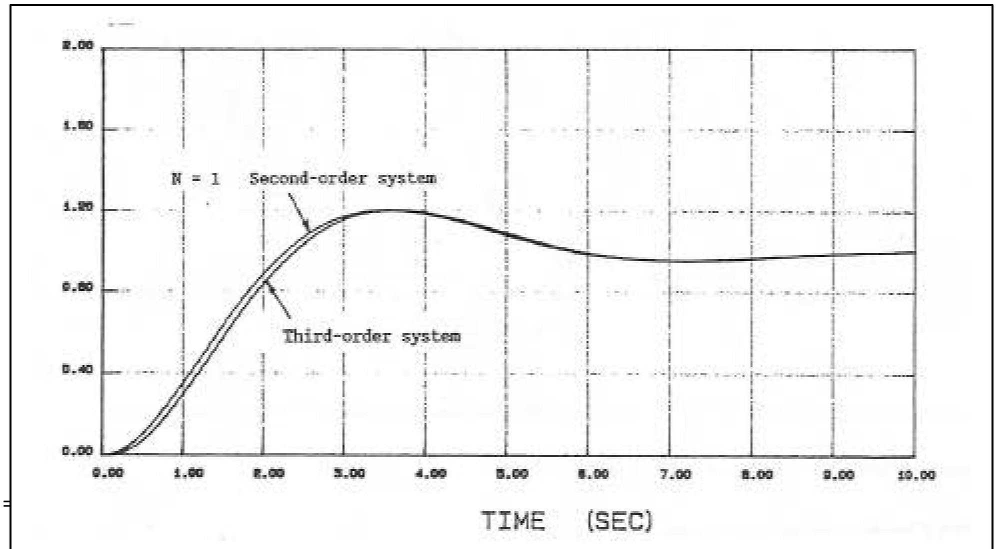
$$M_L(s) = \frac{1}{1 + 0.9s + 1.005s^2} = \frac{0.995}{s^2 + 0.8955s + 0.995} \qquad G_L(s) = \frac{0.995}{s(s + 0.895)}$$

Roots of Characteristic Equation:

Third-order System			**Second-order System**	
-10.11	$-0.4457 + j0.8892$	$-0.4457 - j0.8892$	$-0.4478 + j0.8914$	$-0.4478 - j0.8914$

The real root at -10.11 is dropped by the second-order approximating system, and the two complex roots are slightly preturbed.

Unit-Step Responses:



(b) $N =$

$$M_H(s) = \frac{20}{s^3 + 11s^2 + 10s + 20} = \frac{1}{1 + 0.5s + 0.55s^2 + 0.05s^3}$$

$l_1 = 0.5 \qquad l_2 = 0.55 \qquad l_3 = 0.05 \qquad e_2 = 2d_2 - d_1^2 = f_2 = 2l_2 - l_1^2 = 2 \times 0.55 - (0.5)^2 = 0.85$

$e_4 = d_2^2 = f_4 = -2l_1 l_3 + l_2^2 = -0.05 + 0.3025 = 0.2525$ Thus $d_2^2 = f_4 = 0.2525 \qquad d_2 = 0.5025$

$d_1^2 = 2d_2 - e_2 = 2 \times 0.5025 - 0.85 = 0.155 \qquad d_1 = 0.3937$

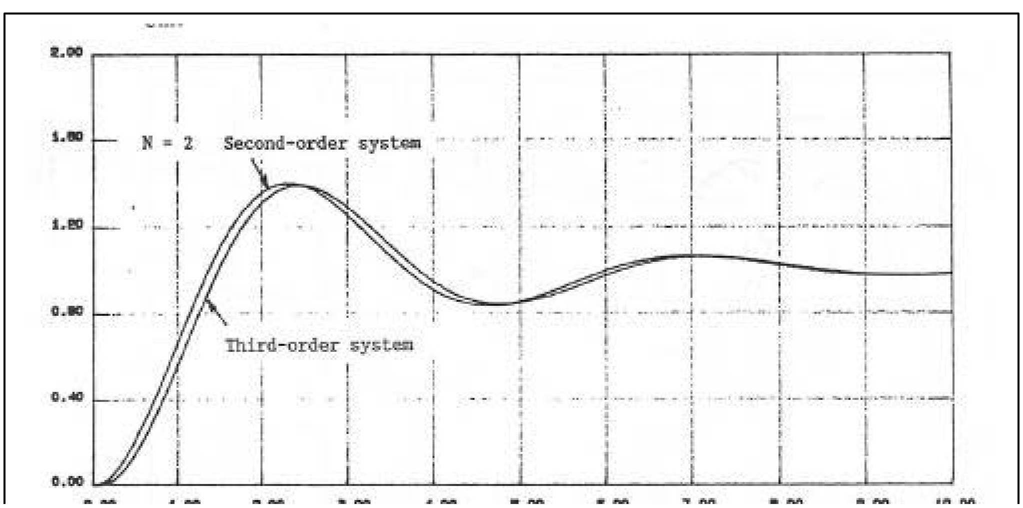
$$M_L(s) = \frac{1}{1 + 0.3937s + 0.5025s^2} = \frac{1.99}{s^2 + 0.7834s + 1.99} \qquad G_L(s) = \frac{1.99}{s(s + 0.7835)}$$

Roots of Characteristic Equations:

Third-order System			Second-order System	
-10.213	$-0.3937 + j1.343$	$-0.3937 - j1.343$	$-0.3917 + j1.3552$	$-0.3917 - j1.3552$

The real root at -10.213 is dropped by the second-order approximating system, and the two complex roots are slightly preturbed.

Unit-Step Responses:



(c) *N=3* **Closed-Loop Transfer Function:**

$$M_H(s) = \frac{30}{s^3 + 11s^2 + 10s + 30} = \frac{1}{1 + 0.333\,s + 0.3667\,s^2 + 0.0333\,s^3}$$

$$l_1 = 0.3333 \qquad l_2 = 0.3667 \qquad l_3 = 0.0333$$

$$e_2 = 2d_2 - d_1^2 = f_2 = 2l_2 - l_1^2 = 0.7333 - 0.1111 = 0.6222 \qquad e_4 = d_2^2 = f_4 = -2l_1l_3 + l_2^2 = 0.1122$$

Thus, $d_2^2 = f_4 = 0.1122 \qquad d_1^2 = 2d_2 - f_2 = 2 \times 0.335 - 0.6222 = 0.0477$

$$d_1 = 0.2186 \qquad d_2 = 0.335$$

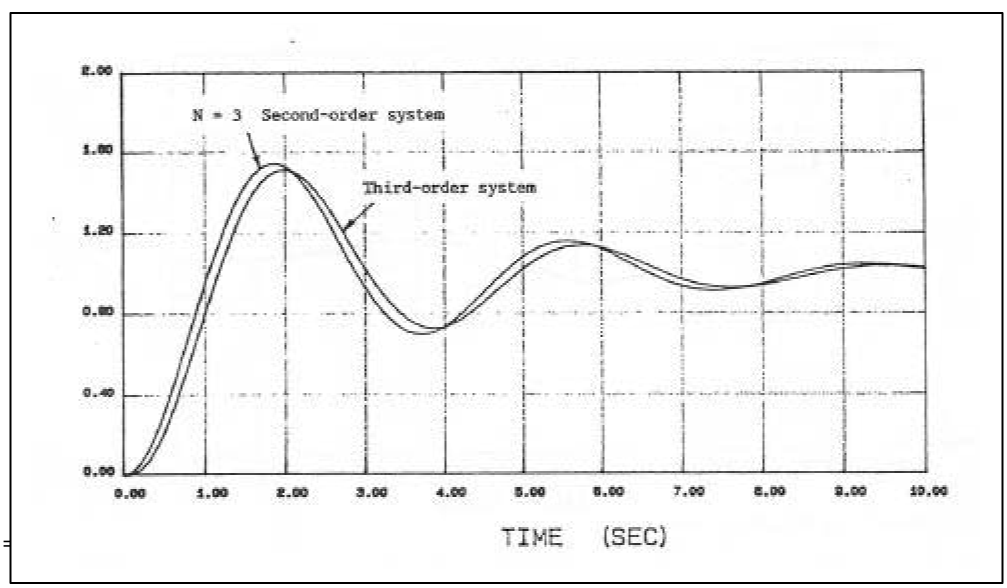
$$M_L(s) = \frac{1}{1 + 0.2186\,s + 0.335\,s^2} = \frac{2.985}{s^2 + 0.6524\,s + 2.985} \qquad G_L(s) = \frac{2.985}{s(s + 0.6525)}$$

Roots of Characteristic Equation:

Third-order System			Second-order System	
-10.312	$-0.3438 + j1.6707$	$-0.3438 - j1.6707$	$-0.3262 + j1.6966$	$-0.3262 - j1.6966$

The real root at -10.312 is dropped by the second-order approximating system, and the complex roots are slightly preturbed.

Unit-Step Responses:



(d) *N* =

$$M_H(s) = \frac{40}{s^3 + 11s^2 + 10s + 40} = \frac{1}{1 + 0.25s + 0.275s^2 + 0.025s^3}$$

$$l_1 = 0.25 \qquad l_2 = 0.275 \qquad l_3 = 0.025$$

$$e_2 = 2d_2 - d_1^2 = f_2 = 2l_2 - l_1^2 = 0.4875 \qquad e_4 = d_2^4 = f_4 = -2l_1l_3 + l_2^2 = 0.06313$$

Thus, $d_2^2 = f_4 = 0.06313 \qquad d_2 = 0.2513 \qquad d_1^2 = 2d_2 - f_2 = 0.5025 - 0.4875 = 0.015 \qquad d_1 = 0.1225$

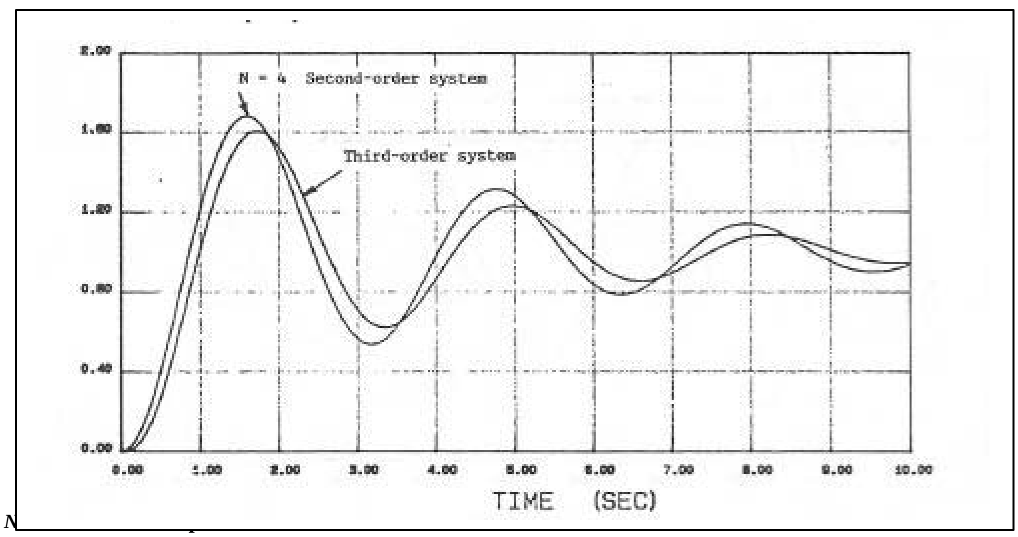
$$M_L(s) = \frac{1}{1 + 0.1225s + 0.2513s^2} = \frac{3.98}{s^2 + 0.4874s + 3.98} \qquad G_L(s) = \frac{3.98}{s(s + 0.4874)}$$

Roots of Characteristic Equation:

Third-order System	Second-order System
$-10.408 \quad -0.2958 + j1.9379 \quad -/2958 - j1.9379$	$-0.2437 + j1.98 \quad -0.2437 - j1.98$

The real root at -10.408 is dropped by the second-order approximating system, but the complex roots are preturbed. As the value of N increases, the gain of the system is increased, and the roots are more preturbed.

Unit-Step Responses:



(e) N

$$M_H(s) = \frac{50}{s^3 + 11s^2 + 10s + 50} = \frac{1}{1 + 0.2s + 0.22s^2 + 0.02s^3}$$

$$l_1 = 0.2 \qquad l_2 = 0.22 \qquad l_3 = 0.02$$

$$e_2 = 2d_1 - d_2^2 = f_2 = 2l_2 - l_1^2 = 0.44 - 0.04 = 0.4$$

$$e_4 = d_2^2 = f_4 = -2l_1l_3 + l_2^2 = -0.008 + 0.0484 = 0.0404$$

Thus,

$$d_2^2 = f_4 = 0.0404 \qquad d_2 = 0.201$$

$$d_1^2 = 2d_2 - e_2 = 0.402 - 0.4 = 0.002 \qquad d_1 = 0.04472$$

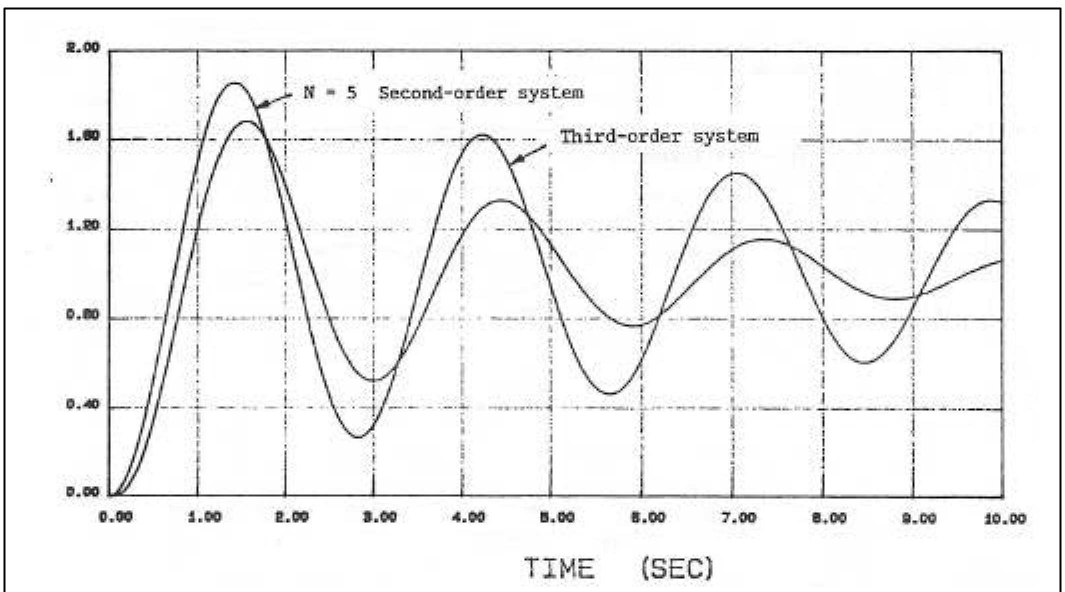
$$M_L(s) = \frac{1}{1 + 0.0447s + 0.201s^2} = \frac{1}{s^2 + 0.2225s + 4.975} \qquad G_L(s) = \frac{4.975}{s(s+0.2225)}$$

Roots of Characteristic Equation:

Third-order System			Second-order System	
-10.501	$-0.2494 + j2.678$	$-0.2494 - j2.678$	$-0.1113 + j2.2277$	$-0.1113 - j2.2277$

The real root at -10.501 is dropped by the second-order approximating system, and the complex roots are changed, especially the real parts.

Unit-Step Responses:



7-33 (a)

$$G(s) = \frac{891100}{s\left(s^2 + 5000s + 566700\right)} = \frac{891100}{s(s+116)(s+4884)}$$

Closed-Loop Transfer Function:

$$M_H(s) = \frac{891100}{s^3 + 5000s^2 + 566700s + 891100} = \frac{1}{1 + 0.636s + 5.611 \times 10^{-3}s^2 + 1.1222 \times 10^{-6}s^3}$$

$$l_1 = 0.636 \qquad l_2 = 5.611 \times 10^{-3} \qquad l_3 = 1.122 \times 10^{-6}$$

$$e_2 = 2d_2 - d_1^2 = f_2 = 2l_2 - l_1^2 = 1.1222 \times 10^{-3} - 0.4045 = -0.3933$$

$$e_4 = d_2^2 = f_4 = -2l_1l_3 + l_2^2 = -2 \times 0.636 \times 1.1222 \times 10^{-6} + (5.611 \times 10^{-3})^2 = 0.000030\;06$$

Thus,

$$d_2^2 = 0.00003006 \qquad d_2 = 0.005482$$

$$d_1^2 = 2d_2 - f_2 = 0.01096 + 0.3933 = 0.4042 \qquad d_1 = 0.6358$$

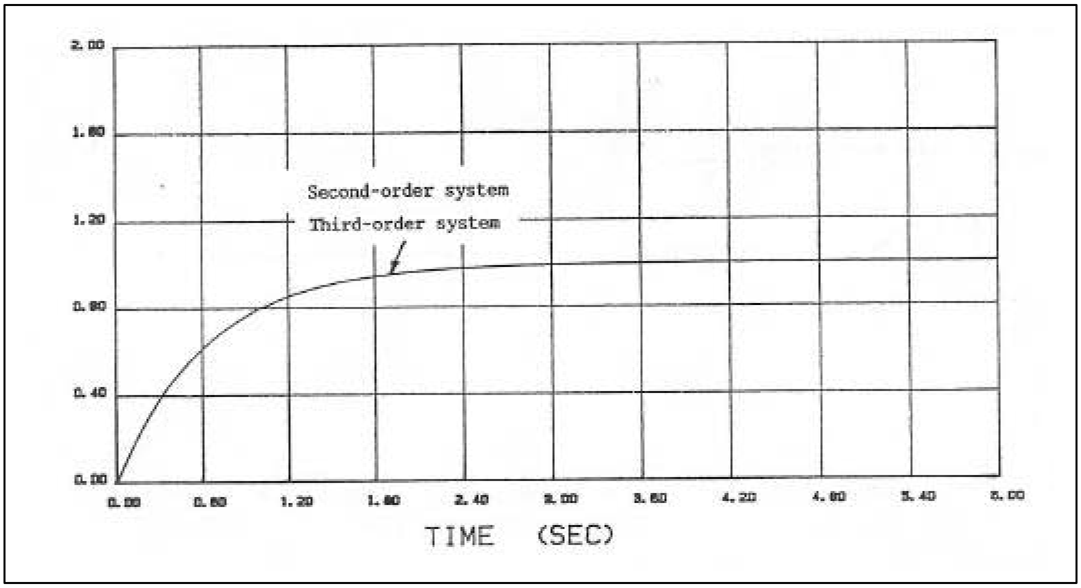
$$M_L(s) = \frac{1}{1 + 0.6358s + 0.005482s^2} = \frac{182.4}{s^2 + 115.97s + 182.4} \qquad G_L(s) = \frac{182.4}{s(s + 115.97)}$$

Roots of Characteristic Equations:

Third-order System	Second-order System
$-1.595 \quad -114.4 \quad -4884$	$-1.5948 \quad -114.38$

The real root at -4884 is dropped by the second-order approximating system. the other two roots are hardly preturbed.

Unit-Step Responses



(b)

$$M_H(s) = \frac{89110000}{s^3 + 5000s^2 + 566700 + 89110000} = \frac{1}{1 + 0.00636s + 5.611 \times 10^{-5}s^2 + 1.1222 \times 10^{-8}s^3}$$

$$l_1 = 0.00636 \qquad l_2 = 5.611 \times 10^{-5} \qquad l_3 = 1.1222 \times 10^{-8}$$

$$e_2 = 2d_2 - d_1^2 = f_2 = 2l_2 - l_1^2 = 11.222 \times 10^{-5} - 4.045 \times 10^{-5} = 0.0000718$$

$$e_4 = d_2^2 = f_4 = 2l_4 - 2l_1l_3 + l_2^2 = -2 \times 0.00636 \times 1.1222 \times 10^{-8} + (5.611 \times 10^{-5})^2 = 3.0056 \times 10^{-9}$$

Thus,

$$d_2^2 = f_4 = 3.0056 \times 10^{-9} \qquad d_2 = 0.0000548$$

$$d_1^2 = 2d_2 - f_2 = 0.0001096 - 0.0000718 = 0.0000548 \qquad d_1 = 0.007403$$

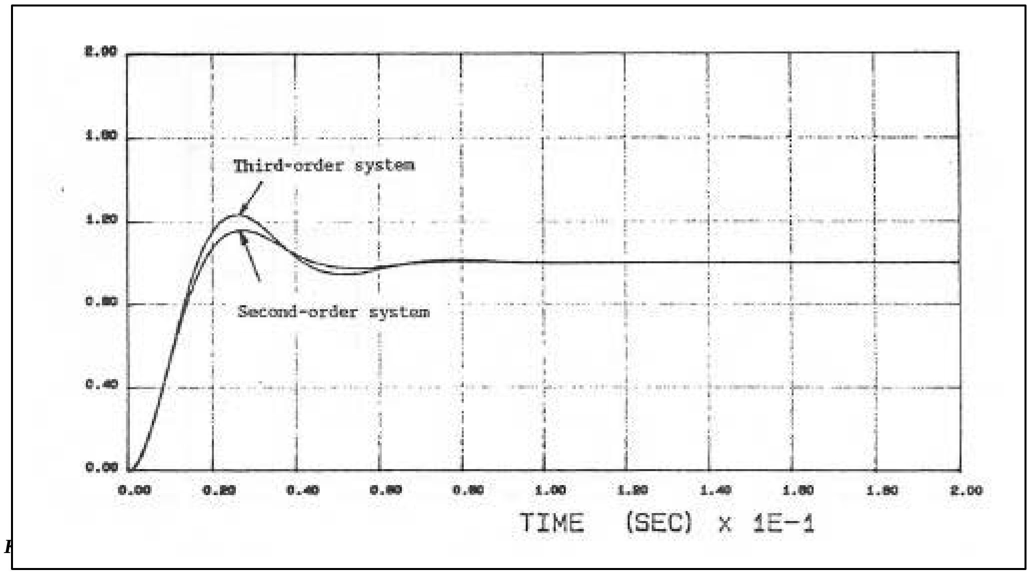
$$M_L(s) = \frac{1}{1 + 0.007403s + 0.0000548s^2} = \frac{18248}{s^2 + 135.1s + 18248} \qquad G_L(s) = \frac{18248}{s(s + 135.1)}$$

Roots of Characteristic Equations:

Third-order System **Second-order System**

$-4887.8 \quad -56.106 + j122.81 \quad -56.106 - j122.81 \qquad -67.55 + j114.98 \quad -67.55 - j114.98$

Unit-Step Responses



(c)

$$M_H(s) = \frac{891100000}{s^3 + 5000s^2 + 566700s + 891100000} = \frac{1}{1 + 0.000636s + 5.611 \times 10^{-6} s^2 + 1.1122 \times 10^{-9} s^3}$$

$$l_1 = 0.000636 \qquad l_2 = 5.611 \times 10^{-6} \qquad l_3 = 1.1222 \times 10^{-9}$$

$$e_2 = 2d_2 - d_1^2 = f_2 = 2l_2 - l_1^2 = 11.222 \times 10^{-6} - 0.000000404 = 0.00001082$$

$$e_4 = d_2^2 = f_4 = -2l_1 l_3 + l_2^2 = -2 \times 0.000636 \times 1.1222 \times 10^{-9} + (5.611 \times 10^{-6})^2 = 3.0056 \times 10^{-11}$$

Thus,

$$d_2^2 = f_4 = 3.0055 \times 10^{-11} \qquad d_2 = 0.000005482$$

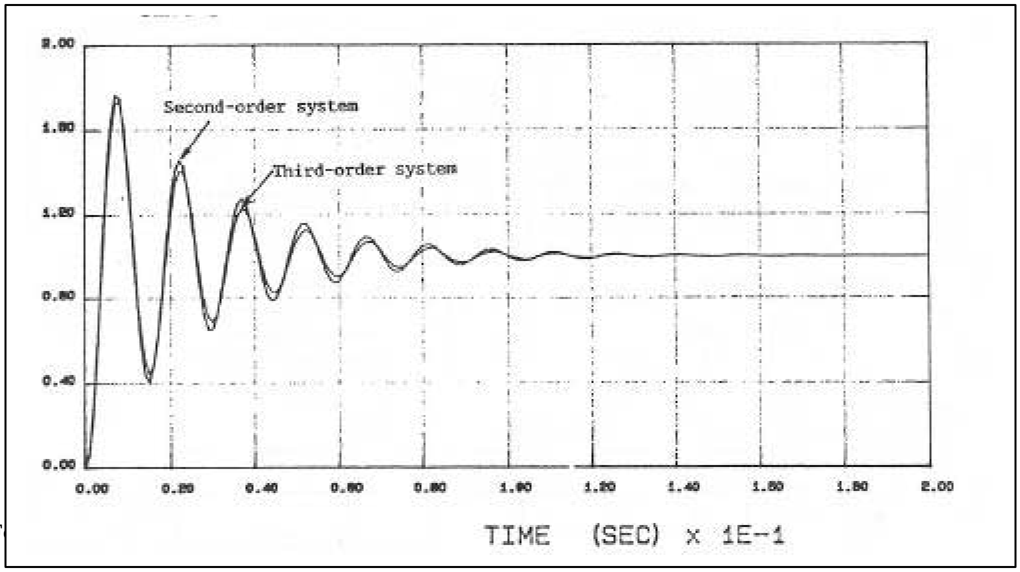
$$d_1^2 = 2d_2 - f_2 = 0.000010965 - 0.000010818 = 0.000000147 \qquad d_1 = 0.0003829$$

$$M_L(s) = \frac{1}{1 + 0.0003829s + 0.000005482s^2} = \frac{182415.177}{s^2 + 69.8555s + 182415.177} \qquad G_L(s) = \frac{182415.177}{s(s + 69.8555)}$$

Roots of Characteristic Equations:

Third-order System	**Second-order System**
$-4921.6 \quad -39.178 + j423.7 \quad -39.178 - j423.7$	$-34.928 + j425.67 \quad -34.928 - j425.67$

Unit-step Responses



7-34 F

$$G(s) = \frac{K(s-1)}{s(s+1)(s+2)} \qquad\qquad M(s) = \frac{-s+1}{s^3+3s^2+s+1}$$

Second-order System:

$$M_L(s) = \frac{1+c_1 s}{1+d_1 s+d_2 s^2}$$

$$\frac{M_H(s)}{M_L(s)} = \frac{(-s+1)\left(1+d_1 s+d_2 s^2\right)}{\left(s^3+3s^2+s+1\right)\left(1+c_1 s\right)} = \frac{1+\left(d_1-1\right)s+\left(d_2-d_1\right)s^2-d_2 s^3}{1+\left(c_1+1\right)s+\left(c_1+3\right)s^2+\left(3c_1+1\right)s^3+c_1 s^4}$$

$$l_1 = 1+c_1 \qquad l_2 = 3+c_1 \qquad l_3 = 1+3c_1 \qquad l_4 = c_1$$

$$m_1 = d_1 - 1 \qquad m_2 = d_2 - d_1 \qquad m_3 = -d_2$$

$$e_2 = f_2 = 2m_2 - m_1^2 = 2\left(d_2-d_1\right)-\left(d_1-1\right)^2 = 2d_2 - d_1^2 - 1$$

$$= 2l_2 - l_1^2 = 2\left(3+c_1\right) - \left(1+c_1\right)^2 = 5 - c_1^2$$

$$e_4 = f_4 = 2m_4 - 2m_1m_3 + m_2^2 = -2\left(d_1-1\right)\left(-d_2\right) + \left(d_2-d_1\right)^2 = d_2^2 - 2d_2 + d_1^2$$

$$= 2l_4 - 2l_1l_3 + l_2^2 = -2\left(1+c_1\right)\left(1+3c_1\right) + \left(3+c_1\right)^2 = 7 - 2c_1 - 5c_1^2$$

$$e_6 = f_6 = 2m_6 - 2m_1m_5 + 2m_2m_4 - m_3^2 = -m_3^2 = -\left(-d_2\right)^2 = -d_2^2$$

$$= 2l_6 - 2l_1l_5 + 2l_2l_4 - l_3^2 = 2l_2l_4 - l_3^2 = 2\left(3+c_1\right)c_1 - \left(1+3c_1\right)^2 = -1 - 7c_1^2$$

Simultaneous equations to be solved: **Solutions:**

$$2d_2 - d_1^2 = 6 - c_1^2 \qquad c_1 = -1.0408$$

$$d_2^2 = 1 + 7c_1^2 \qquad d_1 = 0.971$$

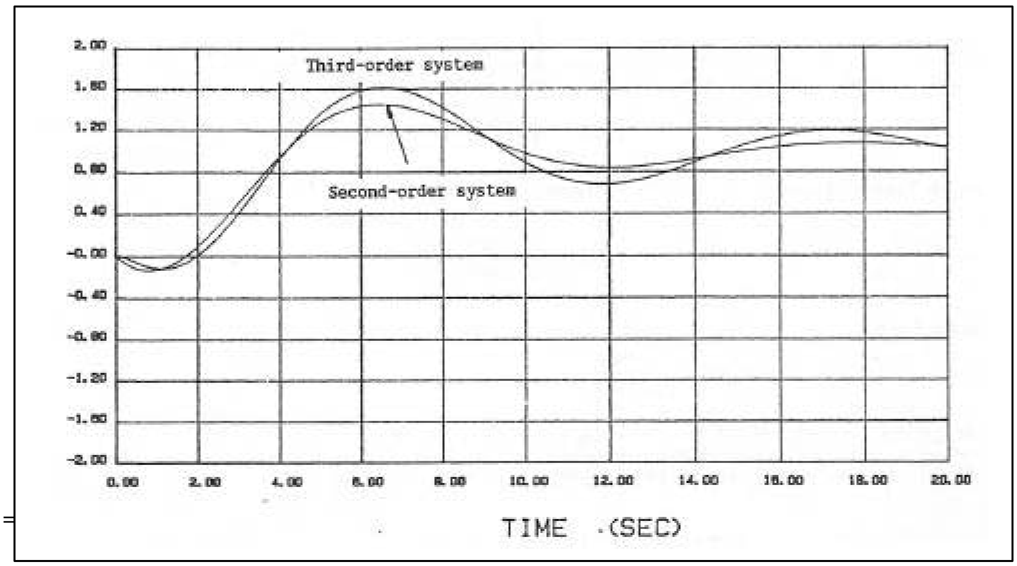
$$d_2^2 - 2d_2 + d_1^2 = 5 - c_1^2 \qquad d_2 = 2.93$$

$$M_L(s) = \frac{1 - 1.0408s}{1 + 0.971s + 2.93s^2} = \frac{-0.3552(s - 0.9608)}{s^2 + 0.3314s + 0.3413} \qquad G_L(s) = \frac{-0.3552(s - 0.9608)}{s(s + 0.69655)}$$

Roots of Characteristic Equations:

Third-order System			Second-order System	
-2.7693	$-0.1154 + j0.5897$	$-0.1154 - j0.5897$	$-0.1657 + j0.5602$	$-0.1657 - j0.5602$

Unit-step Responses



7-35 (a) $K =$

$$M_H(s) = \frac{10}{s^4 + 23s^3 + 62s^2 + 40s + 10} = \frac{1}{1 + 4s + 6.2s^2 + 2.3s^3 + 0.1s^4}$$

Second-order System Approximation:

$$M_L(s) = \frac{1}{1 + d_1 s + d_2 s^2}$$

$$l_1 = 4 \qquad l_2 = 6.2 \qquad l_3 = 2.3 \qquad l_4 = 0.1$$

$$e_2 = f_2 = 2d_2 - d_1^2 = 2l_2 - l_1^2 = 12.4 - 16 = -3.6$$

$$e_4 = f_4 = d_2^2 = 2l_4 - 2l_1 l_3 + l_2^2 = 0.2 - 2 \times 4 \times 2.3 + (6.2)^2 = 20.24$$

$$e_6 = f_6 = 2d_6 - 2d_1 d_5 + 2d_2 d_4 - d_3^2 = -d_3^2 = 2l_2 l_4 - l_3^2 = -4.05$$

Thus,

$$d_2^2 = 20.24 \qquad d_2 = 4.5$$

$$d_1^2 = 2d_2 - f_2 = 9 + 3.6 = 12.6 \qquad d_1 = 3.55$$

$$M_L(s) = \frac{1}{1 + 3.55s + 4.5s^2} = \frac{0.2222}{s^2 + 0.7888s + 0.2222} \qquad G_L(s) = \frac{0.2222}{s(s + 0.7888)}$$

Roots of Characteristic Equations:

Fourth-order system	Second-order system
-2.21 $\quad -20$ $\quad -0.3957 + j0.264$ $\quad -0.3957 - j0.264$	$-0.3944 + j0.258$ $\quad -0.3944 - j0.258$

(b) ***K* = 10 Third-order System Approximation:**

$$M_L(s) = \frac{1}{1 + d_1 s + d_2 s^2 + d_3 s^3} \qquad \frac{M_H(s)}{M_L(s)} = \frac{1 + d_1 s + d_2 s^2 + d_3 s^3}{1 + 4s + 6.2s^2 + 2.3s^3 + 0.1s^4}$$

$$e_2 = f_2 = 2d_2 - d_1^2 = -3.6 \qquad d_3^2 = -f_6 = 4.05 \qquad \text{Thus} \quad d_3 = 2.0125$$

$$e_4 = f_4 = 2d_4 - 2d_1 d_3 + d_2^2 = -2d_1 d_3 + d_2^2 = -4.025d_1 + d_2^2 = f_4 = 20.24$$

Thus,

$$d_2 = 0.5d_1^2 - 1.8 \qquad d_2^2 = 0.25d_1^4 - 1.8d_1^2 + 3.24 = 20.24 + 4.025d_1$$

Solving for d_1, $\quad d_1 = 3.9528, \ -3.0422, \ -0.4553 + j2334, \ -0.4553 - j2334.$

Selecting the positive and real solution, we have $\quad d_1 = 3.9528. \quad d_2 = 0.5d_1^2 - 1.8 = 6.0123$

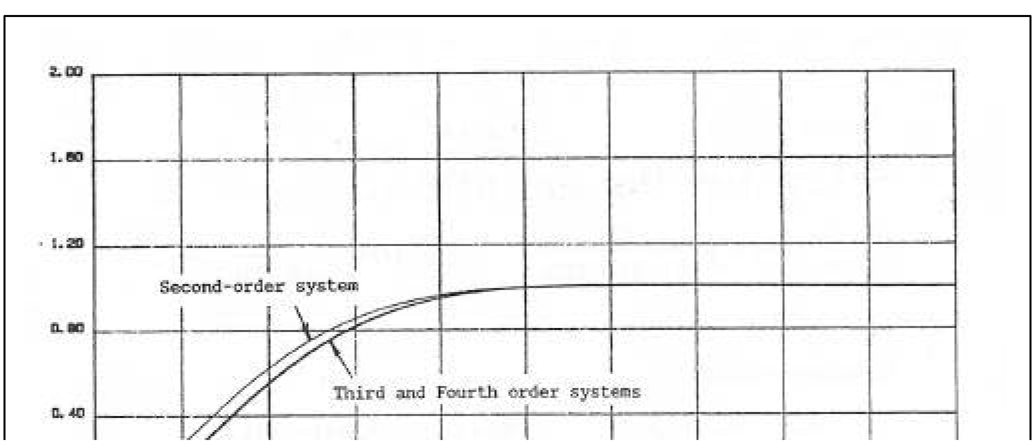
$$M_L(s) = \frac{1}{1 + 3.9528s + 6.0123s^2 + 2.0125s^3} = \frac{0.4969}{s^3 + 2.9875s^2 + 1.964s + 0.4969}$$

$$G_L(s) = \frac{0.4969}{s\left(s^2 + 2.9875s + 1.964\right)}$$

Roots of Characteristic Equations:

Fourth-order System	Third-order System
-2.21 $\quad -20$ $\quad -0.39565 + j0.264$ $\quad -0.39565 - j0.264$	-2.1963 $\quad -0.3956 + j0.264$ $\quad -2.1963 - j0.264$

Unit-step Response



(c) $K = 40$ **Closed-loop Transfer Function:**

$$M_H(s) = \frac{40}{s^4 + 23s^3 + 62s^2 + 40s + 40} = \frac{1}{1 + s + 1.55s^2 + 0.575s^3 + 0.025s^4}$$

$$l_1 = 1 \qquad l_2 = 1.55 \qquad l_3 = 0.575 \qquad l_4 = 0.025$$

Second-order System Approximation:

$$M_L(s) = \frac{1}{1 + d_1 s + d_2 s^2}$$

$$e_2 = f_2 = 2d_2 - d_1^2 = 2l_2 - l_1^2 = 3.1 - 1 = 2.1$$

$$e_4 = f_4 = d_2^2 = 2l_4 - 2l_1 l_3 + l_2^2 = 0.05 - 1.15 + 2.4025 = 1.3025$$

Thus,

$$d_2^2 = 1.3025 \qquad d_2 = 1.1413$$

$$d_1^2 = 2d_2 - f_2 = 2 \times 1.1413 - 2.1 = 0.1825 \qquad d_1 = 0.4273$$

$$M_L(s) = \frac{1}{1 + 0.4273s + 1.1413s^2} = \frac{0.8762}{s^2 + 0.3744s + 0.8762} \qquad G_L(s) = \frac{0.8762}{s(s + 0.3744)}$$

Roots of Characteristic Equations:

Fourth-order System				Second-order System	
-2.5692	-19.994	$-0.2183 + j0.855$	$-0.2183 - j0.855$	$-0.1872 + j0.9172$	$-0.1872 - j0.9172$

Third-order system Approximation:

$$M_L(s) = \frac{1}{1 + d_1 s + d_2 s^2 + d_3 s^3}$$

$$e_2 = f_2 = 2d_2 - d_1^2 = 2l_2 - l_1^2 = 3.1 - 1 = 2.1$$

$$e_4 = f_4 = -2d_1 d_3 + d_2^2 = -1.0062\, d_1 + d_2^2 = 2l_4 - 2l_1 l_3 + l_2^2 = 1.3025$$

$$e_6 = f_6 = -d_3^2 = 2l_2 l_4 - l_3^2 = -0.2531$$

Equations to be Solved Simultaneously:

$$2d_2 - d_1^2 = 2.1 \qquad d_2^2 = -1.00623\, d_1 = 1.3025 \qquad \text{Thus} \qquad d_2 = 0.5d_1^2 + 1.05$$

$$d_2^2 = 0.25d_1^4 + 1.05d_1^2 + 1.1025 \qquad \text{Th us} \qquad d_1^4 + 4.2d_1^2 - 4.0249\, d_1 - 0.8 = 0$$

The roots of the last equation are: $d_1 = -0.1688$, 0.9525 , $-0.392 + j2.196$, $-0.392 - j2.196$

Selecting the positive real solution, we have $d_1 = 0.9525$.

$d_2^2 = 1.00623 \; d_1 + 1.3025 = 2.261$ Thus $d_2 = 1.5037$

$d_3^2 = 0.2531$ Thus $d_3 = 0.5031$

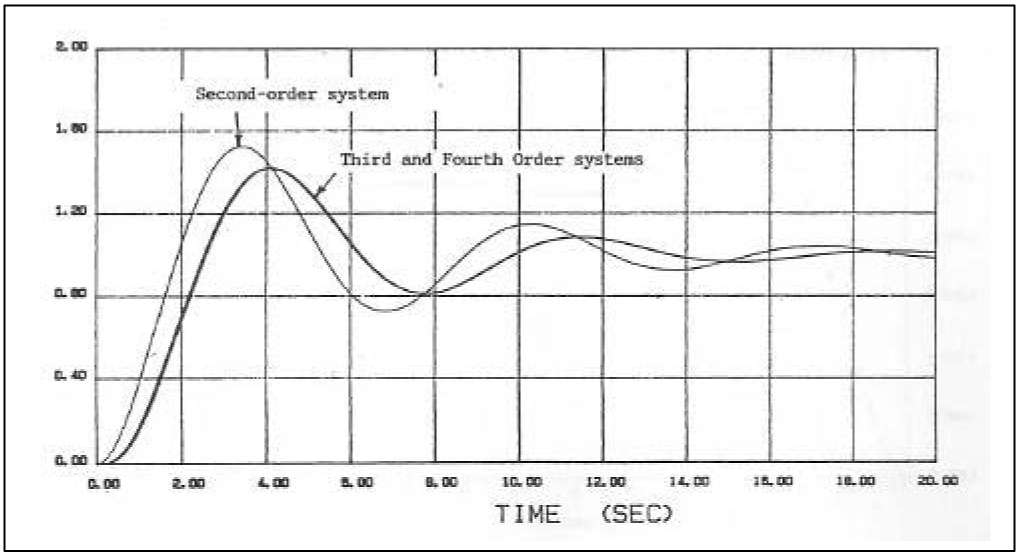
$$M_L(s) = \frac{1}{1 + 0.9525 s + 1.5037 s^2 + 0.5031 s^3} = \frac{1.9876}{s^3 + 2.9886 s^2 + 1.8932 s + 1.9876}$$

$$G_L(s) = \frac{1.9876}{s\left(s^2 + 2.9886s + 1.8932\right)} = \frac{1.9876}{s(s + 2.0772)(s + 0.9114)}$$

Roots of Characteristic Equations:

Fourth-order System	**Third-order System**
-2.5692 $\quad -19.994$ $\quad -0.2183 + j0.855$ $\quad -0.2183 - j0.855$	-2.552 $\quad -0.2183 \pm j0.8551$

Unit-step Responses



Chapter 8 ROOT LOCUS TECHNIQUE

8-1 (a) $P(s) = s^4 + 4s^3 + 4s^2 + 8s \qquad Q(s) = s + 1$

Finite zeros of $P(s)$: $0, \ -3.5098, \ -0.24512 \pm j1.4897$

Finite zeros of $Q(s)$: -1

Asymptotes: **$K > 0$:** $60^o, \ 180^o, \ 300^o$ **$K < 0$:** $0^o, \ 120^o, \ 240^o$

Intersect of Asymptotes:

$$s_1 = \frac{-3.5 - 0.24512 - 0.24512 - (-1)}{4 - 1} = -1$$

(b) $P(s) = s^3 + 5s^2 + s \qquad Q(s) = s + 1$

Finite zeros of $P(s)$: $0, \ -4.7912, \ -0.20871$

Finite zeros of $Q(s)$: -1

Asymptotes: **$K > 0$:** $90^o, \ 270^o$ **$K < 0$:** $0^o, \ 180^o$

Intersect of Asymptotes:

$$s_1 = \frac{-4.7913 - 0.2087 - (-1)}{3 - 1} = -2$$

(c) $P(s) = s^2 \qquad Q(s) = s^3 + 3s^2 + 2s + 8$

Finite zeros of $P(s)$: $0, \ 0$

Finite zeros of $Q(s)$: $-3.156, \ 0.083156 \pm j1.5874$

Asymptotes: **$K > 0$:** 180^o **$K < 0$:** 0^o

(d) $P(s) = s^3 + 2s^2 + 3s \qquad Q(s) = \left(s^2 - 1\right)\left(s + 3\right)$

Finite zeros of $P(s)$: $0, \ -1 \pm j1.414$

Finite zeros of $Q(s)$: $1, -1, -3$

Asymptotes: **There are no asymptotes, since the number of zeros of $P(s)$ and $Q(s)$ are equal.**

(e) $P(s) = s^5 + 2s^4 + 3s^3 \qquad Q(s) = s^2 + 3s + 5$

Finite zeros of $P(s)$: $0, \ 0, \ 0, \ -1 \pm j1.414$

Finite zeros of $Q(s)$: $-1.5 \pm j1.6583$

Asymptotes: **$K > 0$:** $60^o, \ 180^o, \ 300^o$ **$K < 0$:** $0^o, \ 120^o, \ 240^o$

Intersect of Asymptotes:

$$s_1 = \frac{-1 - 1 - (-1.5) - (-1.5)}{5 - 2} = \frac{1}{3}$$

(f) $P(s) = s^4 + 2s^2 + 10 \qquad Q(s) = s + 5$

Finite zeros of $P(s)$: $-1.0398 \pm j1.4426, \ 1.0398 \pm j1.4426$

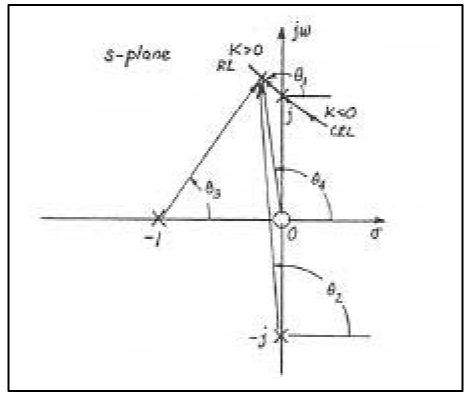
Finite zeros of $Q(s)$: -5

Asymptotes: **$K > 0$:** $60^o, \ 180^o, \ 300^o$ **$K < 0$:** $0^o, \ 120^o, \ 240^o$

Intersect of Asymptotes:

$$s_1 = \frac{-1.0398 \quad -1.0398 \quad +1.0398 \quad +1.0398 \quad -(-5)}{4-1} = \frac{-5}{3}$$

8-2 (a) Angles of departure and arrival.



$K > 0$: $\quad -q_1 - q_2 - q_3 + q_4 = -180^o$

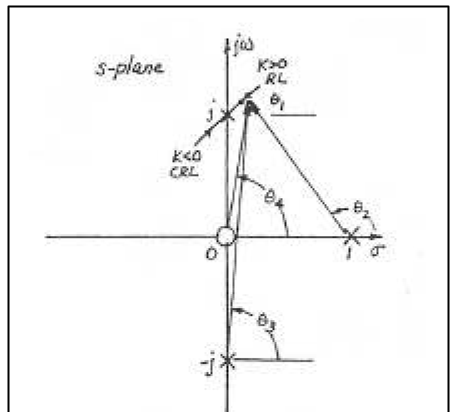
$-q_1 - 90^o - 45^o + 90^o = -180^o$

$q_1 = 135^o$

$K < 0$: $\quad -q_1 - 90^o - 45^o + 90^o = 0^o$

$q_1 = -45^o$

(b) Angles of departure and arrival.

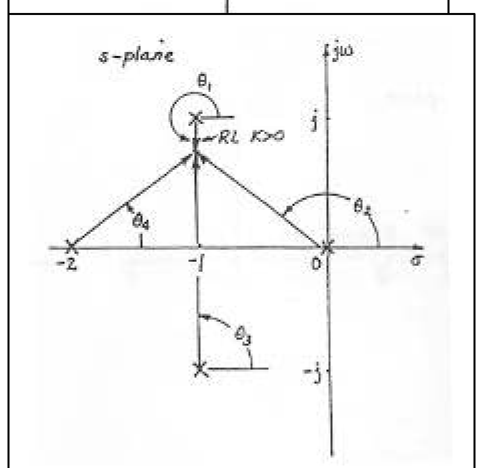


$K > 0$: $\quad -q_1 - q_2 - q_3 + q_4 = -180^o$

$K < 0$: $\quad -q_1 - 135^o - 90^o + 90^o = 0^o$

$q_1 = -135^o$

(c) Angle of departure:

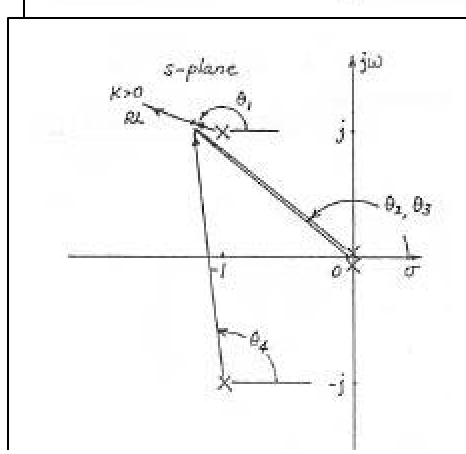


$K > 0$: $\quad -q_1 - q_2 - q_3 + q_4 = -180^o$

$-q_1 - 135^o - 90^o - 45^o = -180^o$

$q_1 = -90^o$

(d) Angle of departure



$K > 0$: $\quad -q_1 - q_2 - q_3 - q_4 = -180^o$

$-q_1 - 135^o - 135^o - 90^o = -180^o$

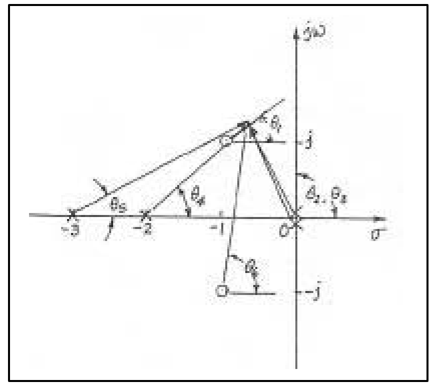
$q_1 = -180^o$

(e) Angle of arrival

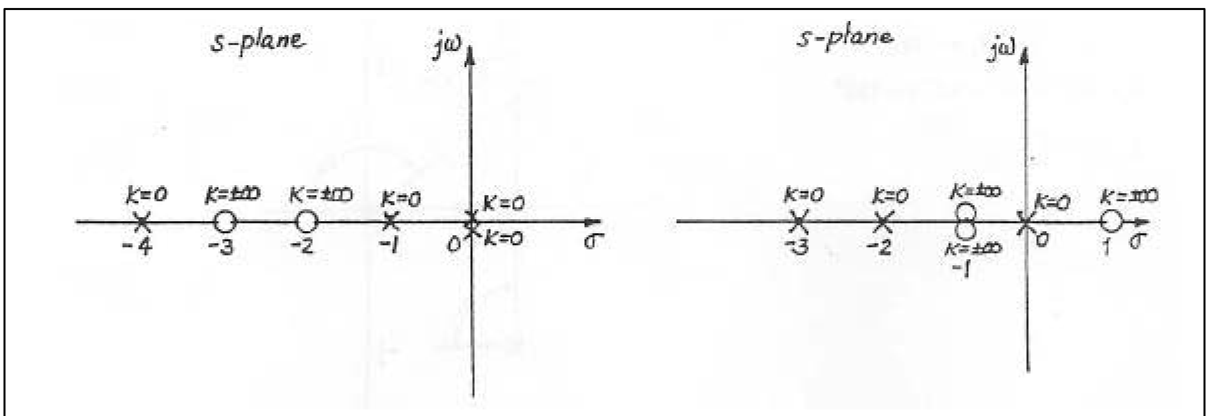
$$K < 0: \quad \theta_1 + \theta_6 - \theta_2 - \theta_3 - \theta_4 - \theta_5 = -360^o$$

$$\theta_1 + 90^o - 135^o - 135^o - 45^o - 26.565^o = -360^o$$

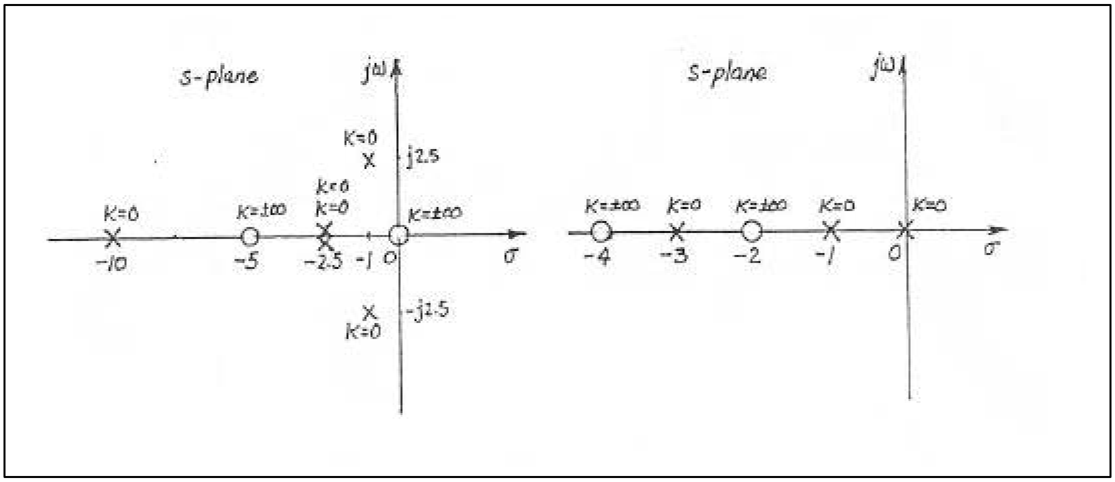
$$\theta_1 = -108.435^o$$



8-3 (a) **(b)**



(c) **(d)**



8-4 (a) Breakaway-point Equation: $2s^5 + 20s^4 + 74s^3 + 110s^2 + 48s = 0$

Breakaway Points: $-0.7275, \quad -2.3887$

(b) Breakaway-point Equation: $3s^6 + 22s^5 + 65s^4 + 100s^3 + 86s^2 + 44s + 12 = 0$

Breakaway Points: $-1, \quad -2.5$

(c) Breakaway-point Equation: $3s^6 + 54s^5 + 347.5s^4 + 925s^3 + 867.2s^2 - 781.25s - 1953 = 0$

Breakaway Points: $-2.5, \quad 1.09$

(d) Breakaway-point Equation: $-s^6 - 8s^5 - 19s^4 + 8s^3 + 94s^2 + 120s + 48 = 0$

Breakaway Points: $-0.6428, \quad 2.1208$

8-5 (a)

$$G(s)H(s) = \frac{K(s+8)}{s(s+5)(s+6)}$$

Asymptotes: $K > 0$**:** 90^o and 270^o $\qquad$ $K < 0$**:** 0^o and 180^o

Intersect of Asymptotes:

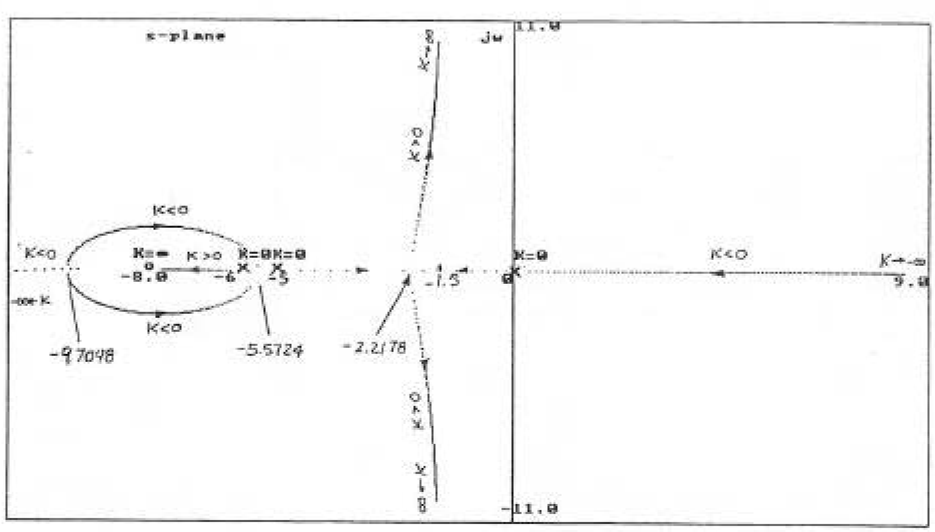
$$s_1 = \frac{0 - 5 - 6 - (-8)}{3 - 1} = -1.5$$

Breakaway-point Equation:

$$2s^3 + 35s^2 + 176s + 240 = 0$$

Breakaway Points: $-2.2178, \quad -5.5724, \quad -9.7098$

Root Locus Diagram:



8-5 (b)

$$G(s)H(s) = \frac{K}{s(s+1)(s+3)(s+4)}$$

Asymptotes: $K > 0$: 45^o, 135^o, 225^o, 315^o $\quad$ $K < 0$: 0^o, 90^o, 180^o, 270^o

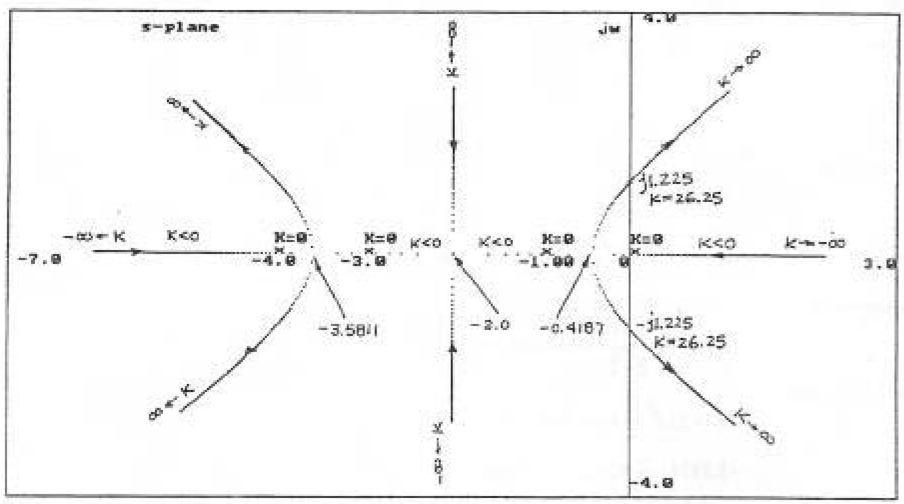
Intersect of Asymptotes:

$$s_1 = \frac{0-1-3-4}{4} = -2$$

Breakaway-point Equation: $4s^3 + 24s^2 + 38s + 12 = 0$

Breakaway Points: -0.4189, -2, -3.5811

Root Locus Diagram:



8-5 (c)

$$G(s)H(s) = \frac{K(s+4)}{s^2(s+2)^2}$$

Asymptotes: $K > 0$: 60^o, 180^o, 300^o $\quad$ $K < 0$: 0^o, 120^o, 240^o

Intersect of Asymptotes:

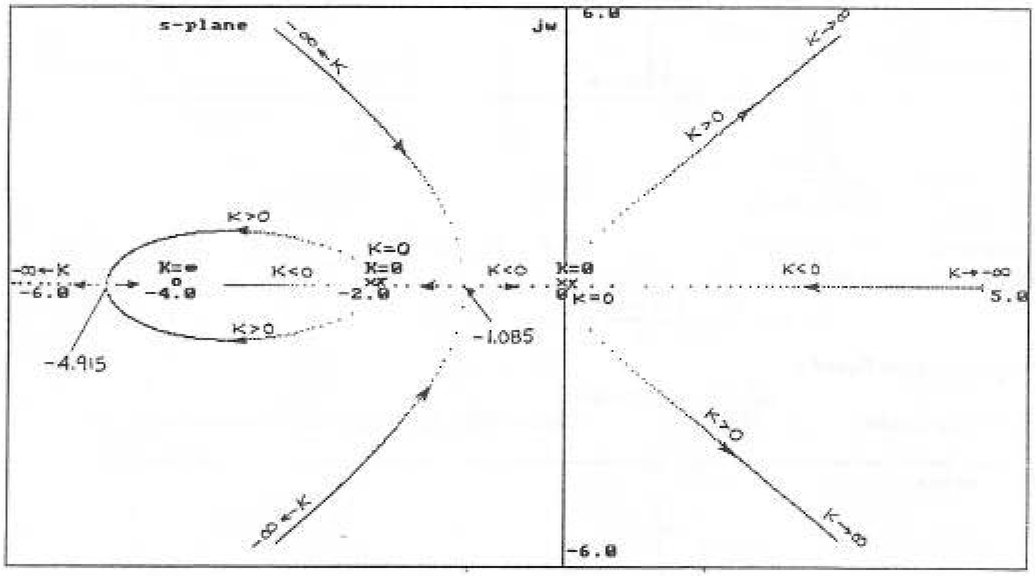
$$s_1 = \frac{0+0-2-2-(-4)}{4-1} = 0$$

Breakaway-point Equation:

$$3s^4 + 24s^3 + 52s^2 + 32s = 0$$

Breakaway Points: 0, -1.085, -2, -4.915

Root Locus Diagram:



8-5 (d)

$$G(s)H(s) = \frac{K(s+2)}{s(s^2+2s+2)}$$

Asymptotes: $\quad$ ***K* > 0:** $\quad 90^o, \quad 270^o$ $\qquad$ ***K* < 0:** $\quad 0^o, \quad 180^o$

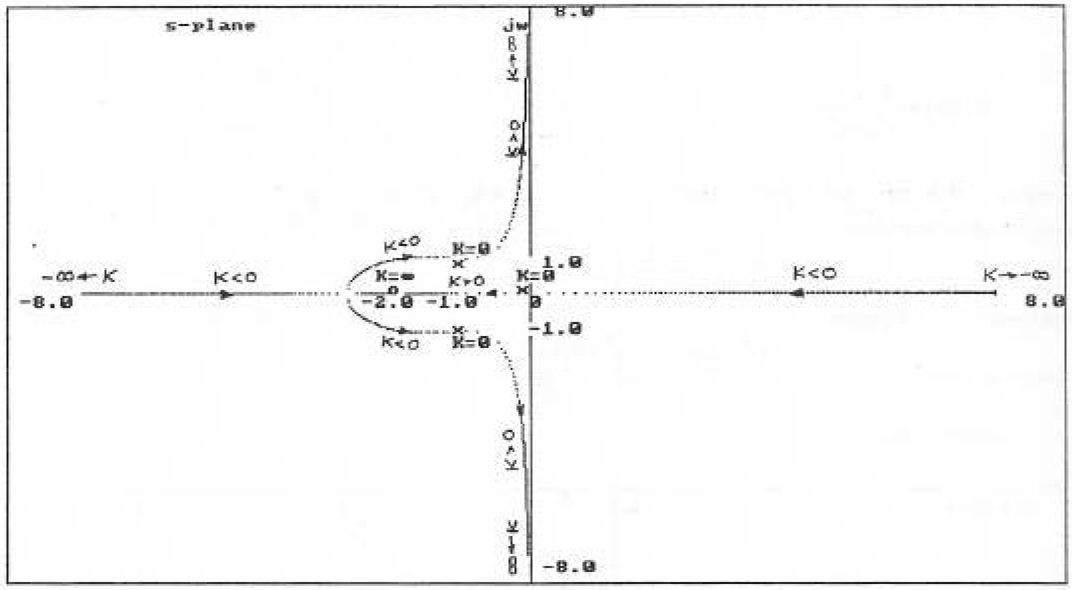
Intersect of Asymptotes:

$$s_1 = \frac{0 - 1 - j - 1 - j - (-2)}{3-1} = 0$$

Breakaway-point Equation: $\quad 2s^3 + 8s^2 + 8s + 4 = 0$

Breakaway Points: $\quad -2.8393 \quad$ The other two solutions are not breakaway points.

Root Locus Diagram



8-5 (e)

$$G(s)H(s) = \frac{K(s+5)}{s\left(s^2+2s+2\right)}$$

Asymptotes: $K > 0$: 90^o, 270^o $K < 0$: 0^o, 180^o

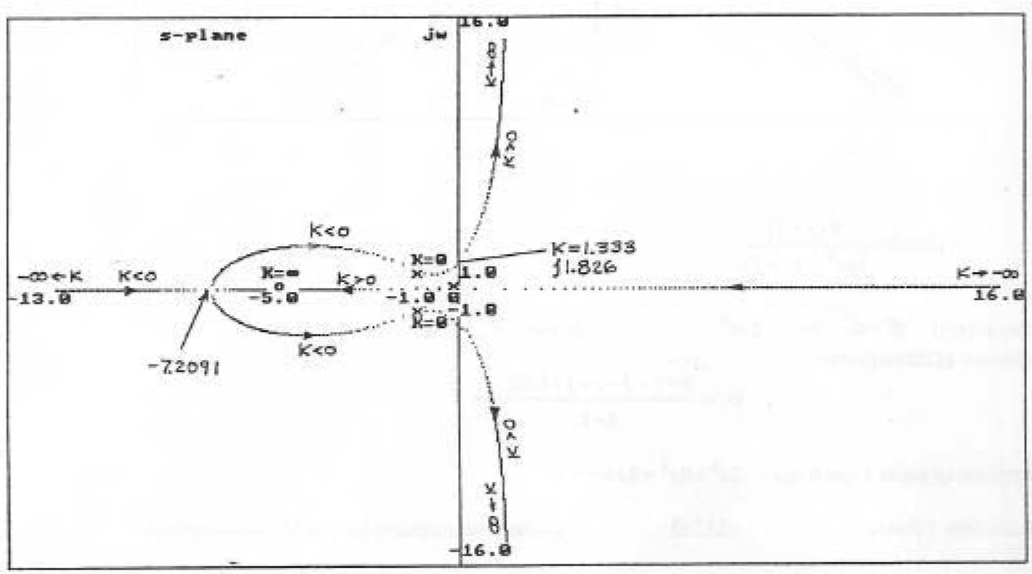
Intersect of Asymptotes:

$$s_1 = \frac{0-1-j-1-j-(-5)}{3-1} = 1.5$$

Breakaway-point Equation:

$$2s^3 + 17s^2 + 20s + 10 = 0$$

Breakaway Points: -7.2091 The other two solutions are not breakaway points.



8-5 (f)

$$G(s)H(s) = \frac{K}{s(s+4)\left(s^2+2s+2\right)}$$

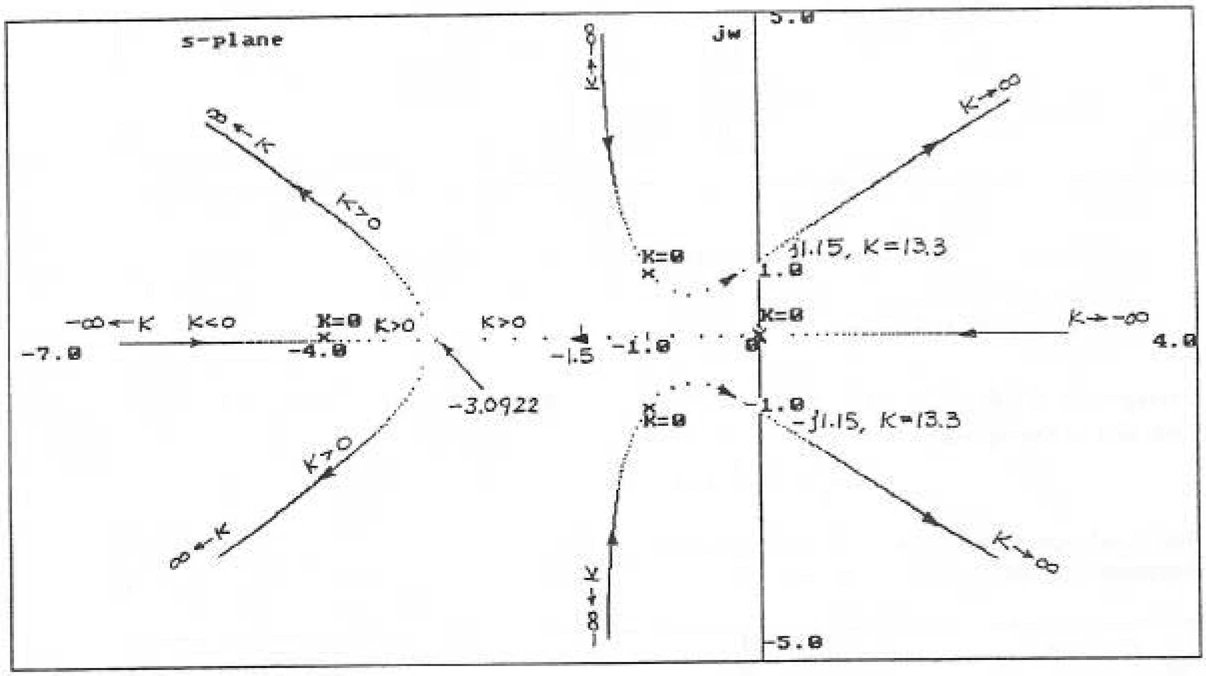
Asymptotes: $K > 0$: 45^o, 135^o, 225^o, 315^o $K < 0$: 0^o, 90^o, 180^o, 270^o

Intersect of Asymptotes:

$$\sigma_1 = \frac{0 - 1 - j - 1 + j - 4}{4} = -1.5$$

Breakaway-point Equation: $4s^3 + 18s^2 + 20s + 8 = 0$

Breakaway Point: -3.0922 The other solutons are not breakaway points.



8-5 (g)

$$G(s)H(s) = \frac{K(s+4)^2}{s^2(s+8)^2}$$

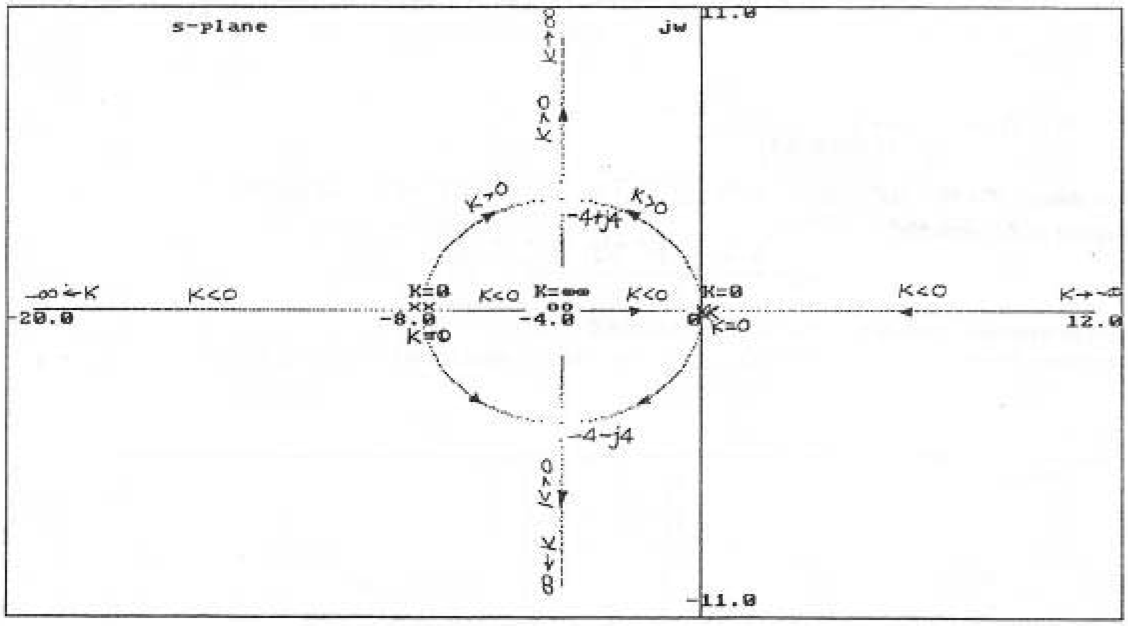
Asymptotes: $K > 0$: 90^o, 270^o $K < 0$: 0^o, 180^o

Intesect of Asymptotes:

$$\sigma_1 = \frac{0 + 0 - 8 - 8 - (-4) - (-4)}{4 - 2}$$

Breakaway-point Equation: $s^5 + 20s^4 + 160s^3 + 640s^2 + 1040s = 0$

Breakaway Points: 0, -4, -8, $-4 - j4$, $-4 + j4$



8-5 (h)

$$G(s)H(s) = \frac{K}{s^2 (s+8)^2}$$

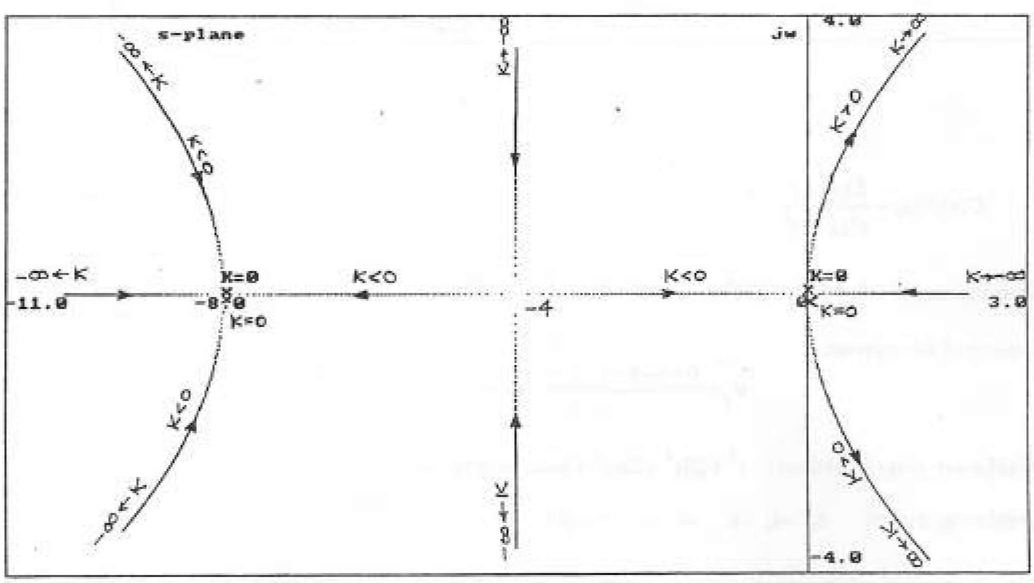
Asymptotes: $K > 0$**:** 45^o, 135^o, 225^o, 315^o $K < 0$**:** 0^o, 90^o, 180^o, 270^o

Intersect of Asymptotes:

$$s_1 = \frac{-8-8}{4} = -4$$

Breakaway-point Equation: $s^3 + 12s^2 + 32s = 0$

Breakaway Point: $0, \ -4, \ -8$



8-5 (i)

$$G(s)H(s)=\frac{K\left(s^2+8s+20\right)}{s^2(s+8)^2}$$

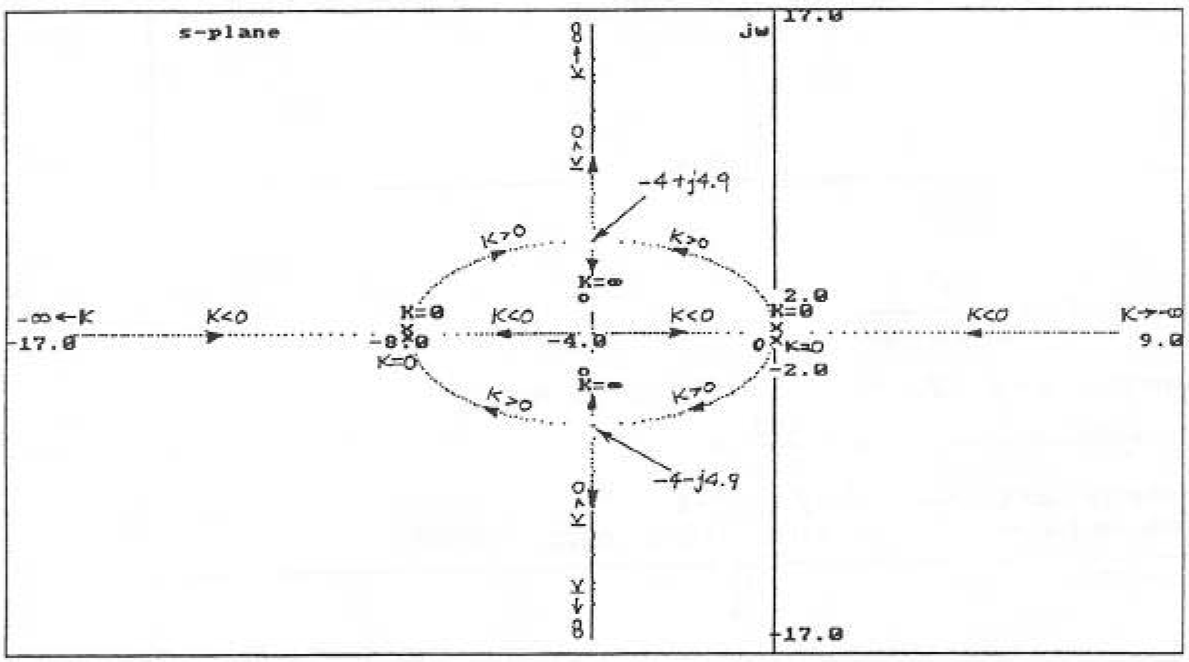
Asymptotes: **$K > 0$:** 90^o, 270^o **$K < 0$:** 0^o, 180^o

Intersect of Asymptotes:

$$s_1=\frac{-8-8-(-4)-(-4)}{4-2}=-4$$

Breakaway-point Equation: $s^5+20s^4+128s^3+736s^2+1280s=0$

Breakaway Points: -4, -8, $-4+j4.9$, $-4-j4.9$



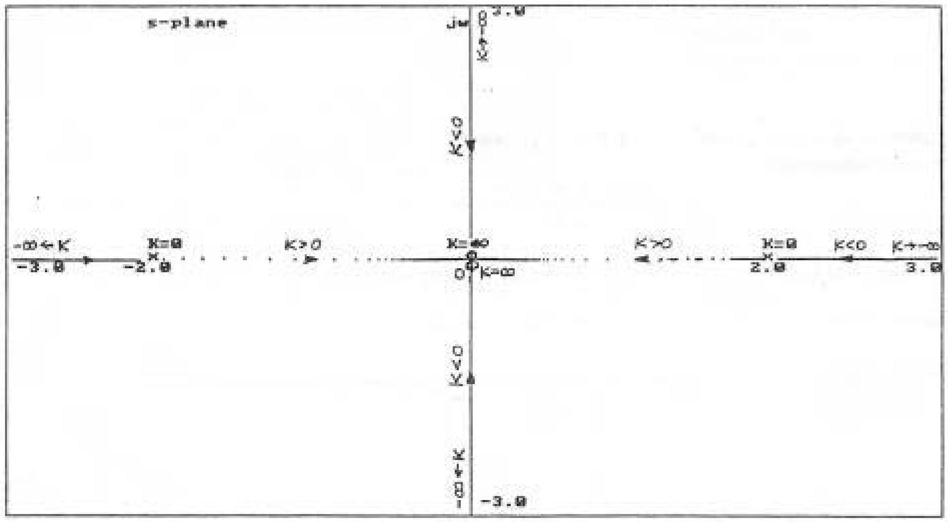
(j)

$$G(s)H(s)=\frac{Ks^2}{\left(s^2-4\right)}$$

Since the number of finite poles and zeros of $G(s)H(s)$ are the same, there are no asymptotes.

Breakaway-point Equation: $8s=0$

Breakaway Points: $s = 0$



8-5 (k)

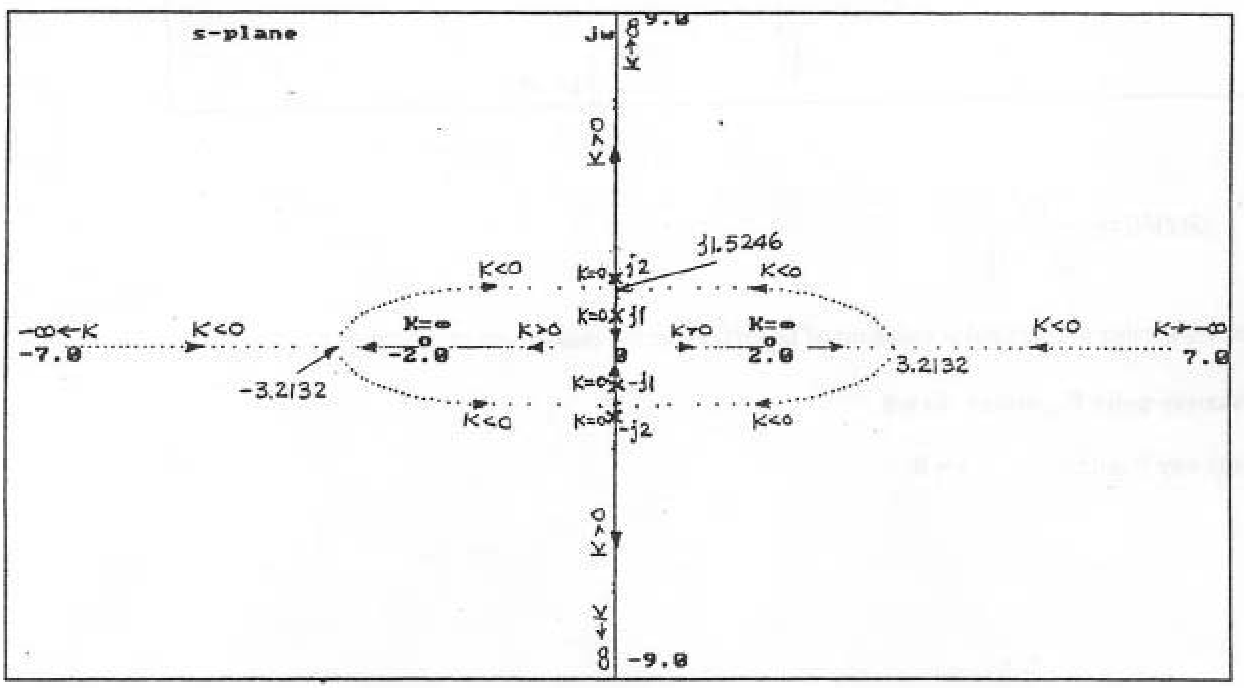
$$G(s)H(s) = \frac{K\left(s^2 - 4\right)}{\left(s^2 + 1\right)\left(s^2 + 4\right)}$$

Asymptotes: **$K > 0$:** 90^o, 270^o **$K < 0$:** 0^o, 180^o

Intersect of Asymptotes: $s_1 = \dfrac{-2+2}{4-2} = 0$

Breakaway-point Equation: $s^6 - 8s^4 - 24s^2 = 0$

Breakaway Points: 0, 3.2132, -3.2132, $j1.5246$, $-j1.5246$



8-5 (l)

$$G(s)H(s) = \frac{K(s^2-1)}{(s^2+1)(s^2+4)}$$

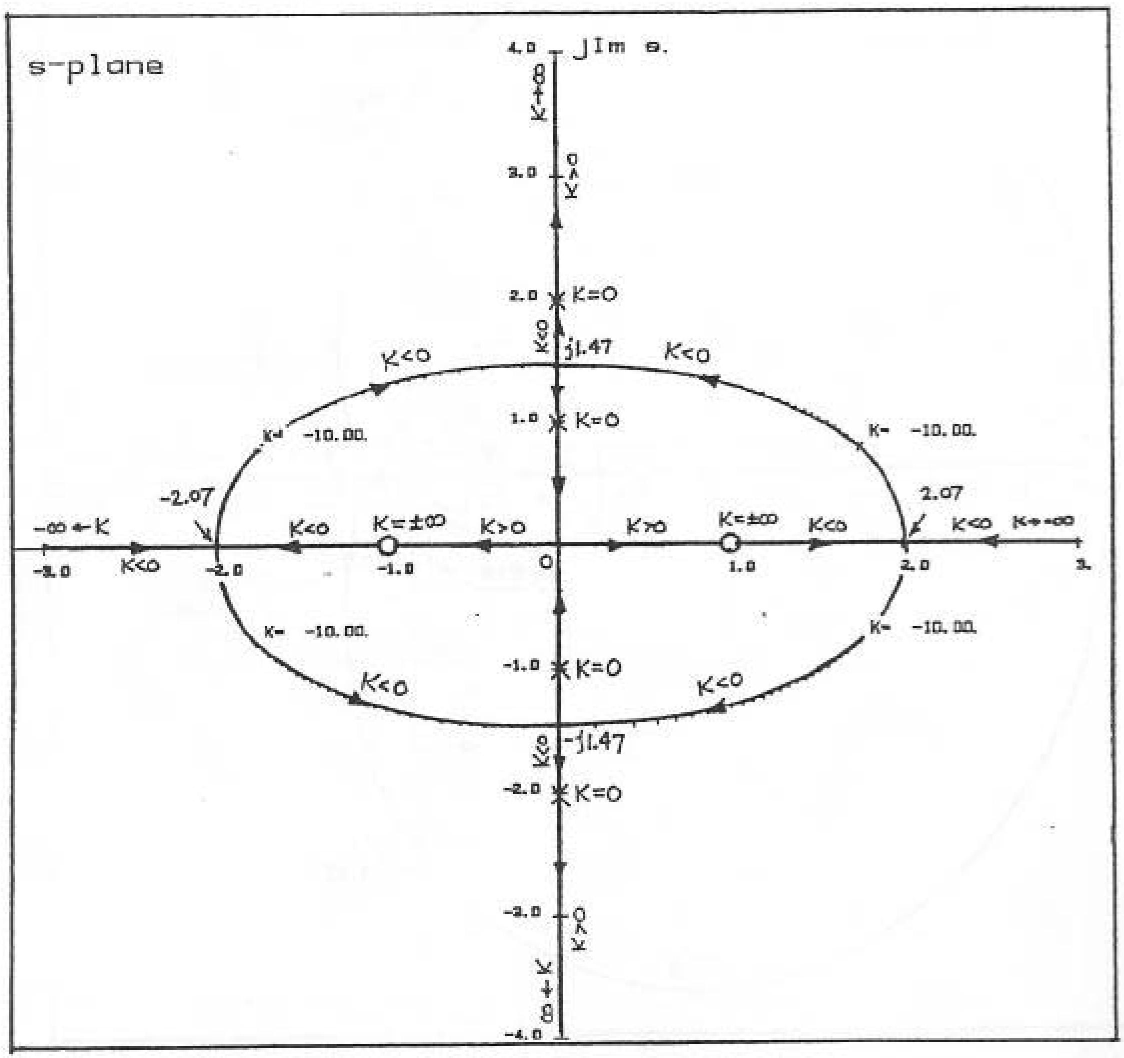
Asymptotes: ***K* > 0:** 90^o, 270^o ***K* < 0:** 0^o, 180^o

Intersect of Asymptotes:

$$s_1 = \frac{-1+1}{4-2} = 0$$

Breakaway-point Equation: $s^5 - 2s^3 - 9s = 0$

Breakaway Points: -2.07, 2.07, $-j1.47$, $j1.47$



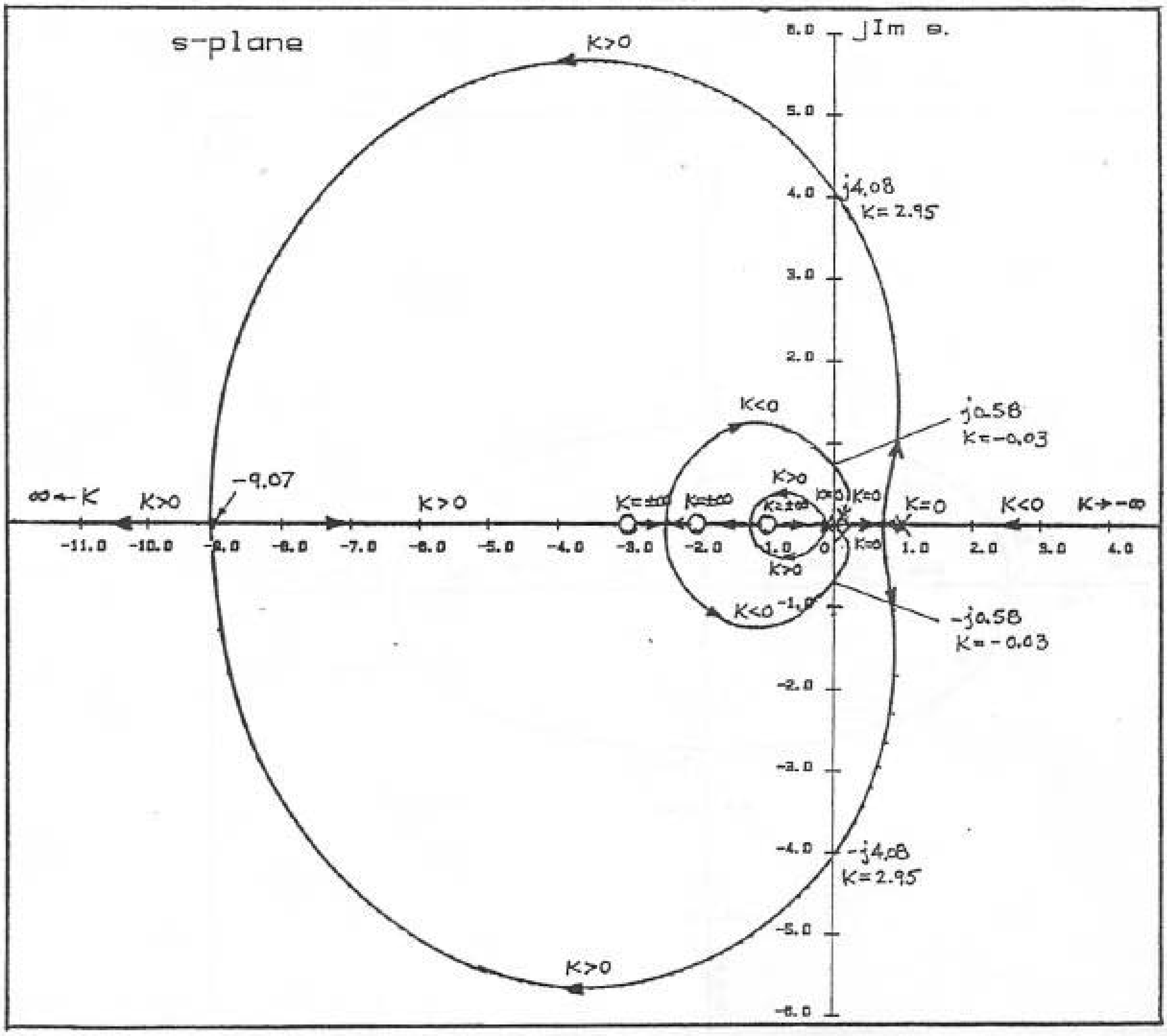
(m)

$$G(s)H(s) = \frac{K(s+1)(s+2)(s+3)}{s^3(s-1)}$$

Asymptotes: $K > 0$**:** 180^o $K < 0$**:** 0^o

Breakaway-point Equation: $s^6 + 12s^5 + 27s^4 + 2s^3 - 18s^2 = 0$

Breakaway Points: $-1.21, \quad -2.4, \quad -9.07, \quad 0.683, \quad 0, \quad 0$



(n)

$$G(s)H(s) = \frac{K(s+5)(s+40)}{s^3(s+250)(s+1000)}$$

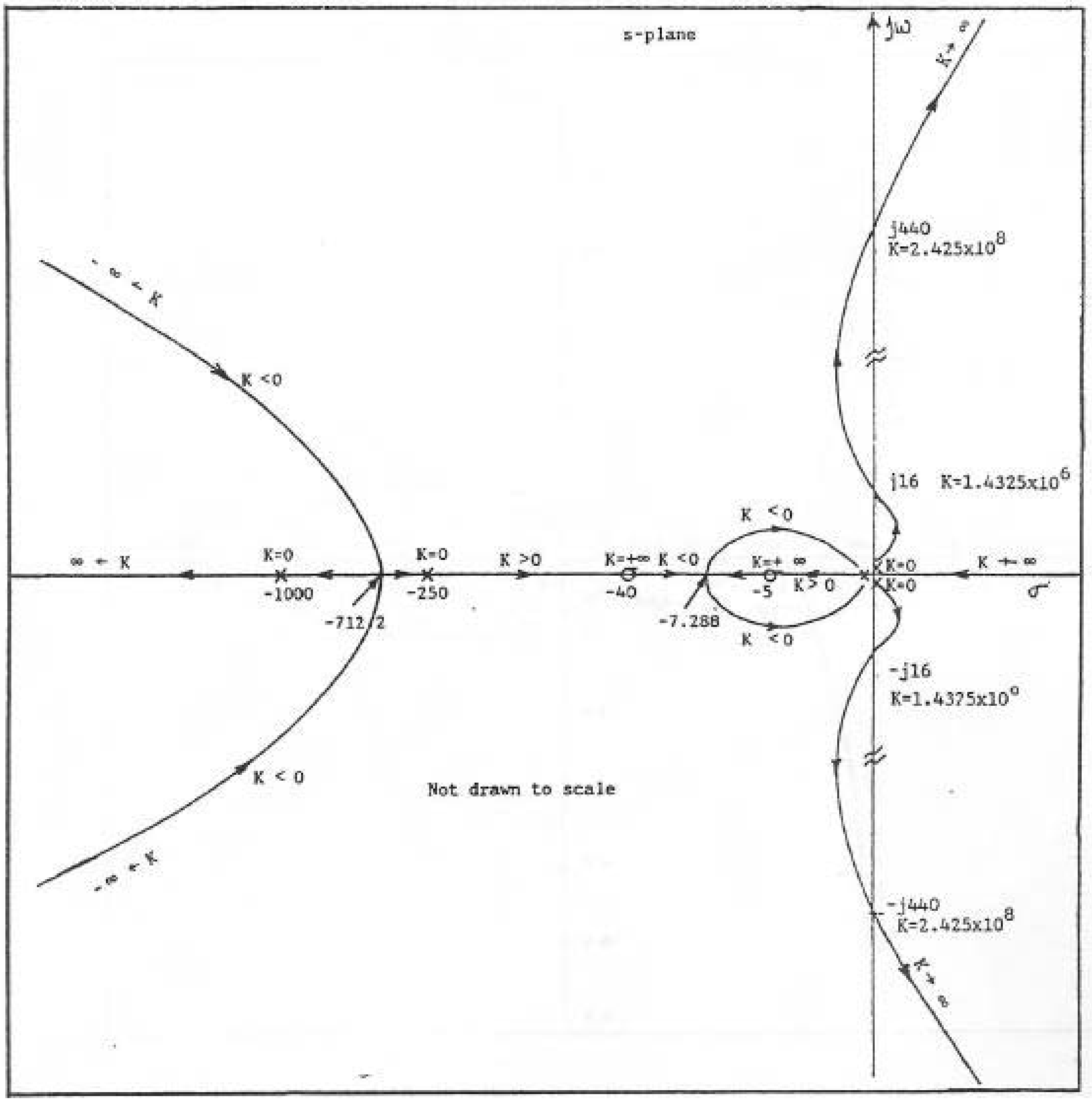
Asymptotes: $K > 0$**:** $60^o, \quad 180^o, \quad 300^o$ $K < 0$**:** $0^o, \quad 120^o, \quad 240^o$

Intersect of asymptotes:

$$s_1 = \frac{0+0+0-250-1000-(-5)-(-40)}{5-2} = -401.67$$

Breakaway-point Equation: $3750s^6 + 335000s^5 + 5.247\times10^8 s^4 + 2.9375\times10^{10}s^3 + 1.875\times10^{11}s^2 = 0$

Breakaway Points: $-7.288,\quad -712.2,\quad 0,\quad 0$



8-5 (o)

$$G(s)H(s) = \frac{K(s-1)}{s(s+1)(s+2)}$$

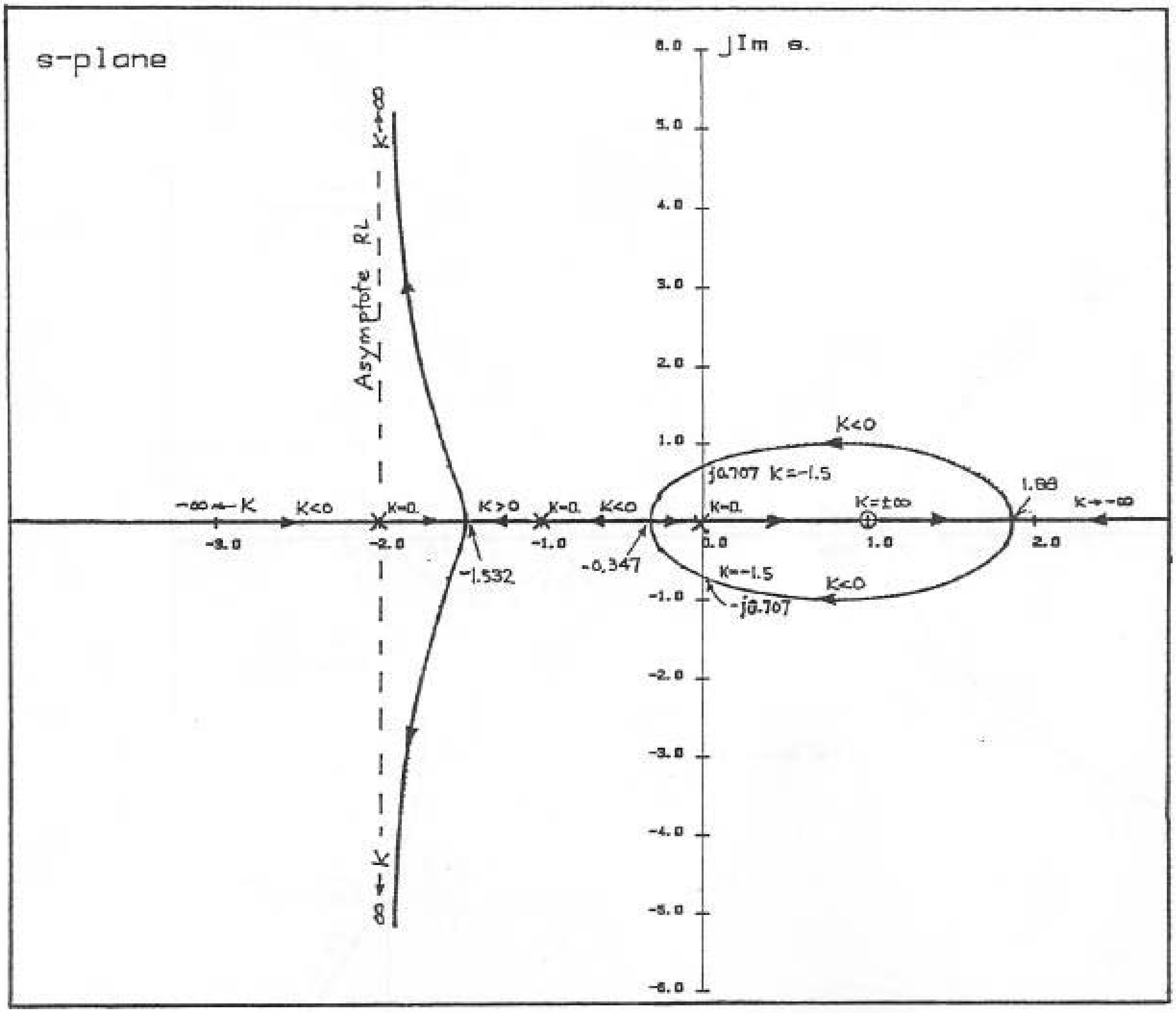
Asymptotes: $K > 0$: $90^o,\quad 270^o$ $\qquad$ $K < 0$: $0^o,\quad 180^o$

Intersect of Asymptotes:

$$s_1 = \frac{-1-2-1}{3-1} = -2$$

Breakaway-point Equation: $s^3 - 3s - 1 = 0$

Breakaway Points; $-0.3473, \quad -1.532, \quad 1.879$



8-6 (a) $Q(s) = s + 5 \qquad P(s) = s\left(s^2 + 3s + 2\right) = s(s+1)(s+2)$

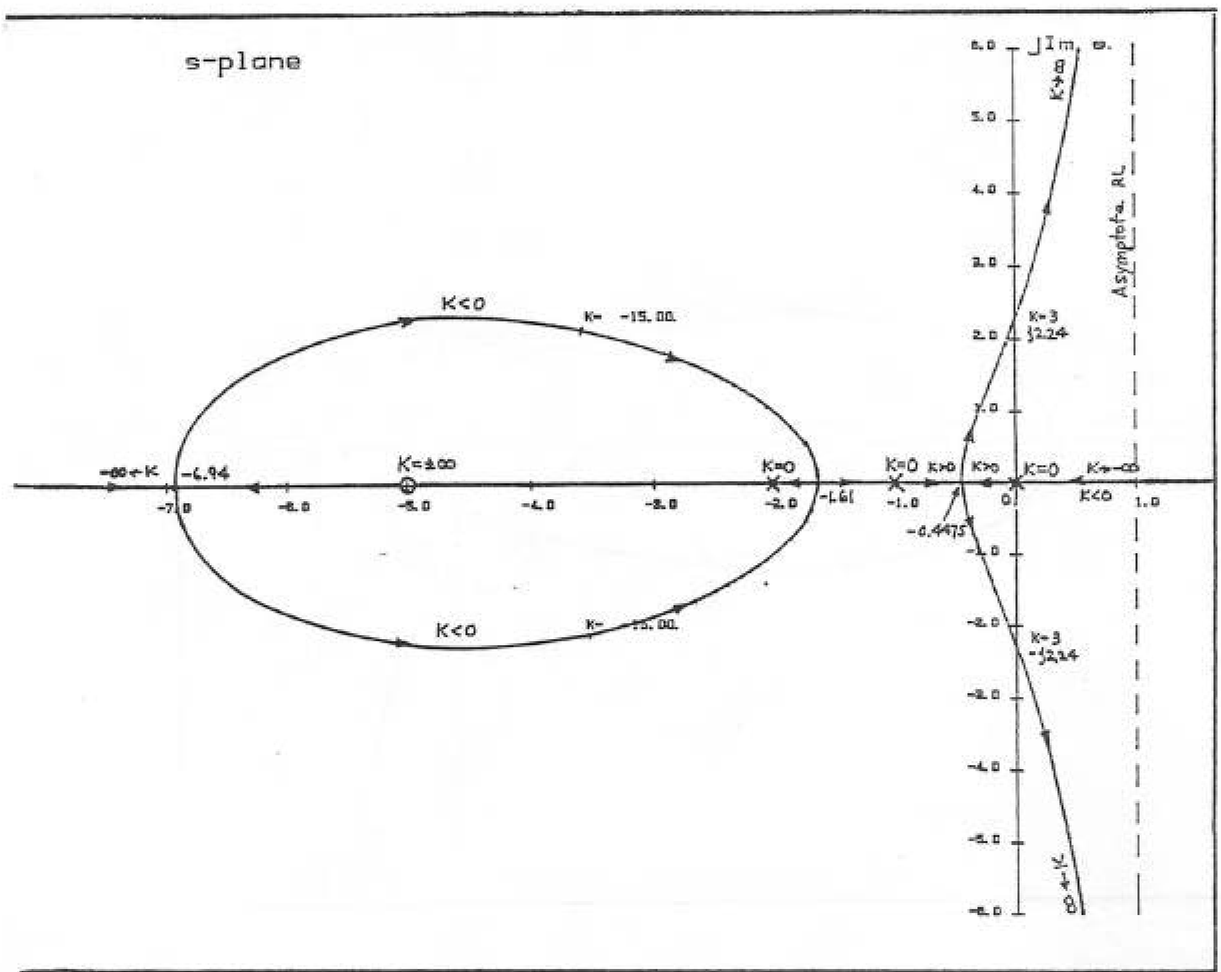
Asymptotes: ***K* > 0:** $90^o, \quad 270^o$ ***K* < 0:** $0^o, \quad 180^o$

Intersect of Asymptotes:

$$s_1 = \frac{-1-2-(-5)}{3-1} = 1$$

Breakaway-point Equation: $s^3 + 9s^2 + 15s + 5 = 0$

Breakaway Points: $-0.4475, \quad -1.609, \quad -6.9434$



8-6 (b) $Q(s) = s + 3 \qquad P(s) = s\left(s^2 + s + 2\right)$

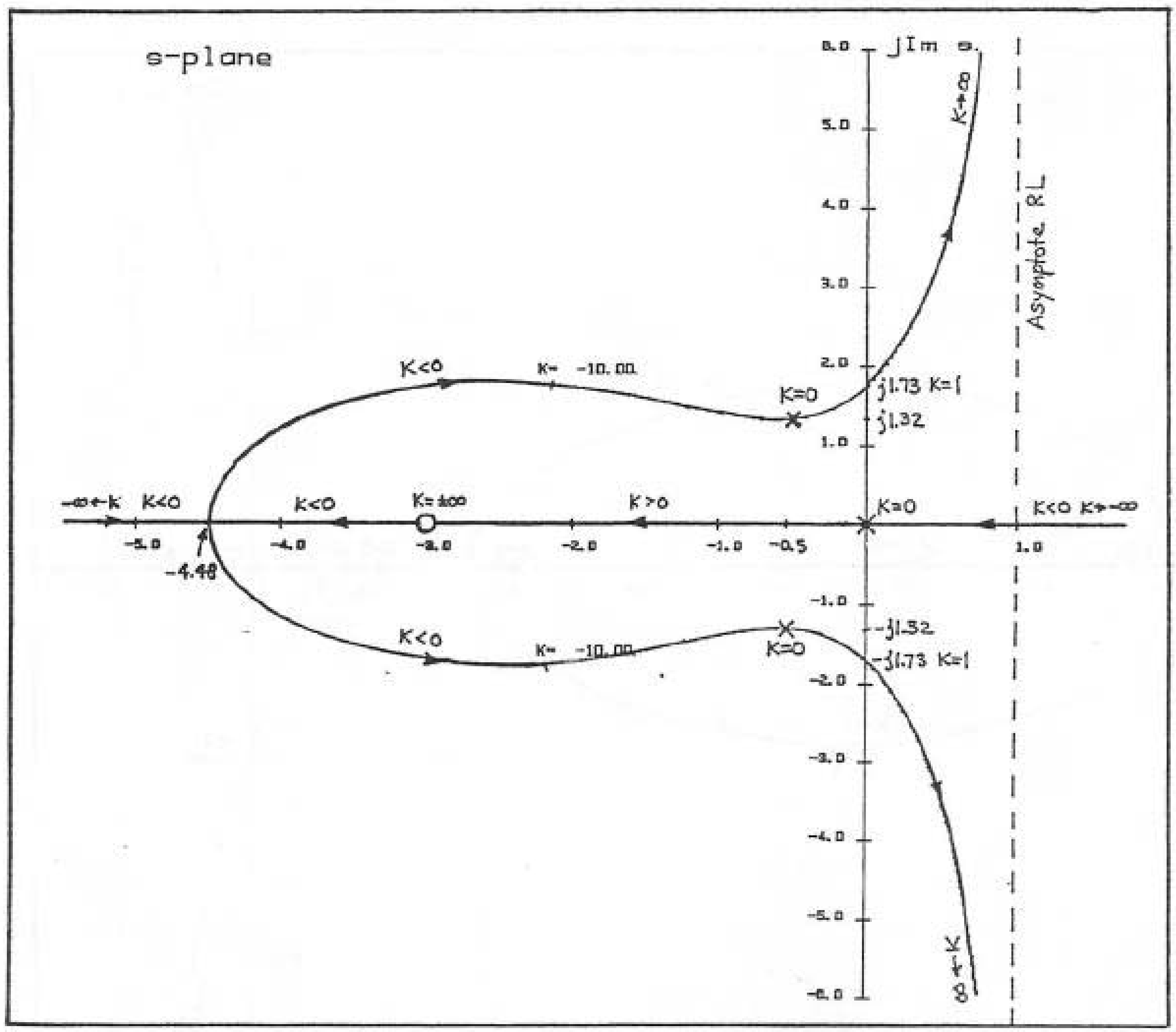
Asymptotes: $\quad$ ***K* > 0:** $90^o, \quad 270^o$ $\qquad$ ***K* < 0:** $0^o, \quad 180^o$

Intersect of Asymptotes:

$$s_1 = \frac{-1-(-3)}{3-1} = 1$$

Breakaway-point Equation: $s^3 + 5s^2 + 3s + 3 = 0$

Breakaway Points: -4.4798 The other solutions are not breakaway points.

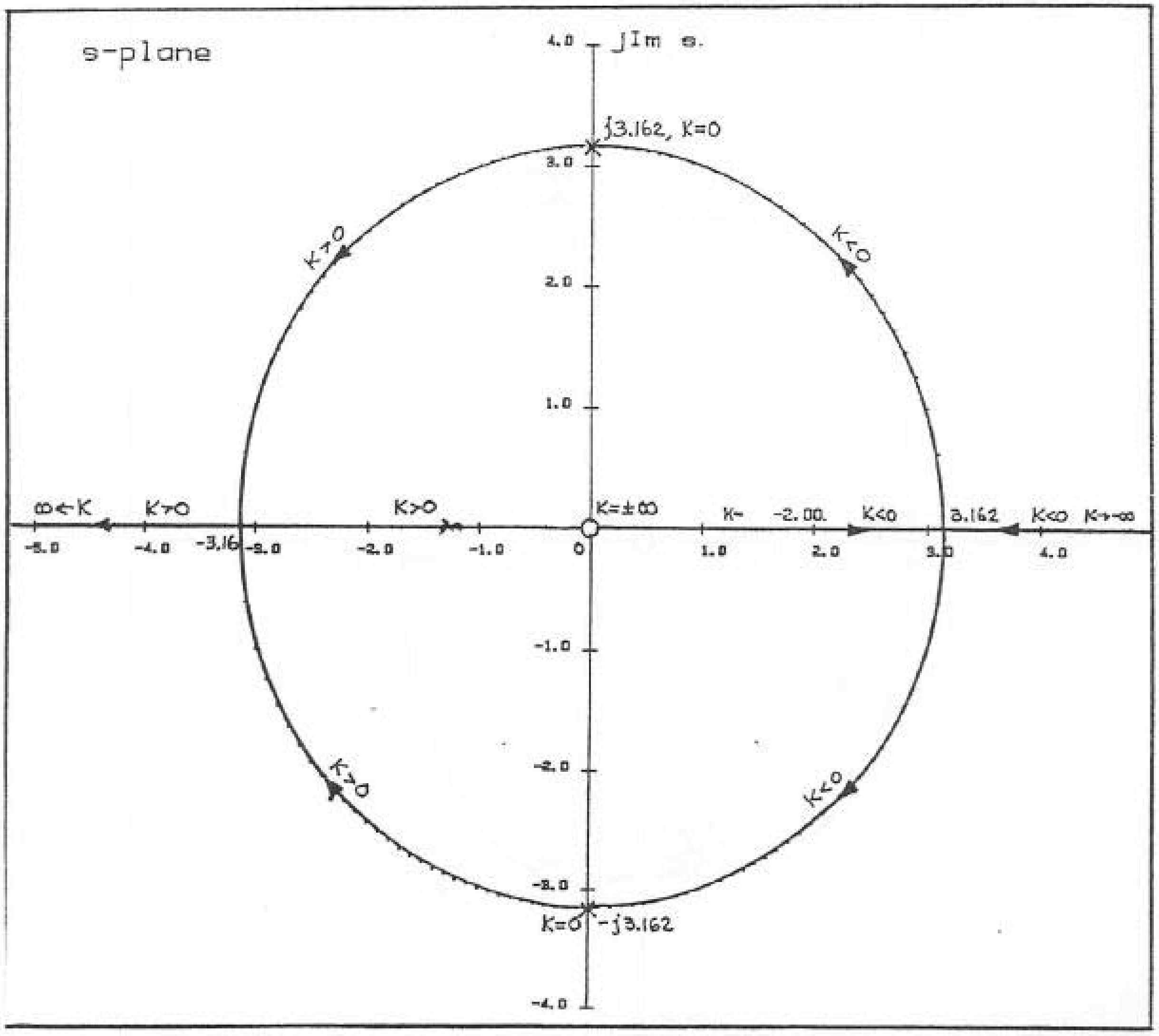


8-6 (c) $Q(s) = 5s \qquad P(s) = s^2 + 10$

Asymptotes: $\boldsymbol{K > 0}$**:** 180^o $\boldsymbol{K < 0}$**:** 0^o

Breakaway-point Equation: $5s^2 - 50 = 0$

Breakaway Points: -3.162, 3.162

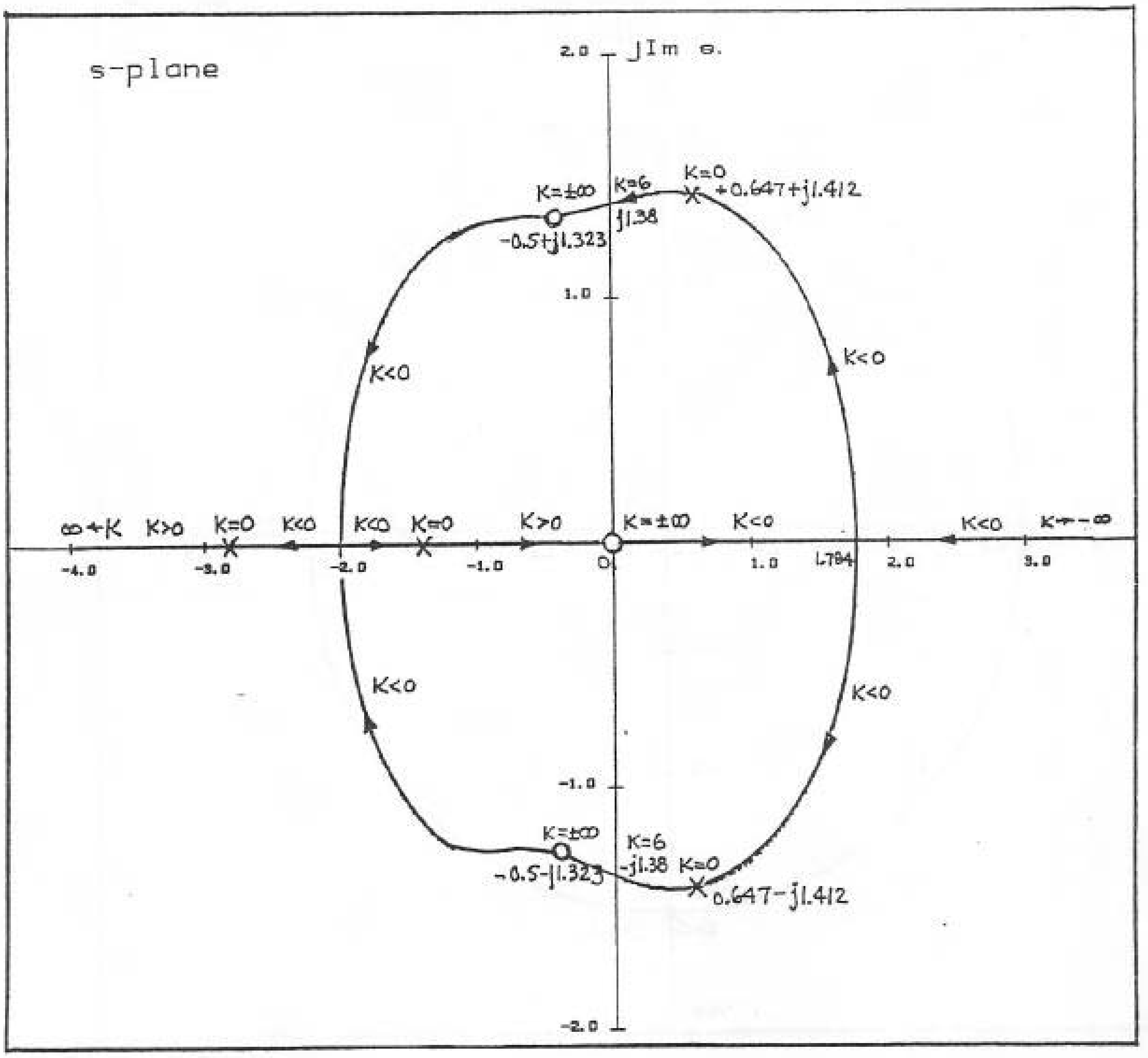


8-6 (d) $Q(s) = s\left(s^2 + s + 2\right)$ $\qquad P(s) = s^4 + 3s^3 + s^2 + 5s + 10$

Asymptotes: $\boldsymbol{K > 0}$**:** 180^o $\qquad$ $\boldsymbol{K < 0}$**:** 0^o

Breakaway-point Equation: $s^6 + 2s^5 + 8s^4 + 2s^3 - 33s^2 - 20s - 20 = 0$

Breakaway Points: -2, 1.784. The other solutions are not breakaway points.

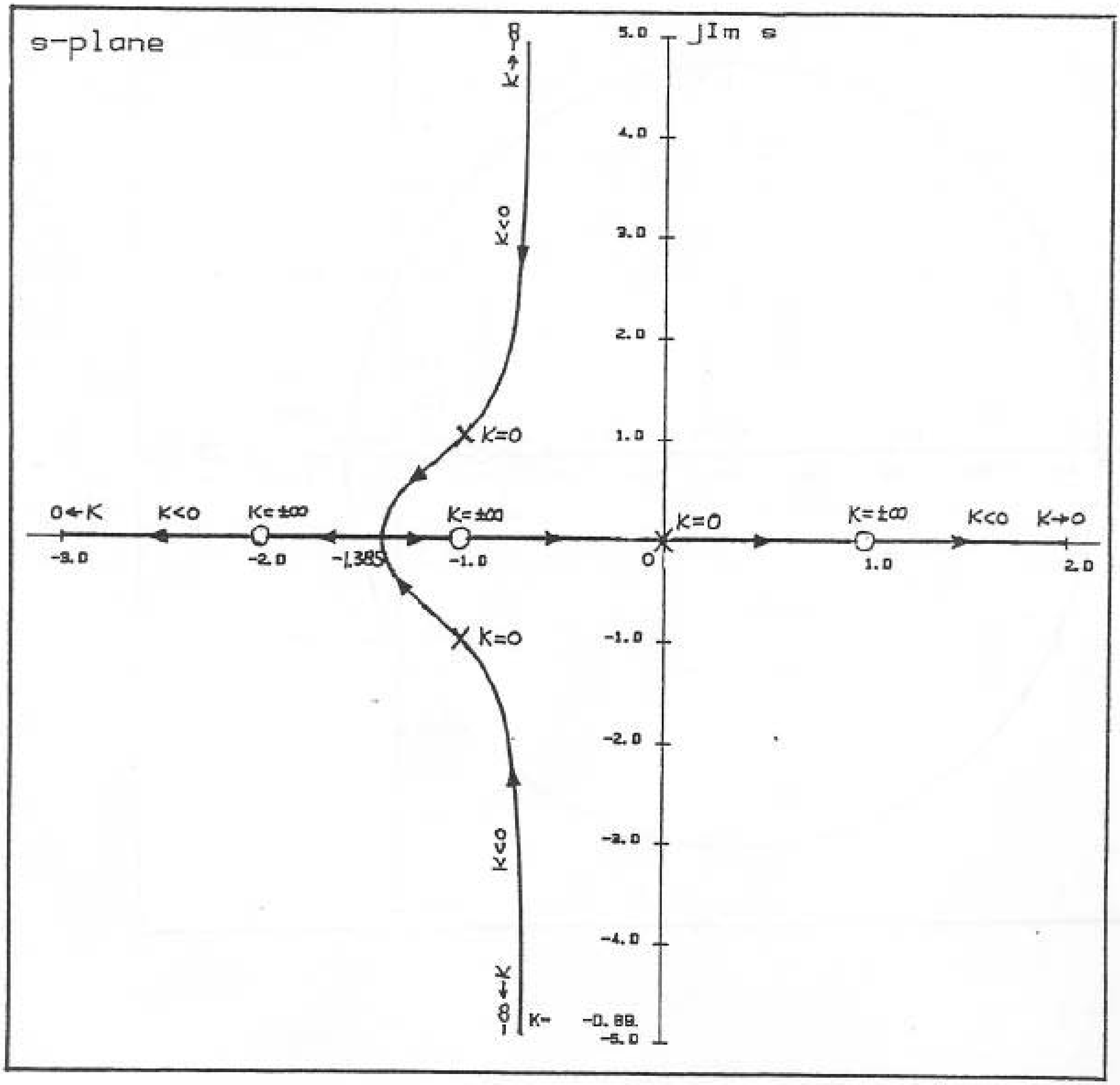


8-6 (e) $Q(s) = \left(s^2 - 1\right)\left(s + 2\right) \qquad P(s) = s\left(s^2 + 2s + 2\right)$

Since $Q(s)$ and $P(s)$ are of the same order, there are no asymptotes.

Breakaway-point Equation: $6s^3 + 12s^2 + 8s + 4 = 0$

Breakaway Points: -1.3848

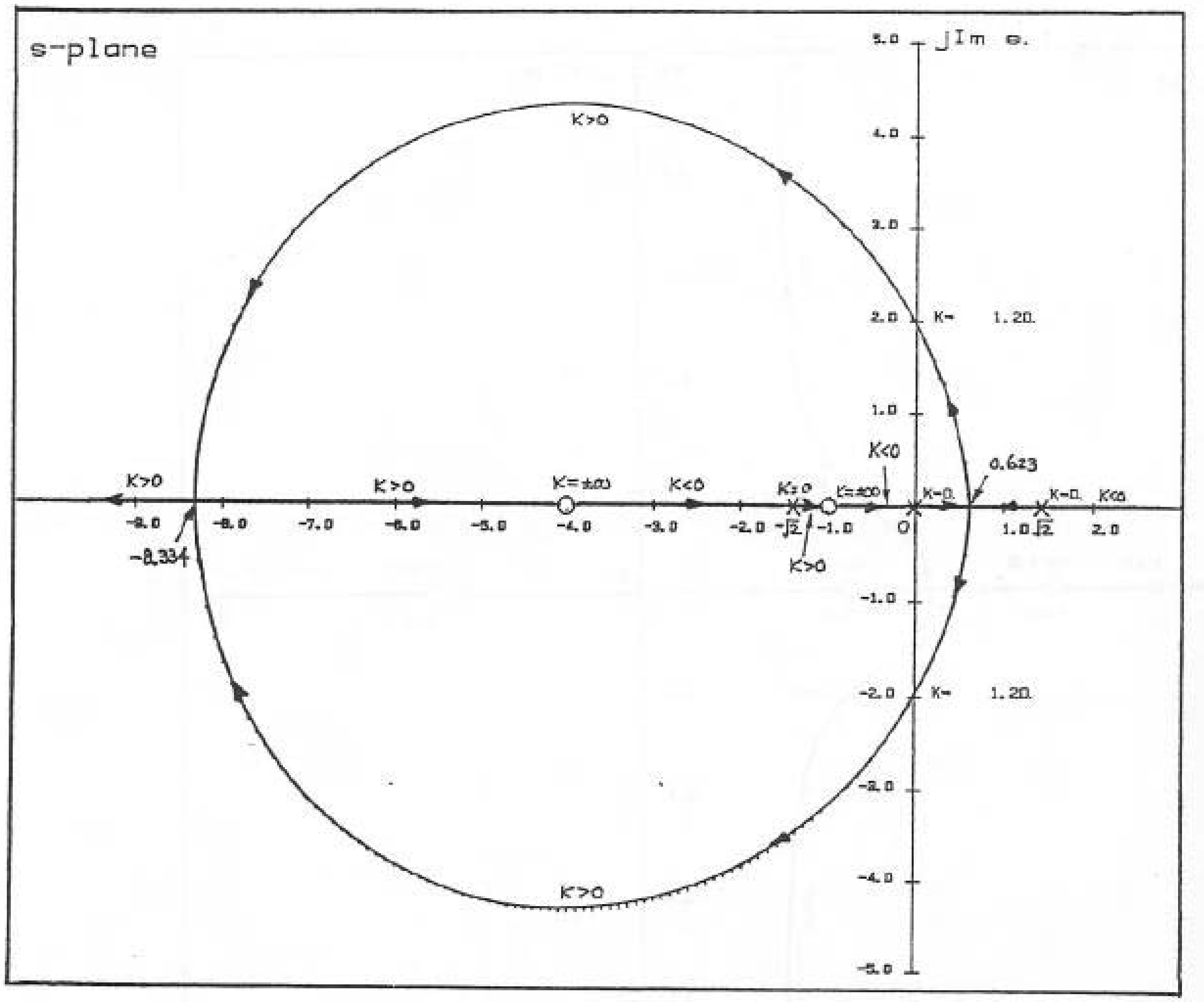


8-6 (f) $Q(s) = (s+1)(s+4) \qquad P(s) = s\left(s^2 - 2\right)$

Asymptotes: $\quad$ ***K* > 0:** 180^o $\qquad$ ***K* < 0:** 0^o

Breakaway-point equations: $\quad s^4 + 10s^3 + 14s^2 - 8 = 0$

Breakaway Points: $\quad -8.334, \quad 0.623$



8-6 (g) $Q(s) = s^2 + 4s + 5 \qquad P(s) = s^2\left(s^2 + 8s + 16\right)$

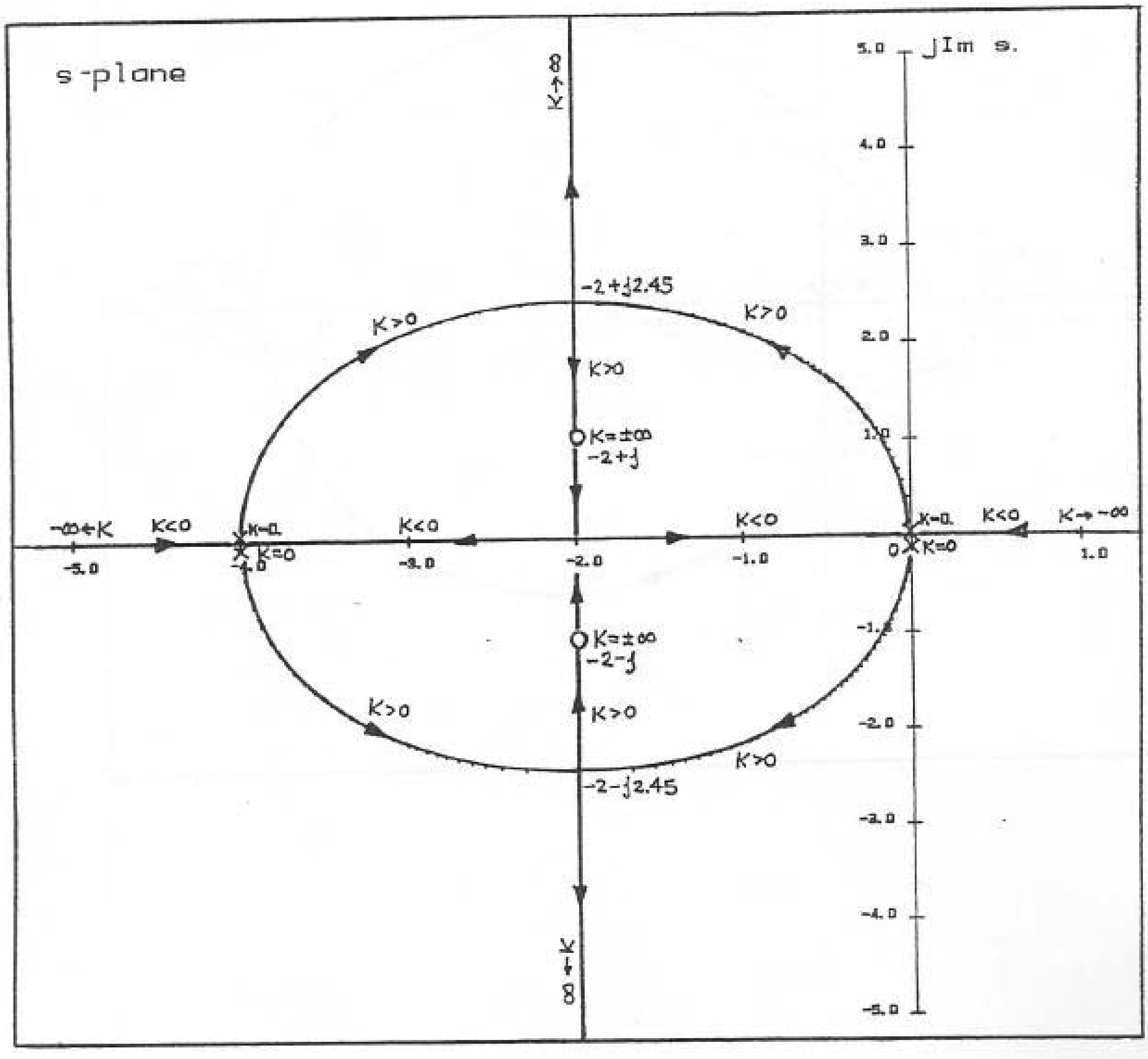
Asymptotes: $\boldsymbol{K > 0}$**:** $90^o, \quad 270^o$ $\qquad$ $\boldsymbol{K < 0}$**:** $0^o, \quad 180^o$

Intersect of Asymptotes:

$$s_1 = \frac{-8 - (-4)}{4 - 2} = -2$$

Breakaway-point Equation: $s^5 + 10s^4 + 42s^3 + 92s^2 + 80s = 0$

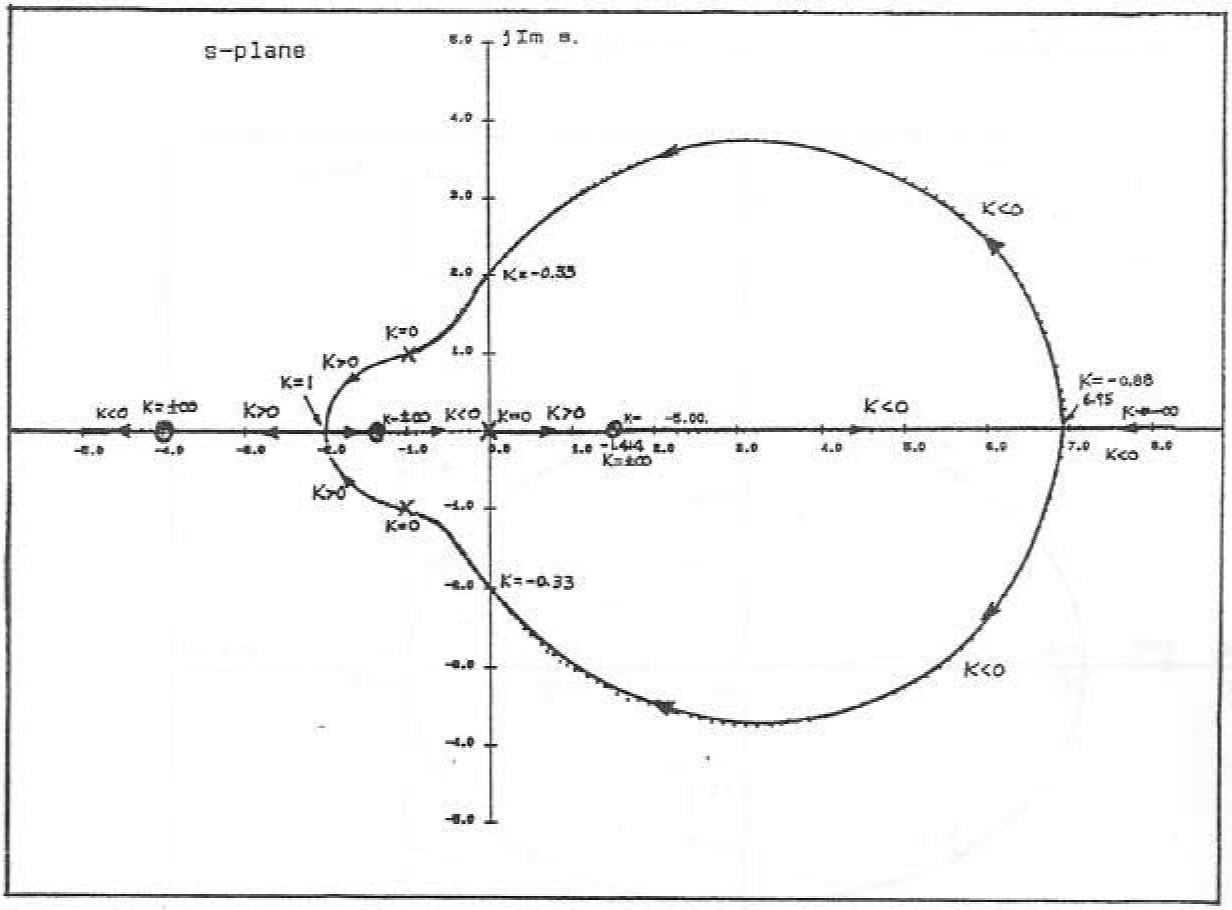
Breakaway Points: $0, \quad -2, \quad -4, \quad -2 + j2.45, \quad -2 - j2.45$



8-6 (h) $Q(s) = \left(s^2 - 2\right)\left(s+4\right) \qquad P(s) = s\left(s^2 + 2s + 2\right)$

Since $Q(s)$ and $P(s)$ are of the same order, there are no asymptotes.

Breakaway Points: -2, 6.95

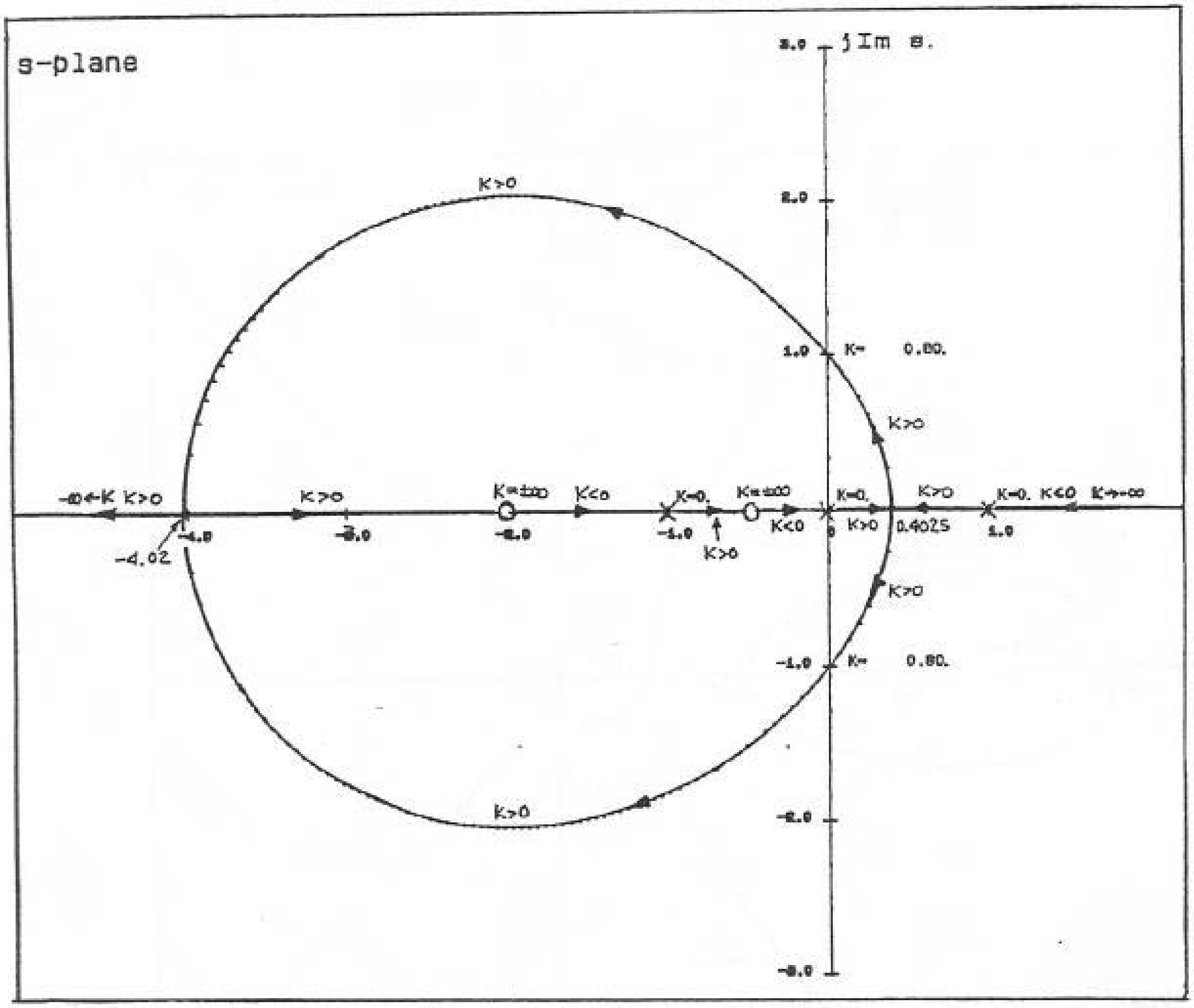


8-6 (i) $Q(s) = (s+2)(s+0.5)$ $\qquad P(s) = s(s^2-1)$

Asymptotes: $\quad$ ***K* > 0:** 180^o $\qquad$ ***K* < 0:** 0^o

Breakaway-point Equation: $\quad s^4 + 5s^3 + 4s^2 - 1 = 0$

Breakaway Points: $\quad -4.0205, \quad 0.40245$ The other solutions are not breakaway points.



8-6 (j) $Q(s) = 2s + 5 \qquad P(s) = s^2\left(s^2 + 2s + 1\right) = s^2\left(s+1\right)^2$

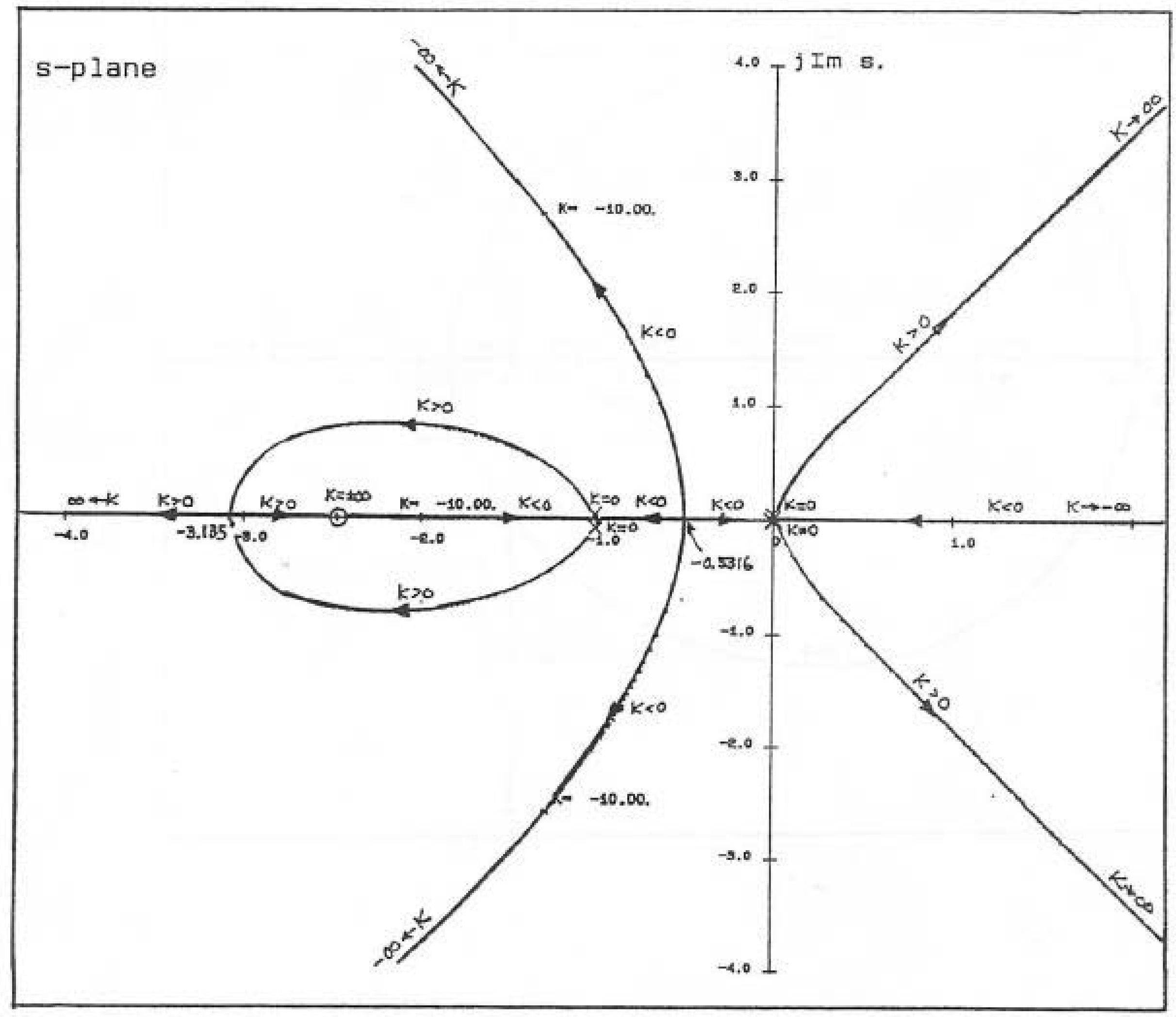
Asymptotes: ***K* > 0:** $60^o, \quad 180^o, \quad 300^o$ ***K* < 0:** $0^o, \quad 120^o, \quad 240^o$

Intersect of Asymptotes;

$$s_1 = \frac{0 + 0 - 1 - 1 - (-2.5)}{4 - 1} = \frac{0.5}{3} = 0.167$$

Breakaway-point Equation: $6s^4 + 28s^3 + 32s^2 + 10s = 0$

Breakaway Points: $0, \quad -0.5316, \quad -1, \quad -3.135$



8-7 (a) Asymptotes: $K > 0$: $45^o, \quad 135^o, \quad 225^o, \quad 315^o$

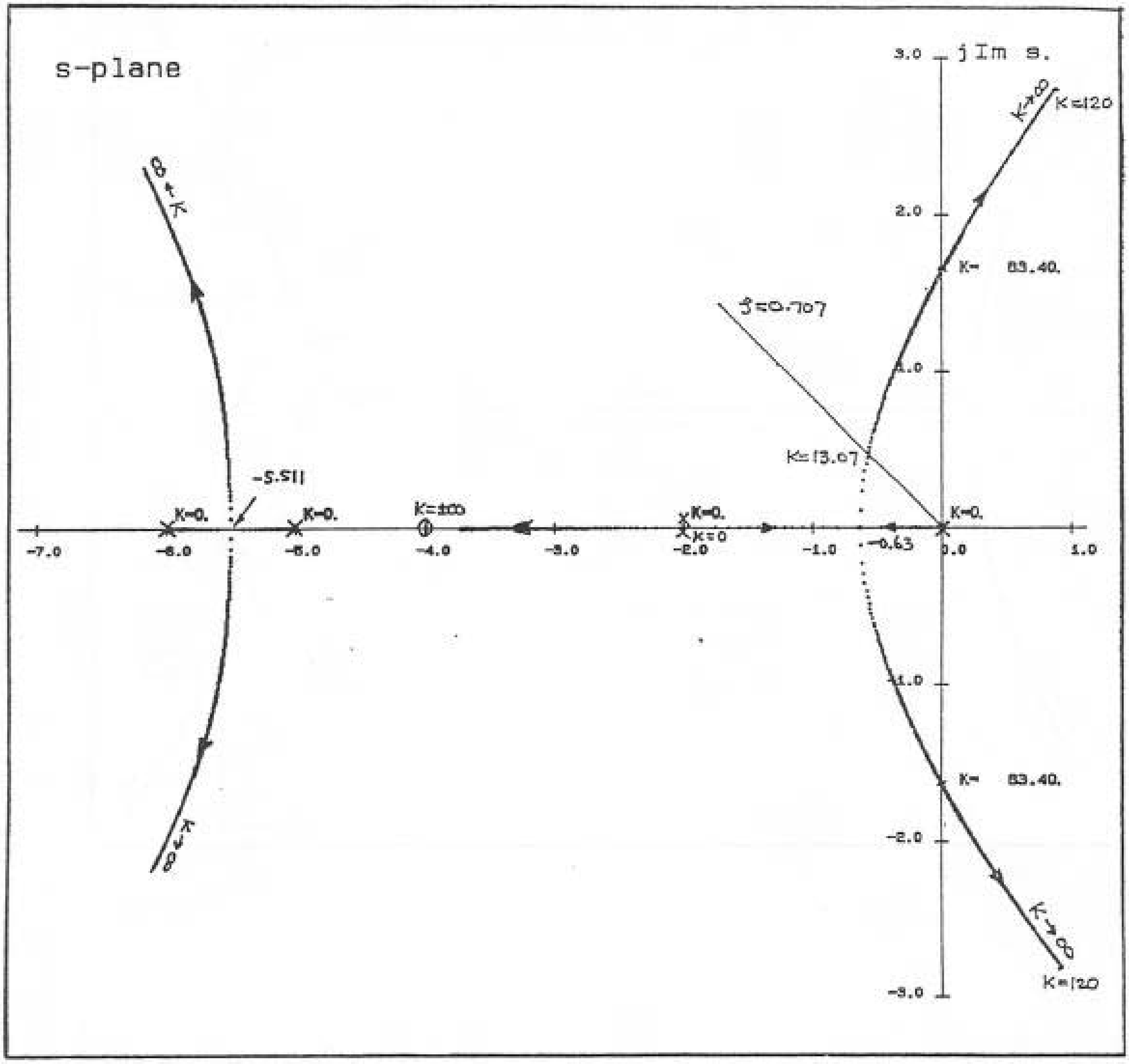
Intersect of Asymptotes:

$$s_1 = \frac{-2-2-5-6-(-4)}{5-1} = -2.75$$

Breakaway-point Equation: $4s^5 + 65s^4 + 396s^3 + 1100s^2 + 1312s + 480 = 0$

Breakaway Points: $-0.6325, \quad -5.511$ (on the RL)

When $Z = 0.707$, $\mathbf{K = 13.07}$



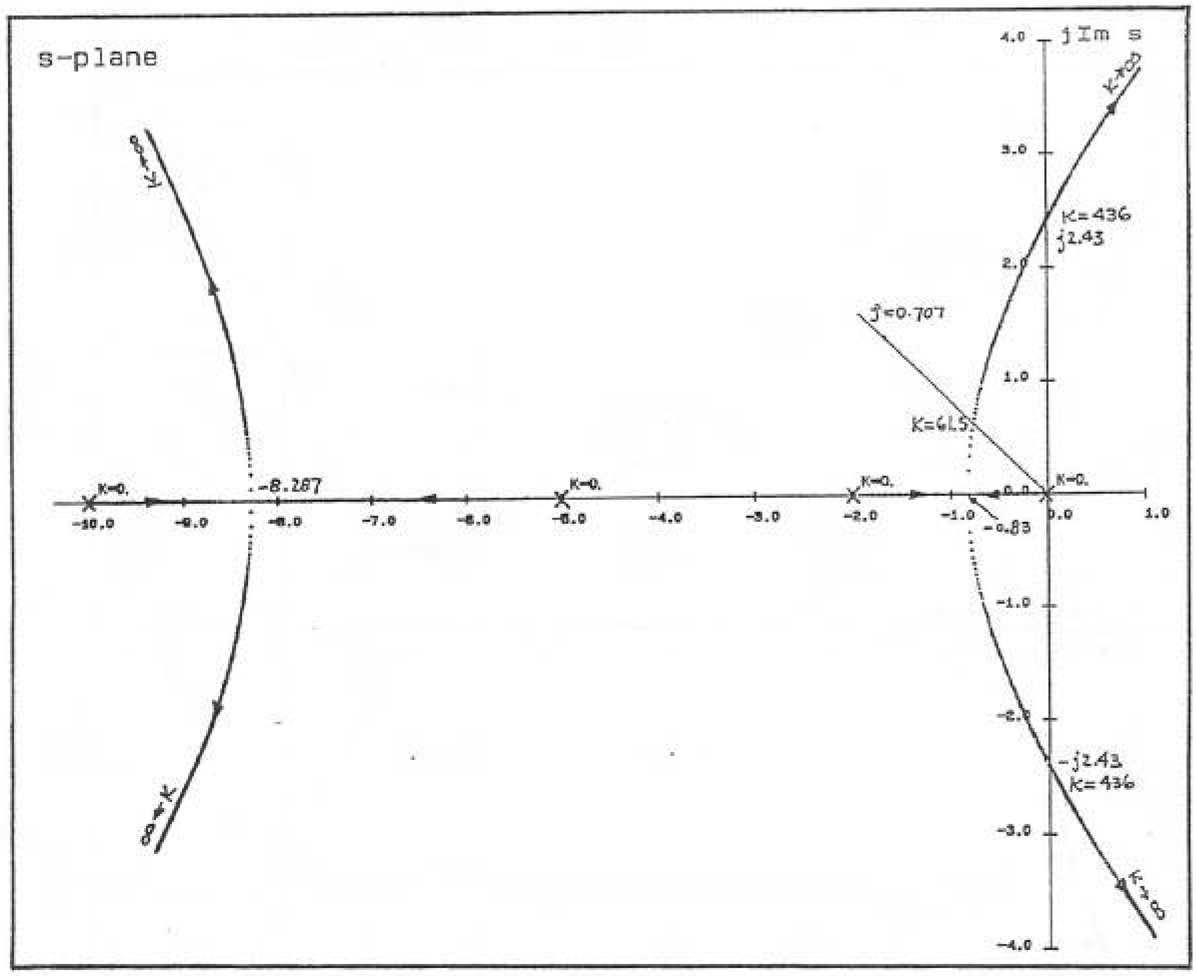
8-7 (b) **Asymptotes:** $K > 0$: 45^o, 135^o, 225^o, 315^o

Intersect of Asymptotes:

$$s_1 = \frac{0-2-5-10}{4} = -4.25$$

Breakaway-point Equation: $4s^3 + 51s^2 + 160s + 100 = 0$

When $\zeta = 0.707$, $K = 61.5$

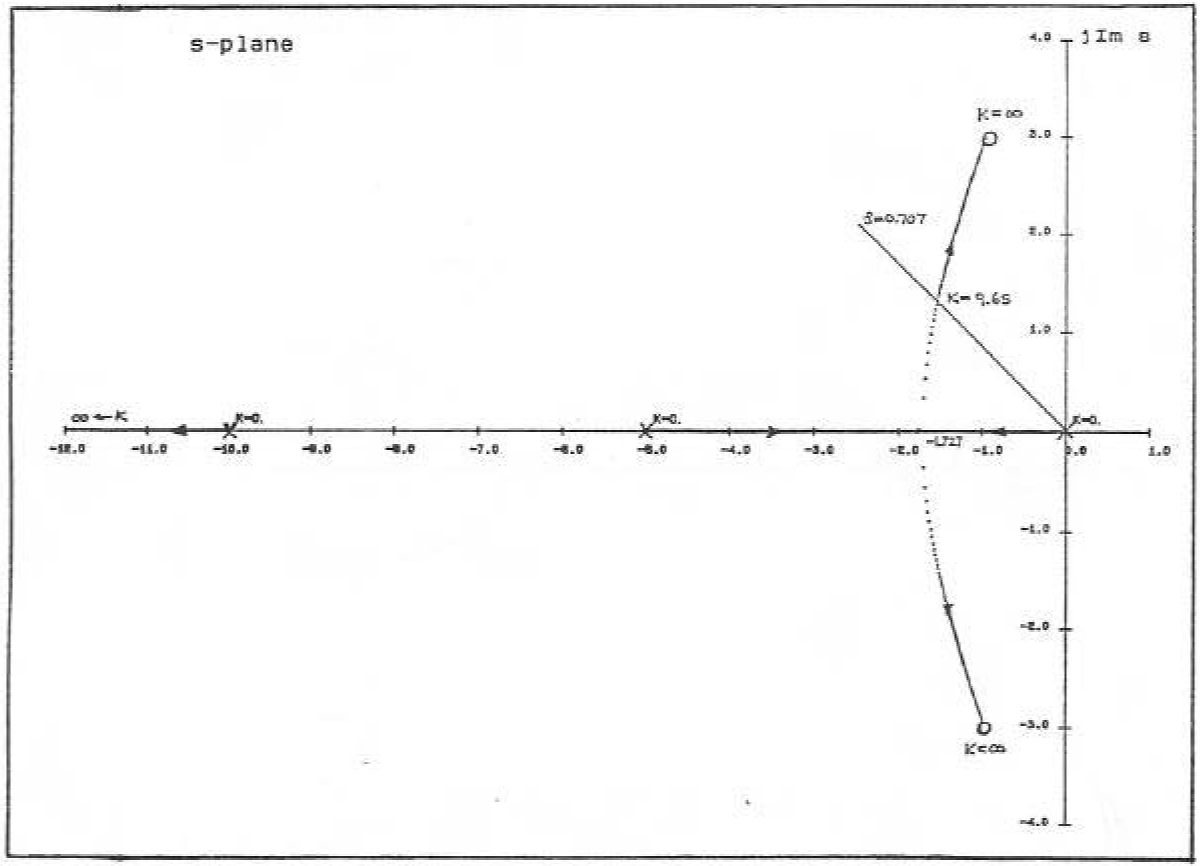


8-7 (c) **Asymptotes:** $K > 0$: 180^o

Breakaway-point Equation: $s^4 + 4s^3 + 10s^2 + 300s + 500 = 0$

Breakaway Points: -1.727 (on the RL)

When $\zeta = 0.707$, $\boldsymbol{K = 9.65}$

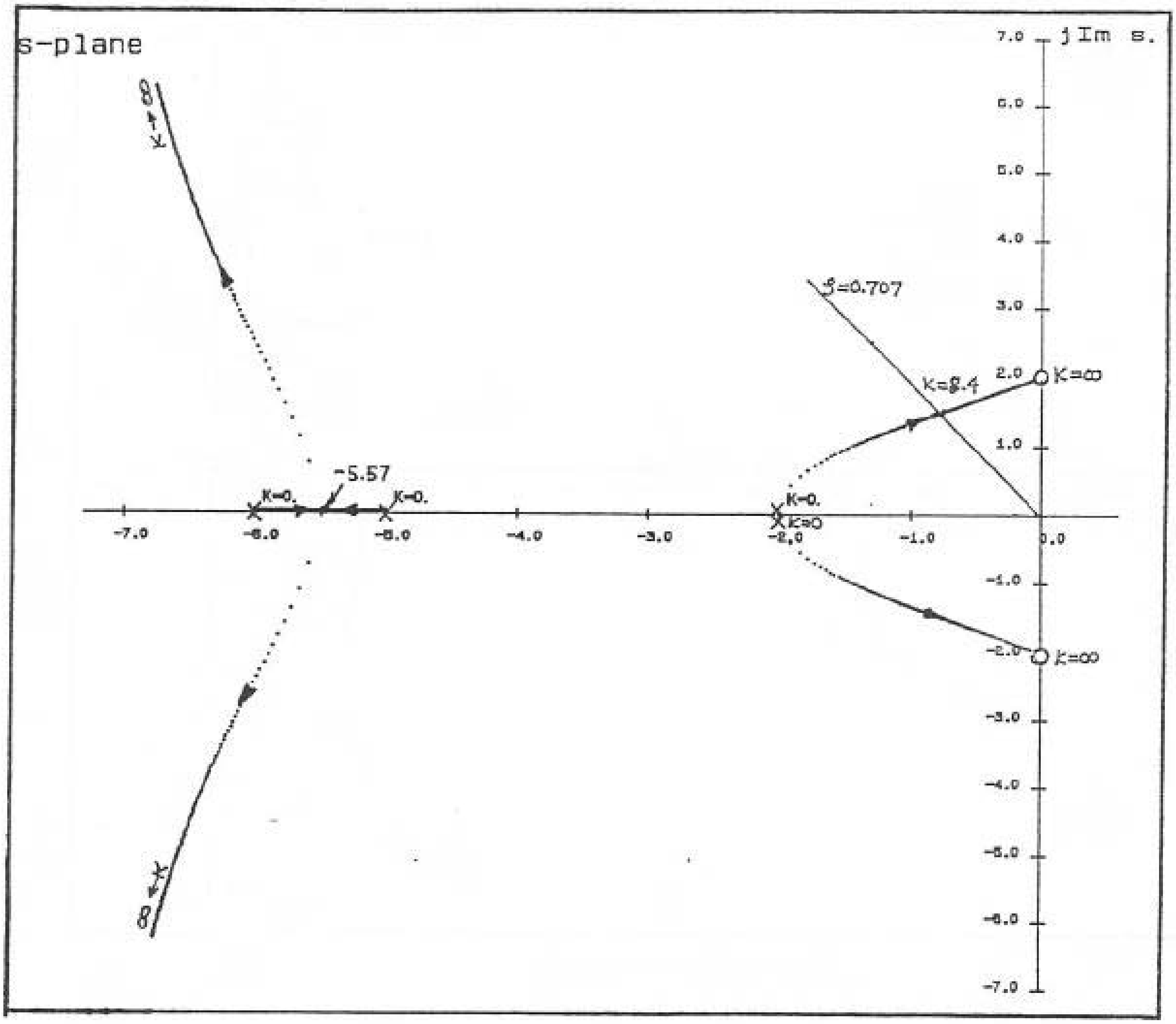


8-7 (d) $K > 0$: 90^o, 270^o

Intersect of Asymptotes:

$$s_1 = \frac{-2-2-5-6}{4-2} = -7.5$$

When $z = 0.707$, $K = 8.4$

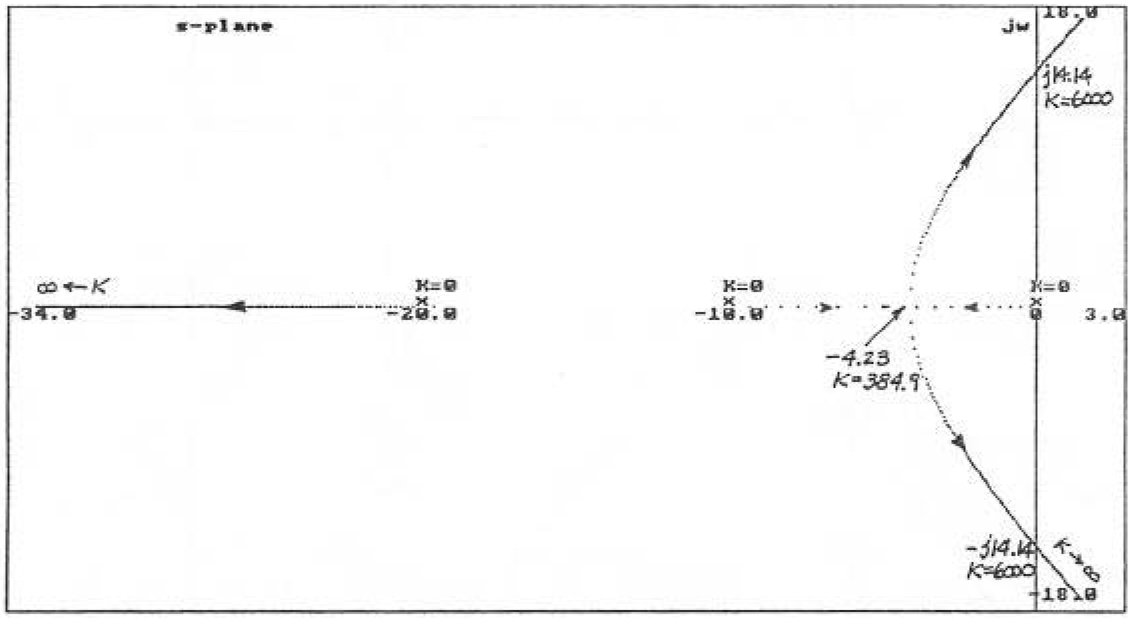


8-8 (a) **Asymptotes:** ***K* > 0:** 60^o, 180^o, 300^o

Intersect of Asymptotes:

$$s_1 = \frac{0 - 10 - 20}{3} = -10$$

Breakaway-point Equation: $3s^2 + 60s + 200 = 0$ **Breakaway Point: (RL)** -4.2265, ***K* = 384.9**



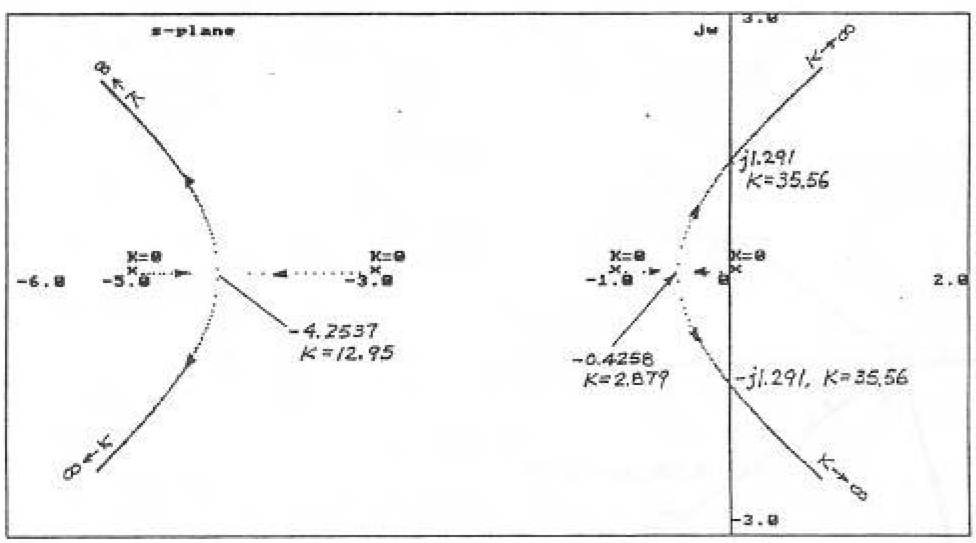
(b) Asymptotes: $K > 0$: 45^o, 135^o, 225^o, 315^o

Intersect of Asymptotes:

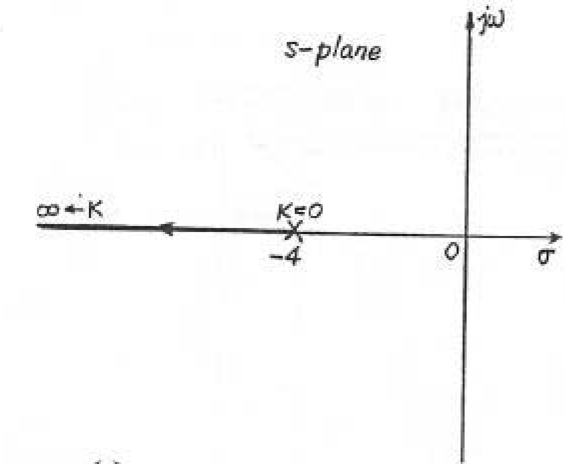
$$s_1 = \frac{0 - 1 - 3 - 5}{4} = -2.25$$

Breakaway-point Equation: $4s^3 + 27s^2 + 46s + 15 = 0$

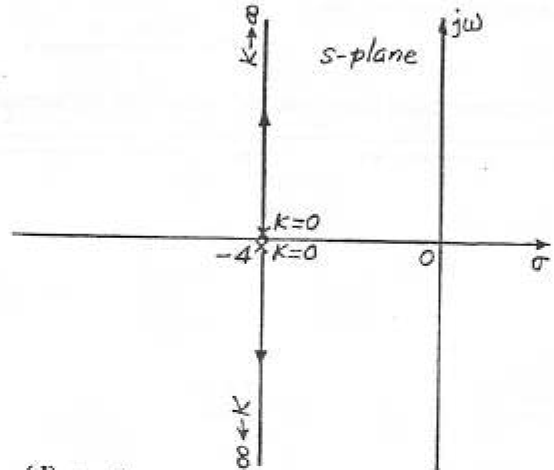
Breakaway Points: (RL) -0.4258 $\quad$ K **= 2.879**, $\quad$ -4.2537 $\quad$ K **= 12.95**



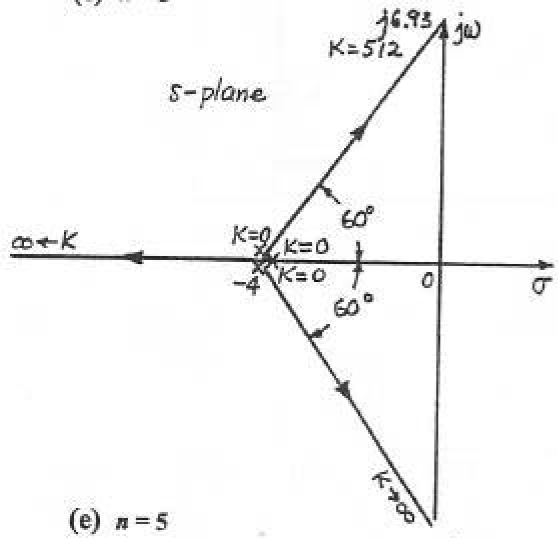
8-9 (a) $n = 1$



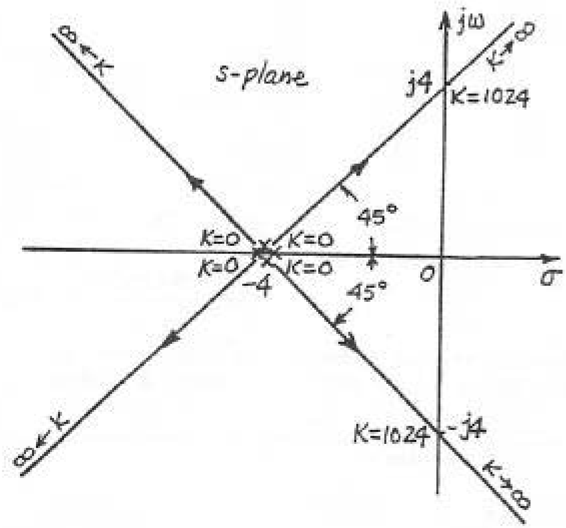
(b) $n = 2$



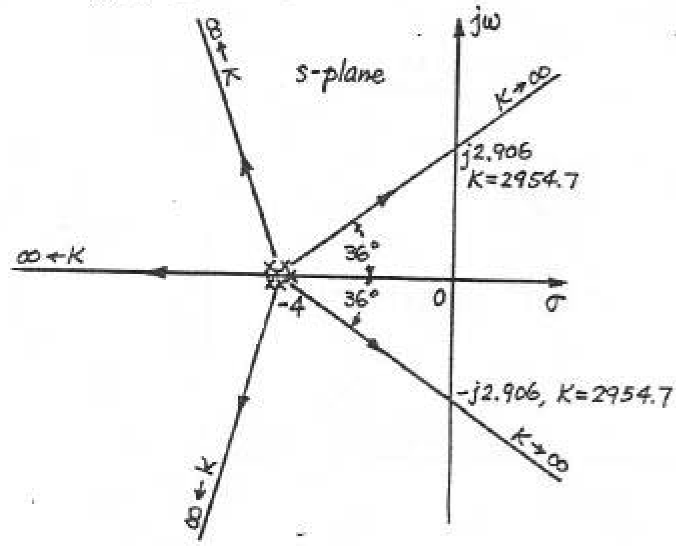
(c) $n = 3$



(d) $n = 4$



(e) $n = 5$



8-10 $P(s) = s^3 + 25s^2 + 2s + 100 \qquad Q(s) = 100s$

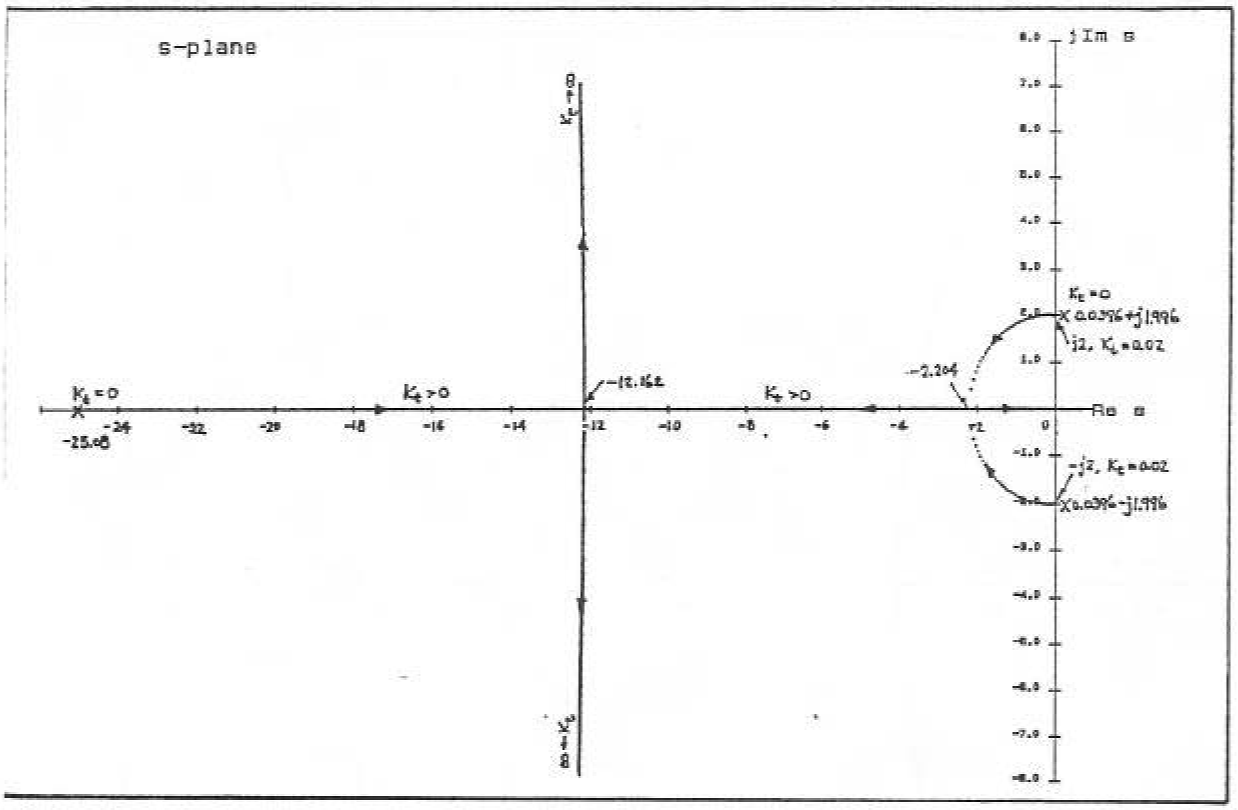
Asymptotes: $K_t > 0$: 90^o, 270^o

Intersect of Asymptotes:

$$s_1 = \frac{-25 - 0}{3 - 1} = -12.5$$

Breakaway-point Equation: $s^3 + 12.5s^2 - 50 = 0$

Breakaway Points: **(RL)** -2.2037, -12.162



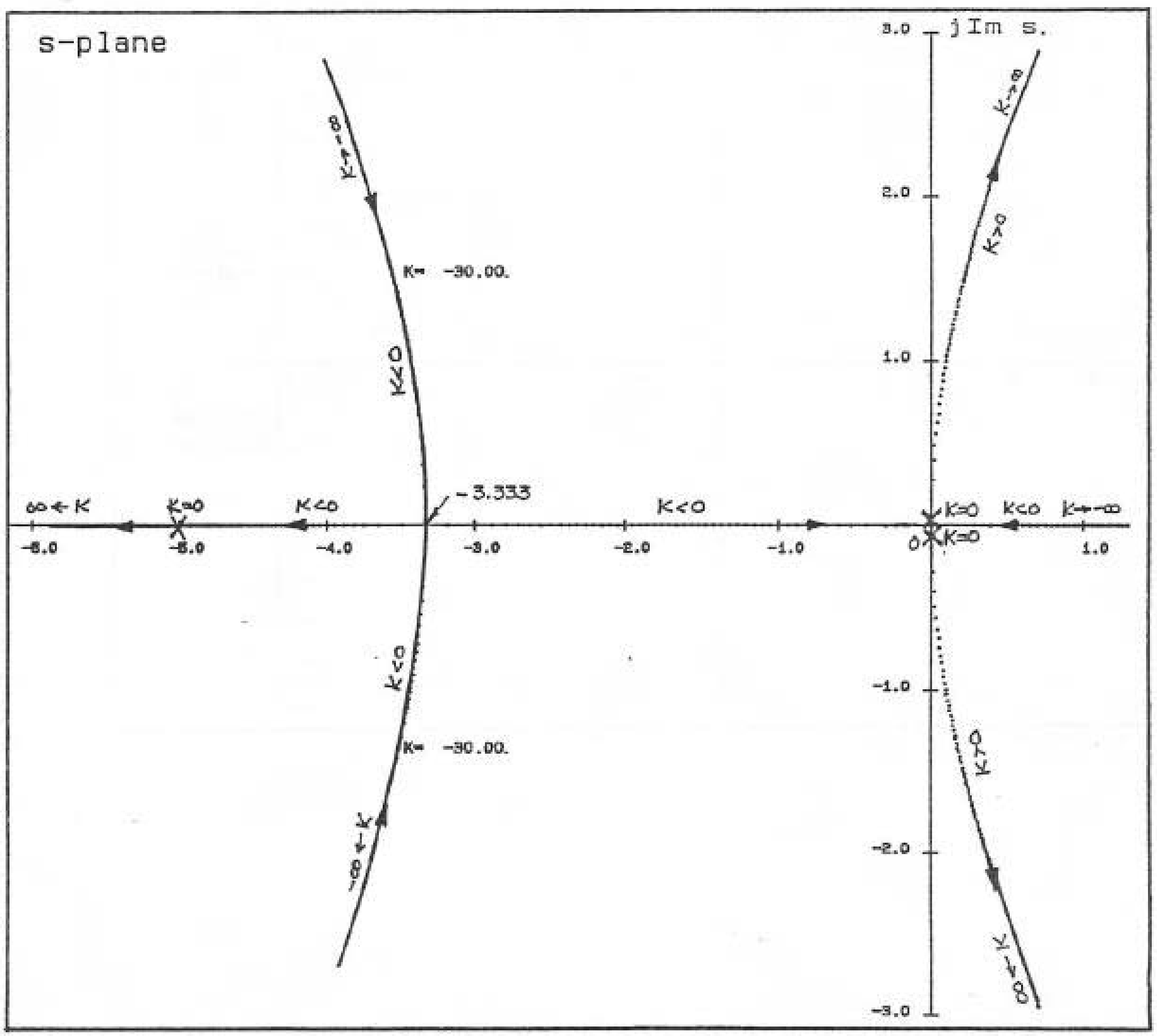
8-11 Characteristic equation: $s^3 + 5s^2 + K_t s + K = 0$

(a) $K_t = 0$: $P(s) = s^2(s+5)$ $Q(s) = 1$

Asymptotes: $K > 0$: 60^o, 180^o, 300^o
Intersect of Asymptotes:

$$s_1 = \frac{-5-0}{3} = -1.667$$

Breakaway-point Equation: $3s^2 + 10s = 0$ **Breakaway Points:** $0, \; -3.333$

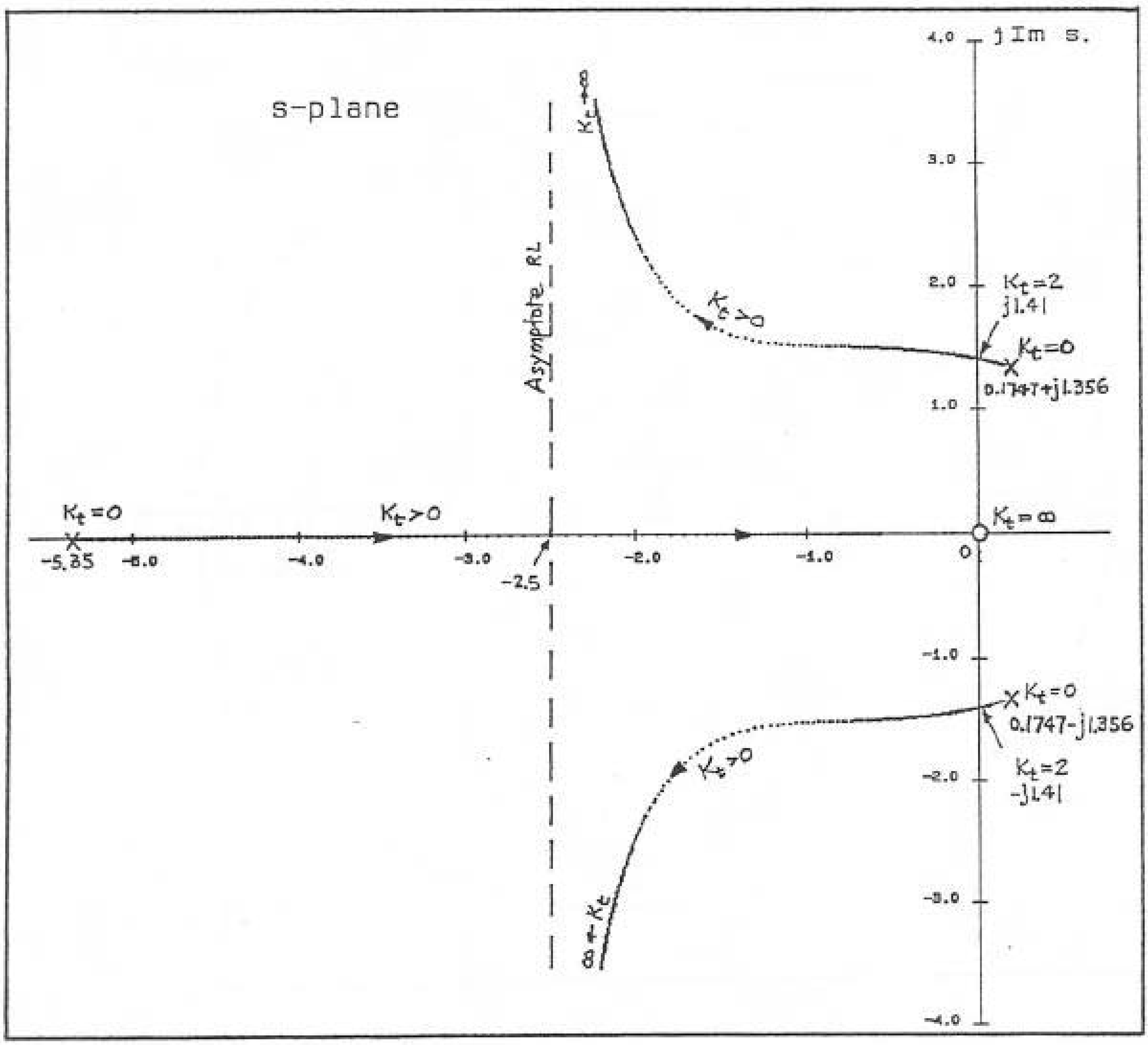


8-11 (b) $P(s) = s^3 + 5s^2 + 10 = 0$ $Q(s) = s$

Asymptotes: $K > 0$: 90^o, 270^o
Intersect of Asymptotes:

$$s_1 = \frac{-5-0}{2-1} = 0$$

Breakaway-point Equation: $2s^3 + 5s - 10 = 0$
There are no breakaway points on RL.

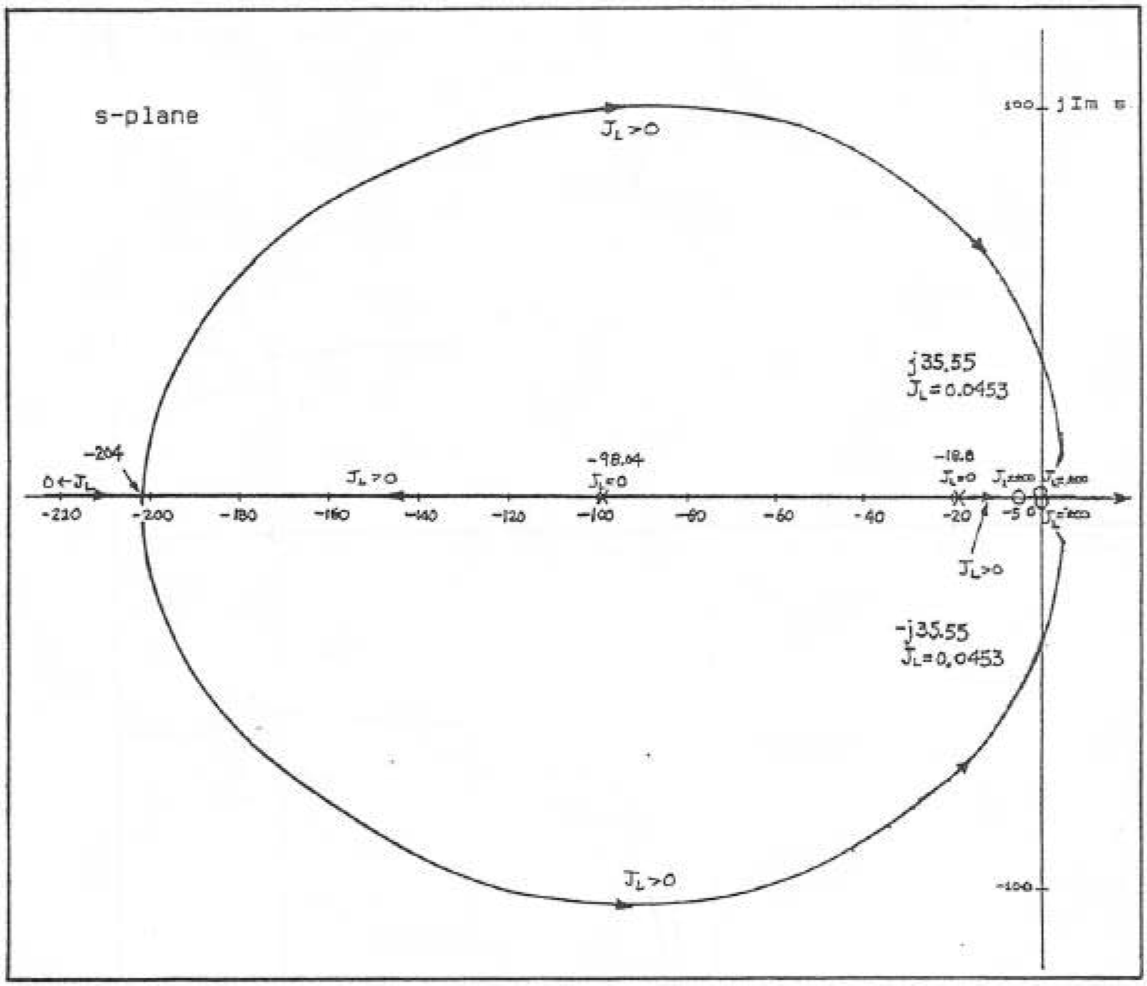


8-12 $P(s) = s^2 + 116.84s + 1843 \qquad Q(s) = 2.05s^2(s+5)$

Asymptotes: $J_L = 0$: 180^o

Breakaway-point Equation: $-2.05s^4 - 479s^3 - 12532s^2 - 37782s = 0$

Breakaway Points: **(RL)** $0, \quad -204.18$

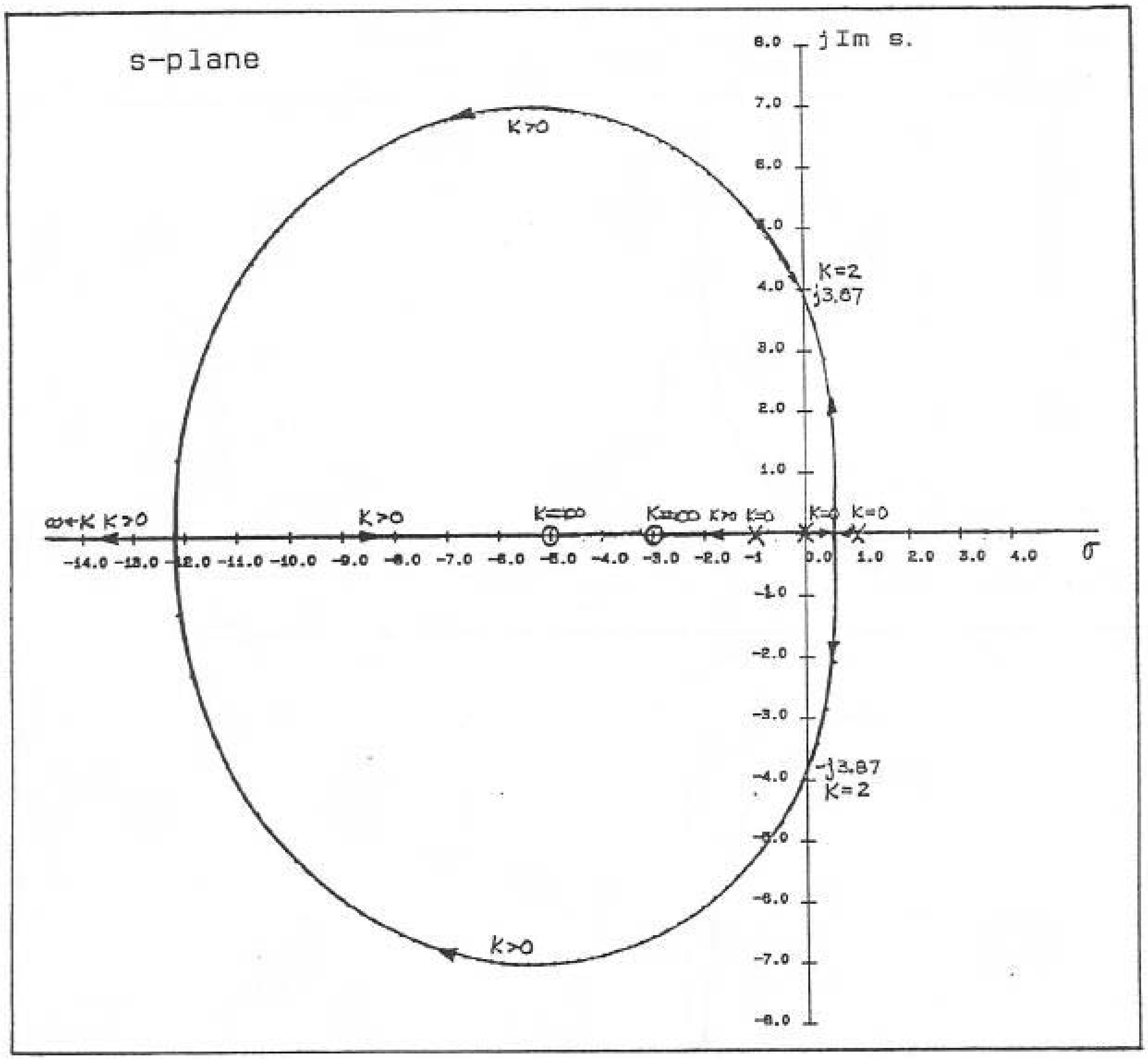


8-13 (a) $P(s) = s\left(s^2 - 1\right)$ $\qquad Q(s) = (s+5)(s+3)$

Asymptotes: $\quad$ ***K* > 0:** $\quad 180^o$

Breakaway-point Equation: $\quad s^4 + 16s^3 + 46s^2 - 15 = 0$

Breakaway Points: $\quad$ **(RL)** $\quad 0.5239, \quad -12.254$



8-13 (b) $P(s) = s\left(s^2 + 10s + 29\right) \qquad Q(s) = 10(s+3)$

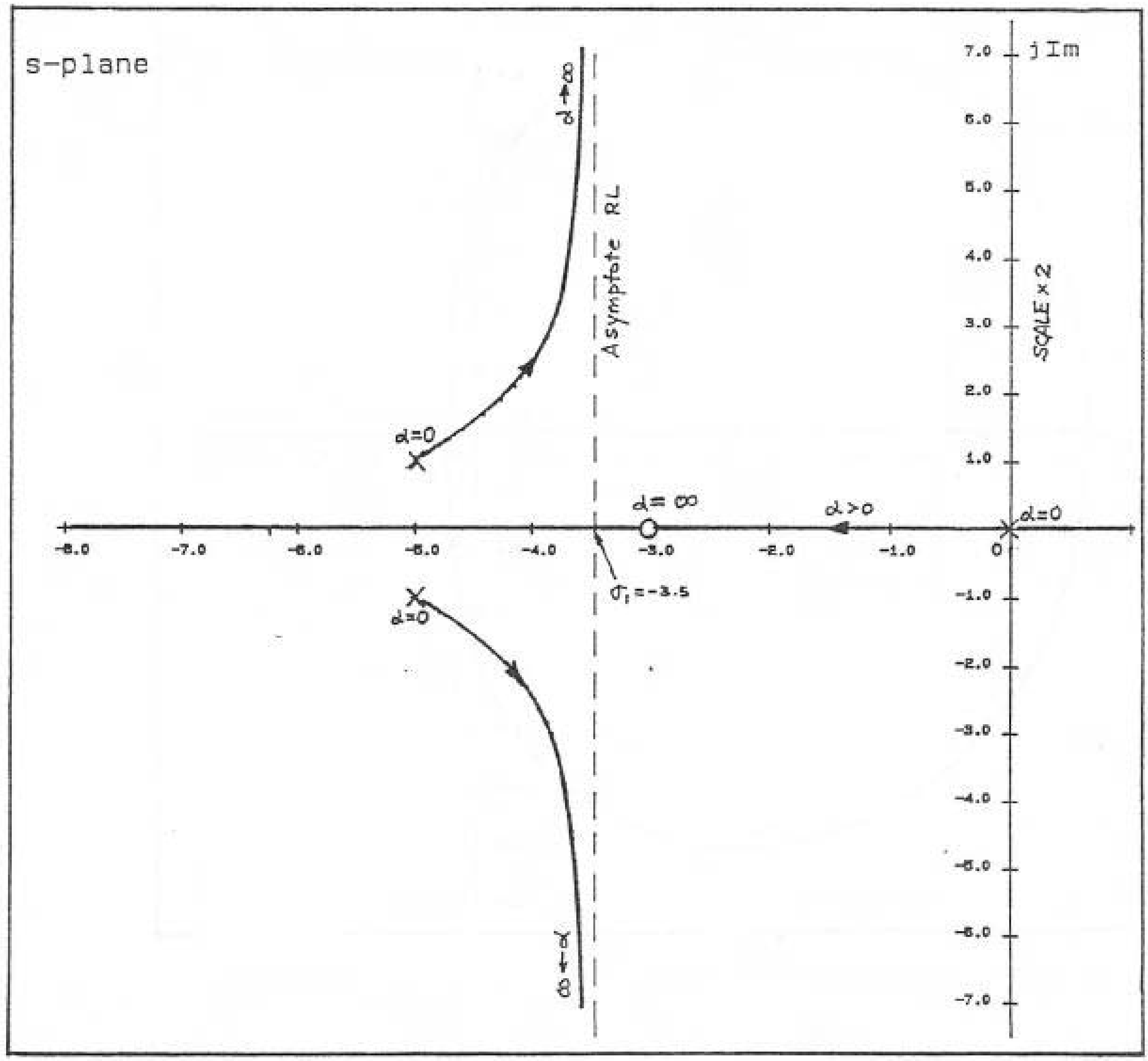
Asymptotes: $\quad$ ***K* > 0:** $\quad 90^o, \quad 270^o$

Intersect of Asymptotes:

$$s_1 = \frac{0 - 10 - (-3)}{3 - 1} = -3.5$$

Breakaway-point Equation: $\quad 20s^3 + 190s^2 + 600s + 870 = 0$

There are no breakaway points on the RL.



8-14 (a) $P(s) = s(s+12.5)(s+1) \qquad Q(s) = 83.333$

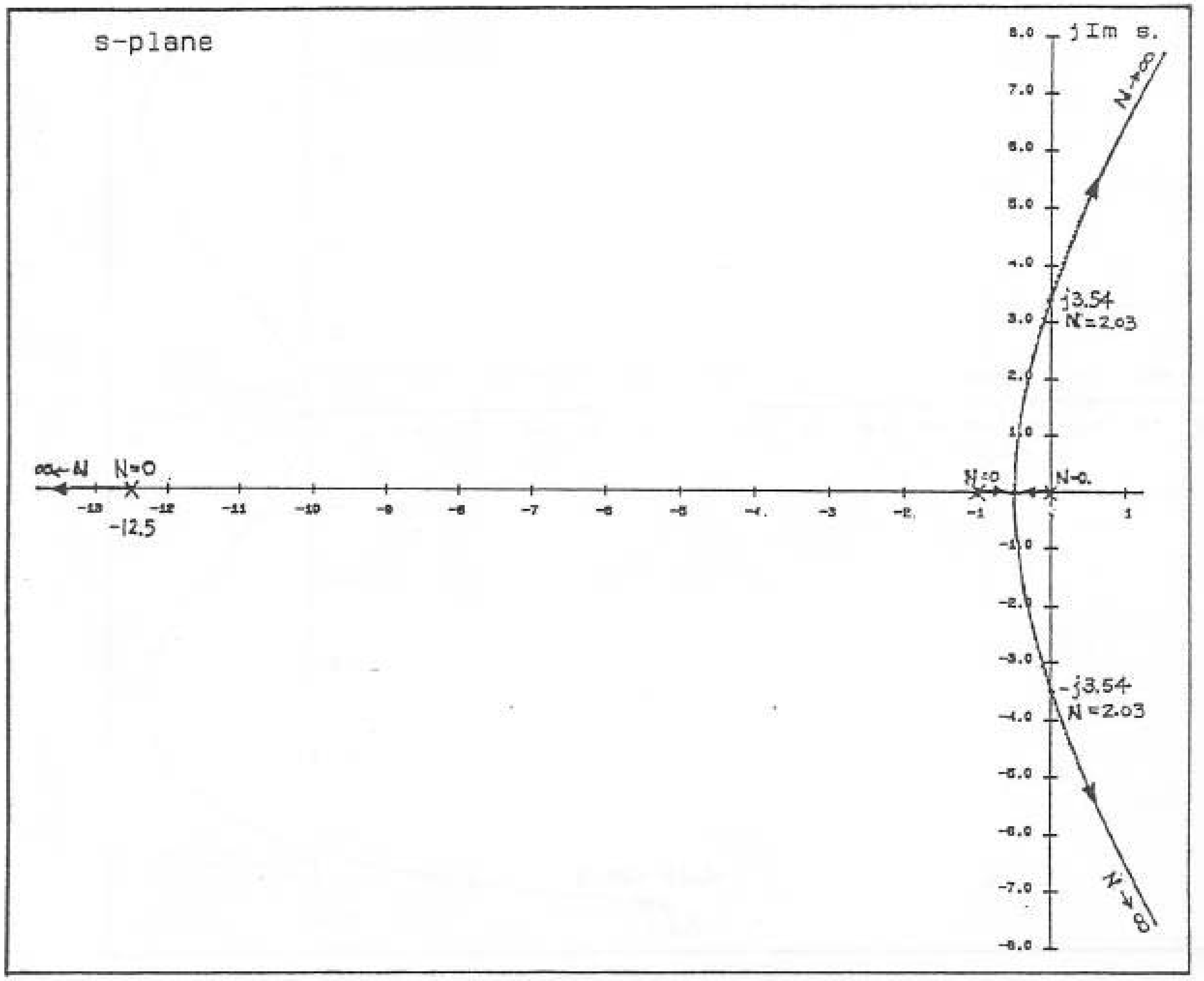
Asymptotes: $\quad$ ***N* > 0:** $\quad 60^o, \quad 180^o, \quad 300^o$

Intersect of Asymptotes:

$$s_1 = \frac{0 - 12.5 - 1}{3} = -4.5$$

Breakaway-point Equation: $3s^2 + 27s\_12.5 = 0$

Breakaway Point: (RL) $\quad -0.4896$

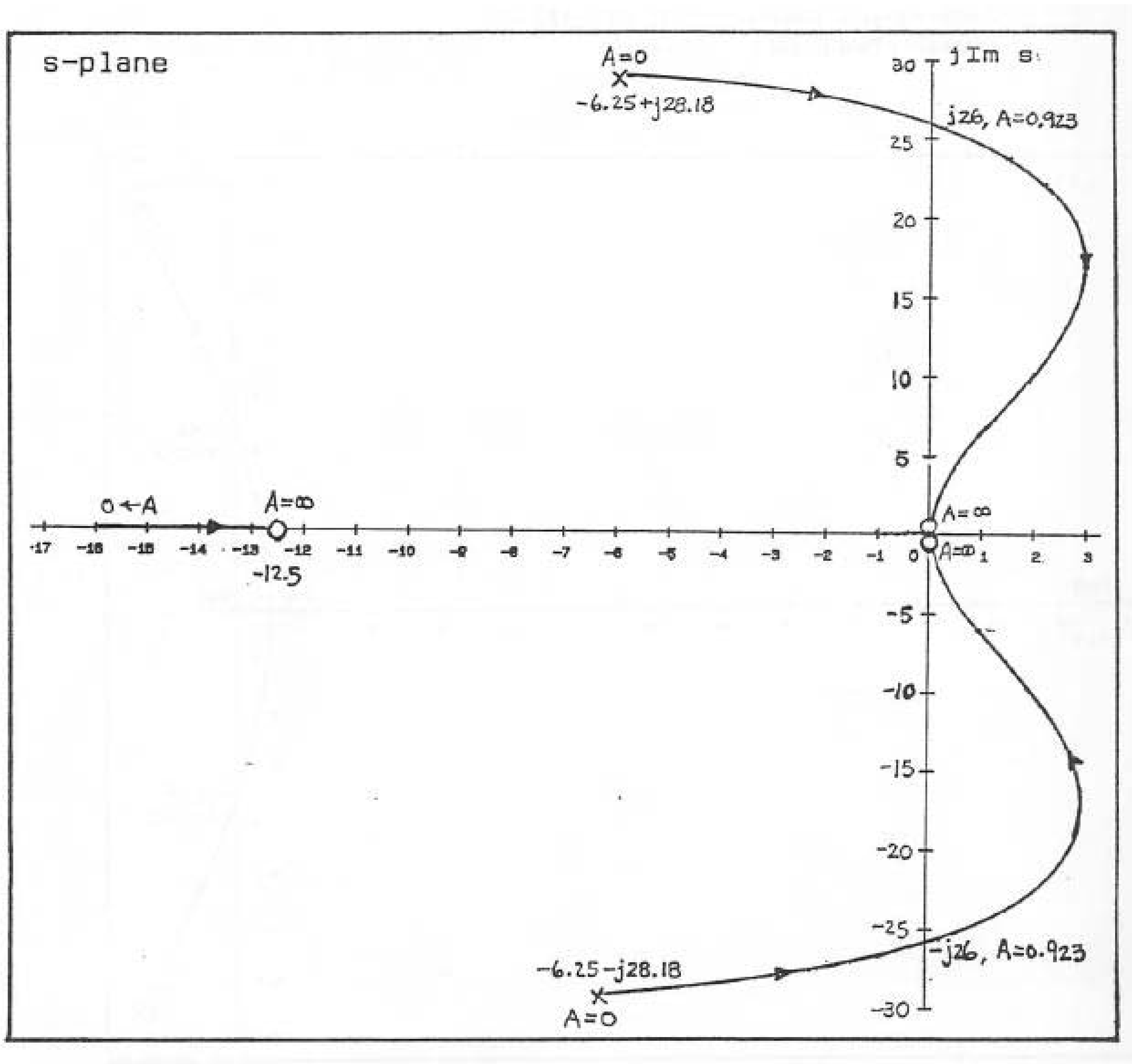


8-14 (b) $P(s) = s^2 + 12.5s + 833.333 \qquad Q(s) = 0.02s^2(s + 12.5)$

A > 0: 180^o

Breakaway-point Equation: $0.02s^4 + 0.5s^3 + 53.125s^2 + 416.67s = 0$

Breakaway Points: **(RL)** 0



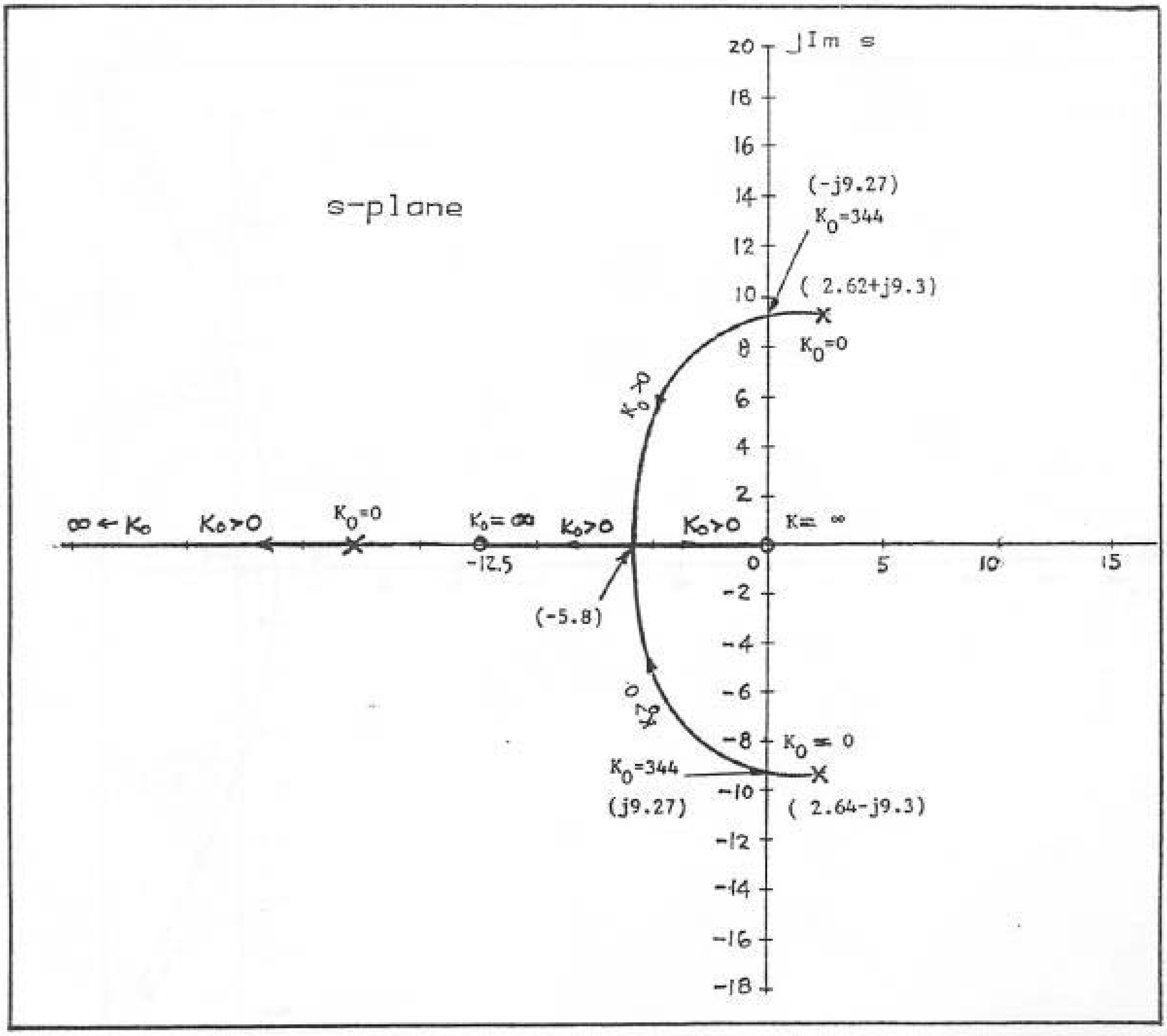
8-14 (c) $P(s) = s^3 + 12.5s^2 + 1666.67 = (s + 17.78)(s - 2.64 + j9.3)(s - 2.64 - j9.3)$

$Q(s) = 0.02\,s(s + 12.5)$

Asymptotes: $K_o > 0$: 180^o

Breakaway-point Equation: $0.02s^4 + 0.5s^3 + 3.125s^2 - 66.67s - 416.67 = 0$

Breakaway Point: (RL) -5.797



8-15 (a) $A = K_o = 100$: $\qquad P(s) = s(s + 12.5)(s + 1) \qquad Q(s) = 41.67$

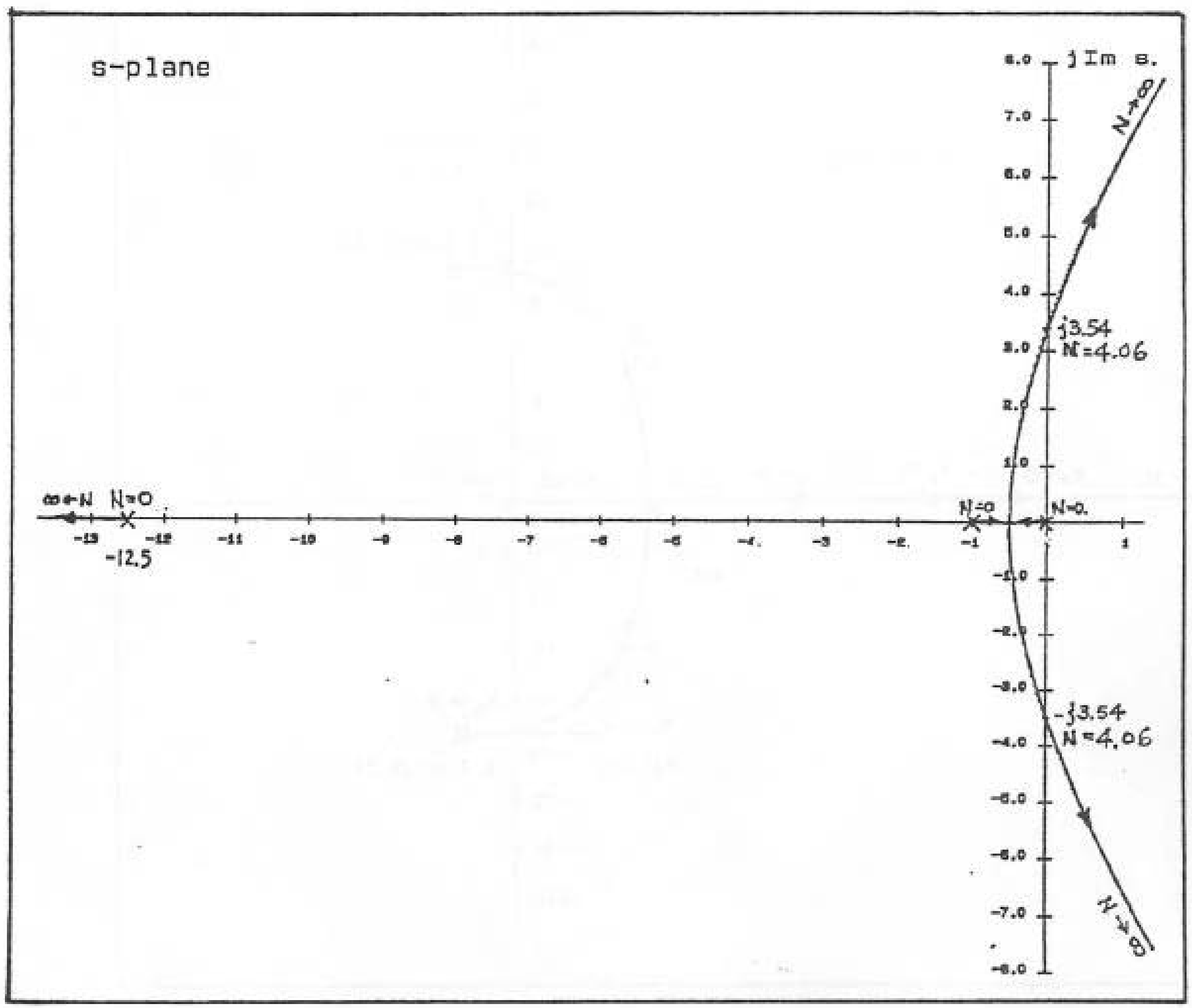
Asymptotes: $\qquad$ ***N* > 0:** $\qquad 60^o \quad 180^o \quad 300^o$

Intersect of Asymptotes:

$$s_1 = \frac{0 - 1 - 12.5}{3} = -4.5$$

Breakaway-point Equation: $\qquad 3s^2 + 27s + 12.5 = 0$

Breakaway Points: (RL) $\qquad -0.4896$



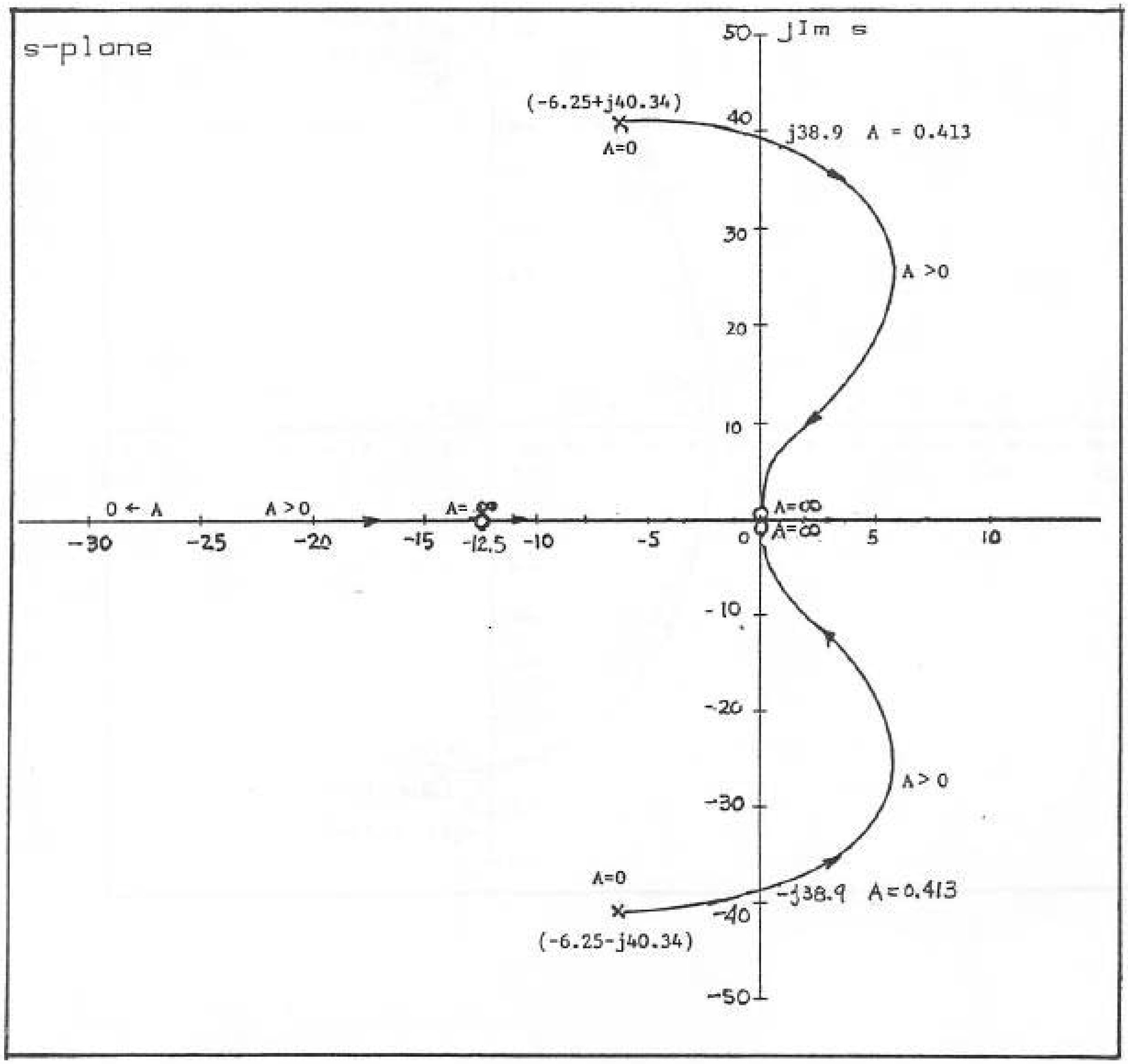
8-15 (b) $P(s) = s^2 + 12.5 + 1666.67 = (s + 6.25 + j40.34)(s + 6.25 - j40.34)$

$Q(s) = 0.02s^2(s + 12.5)$

Asymptotes: ***A* > 0:** 180^o

Breakaway-point Equation: $0.02s^4 + 0.5s^3 + 103.13s^2 + 833.33s = 0$

Breakaway Points: **(RL)** 0



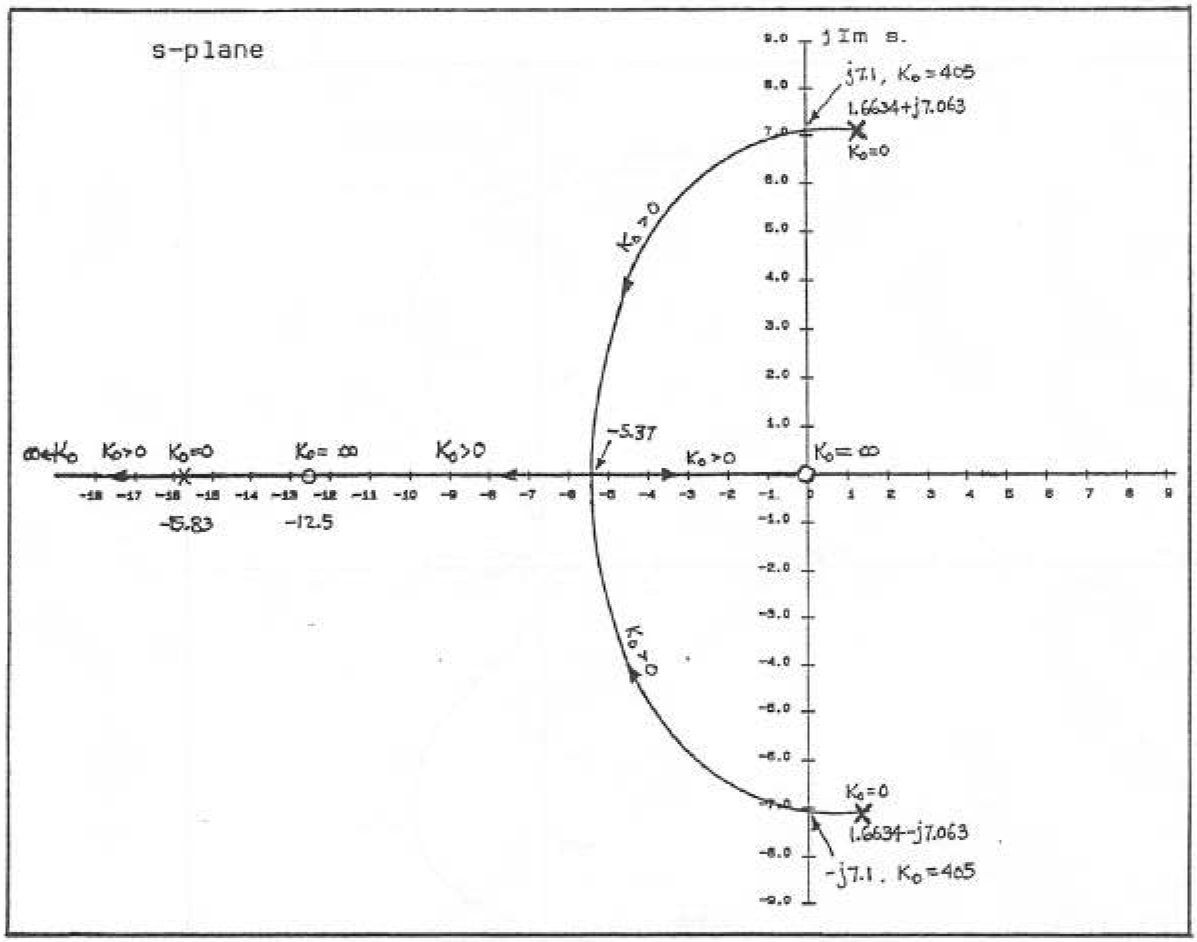
8-15 (c) $P(s) = s^3 + 12.5s^2 + 833.33 = (s + 15.83)(s - 1.663 + j7.063)(s - 1.663 - j7.063)$

$Q(s) = 0.01s(s + 12.5)$

Asymptotes: $K_o > 0$: 180^o

Breakaway-point Equation: $0.01s^4 + 0.15s^3 + 1.5625s^2 - 16.67s - 104.17 = 0$

Breakaway Point: **(RL)** -5.37



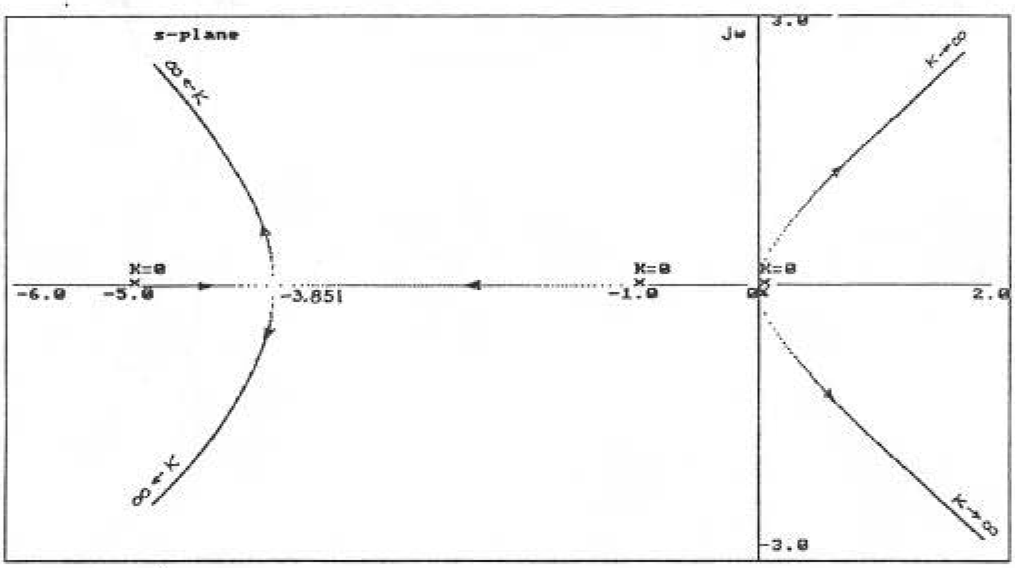
8-16 (a) $P(s) = s^2(s+1)(s+5) \qquad Q(s) = 1$

Asymptotes: $K > 0$: $45^o, \quad 135^o, \quad 225^o, \quad 315^o$

Intersect of Asymptotes:

$$s_1 = \frac{0+0-1-5}{4} = -1.5$$

Breakaway-point Equation: $4s^3 + 18s^2 + 10s = 0$ **Breakaway point: (RL)** $0, \; -3.851$



(b) $P(s) = s^2(s+1)(s+5) \qquad Q(s) = 5s+1$

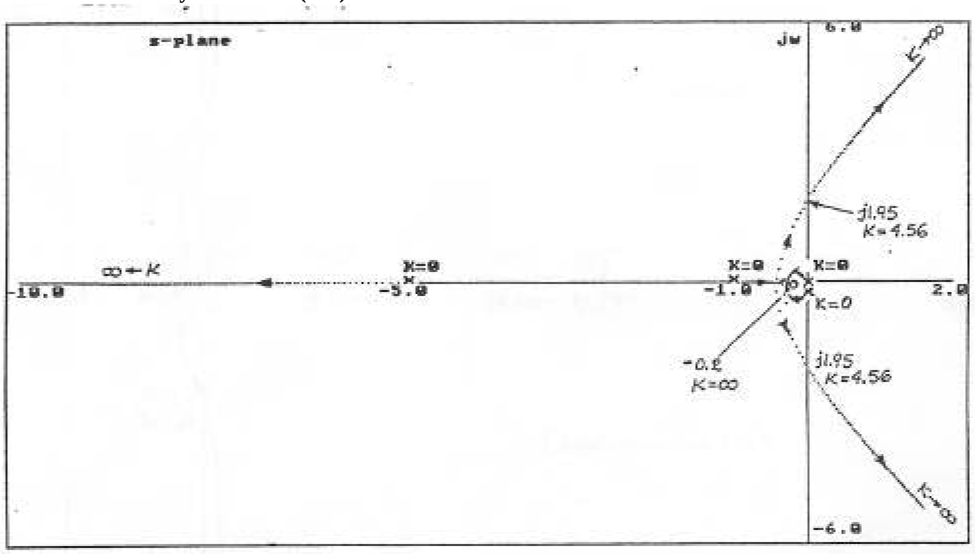
Asymptotes: $\quad$ ***K* > 0:** $\quad 60^o, \quad 180^o, \quad 300^o$

Intersect of Asymptotes:

$$s_1 = \frac{0+0-1-5-(-0.2)}{4-1} = -\frac{5.8}{3} = -1.93$$

Breakaway-point Equation: $15s^4 + 64s^3 + 43s^2 + 10s = 0$

Breakaway Points: $\quad$ **(RL)** $\; -3.5026$



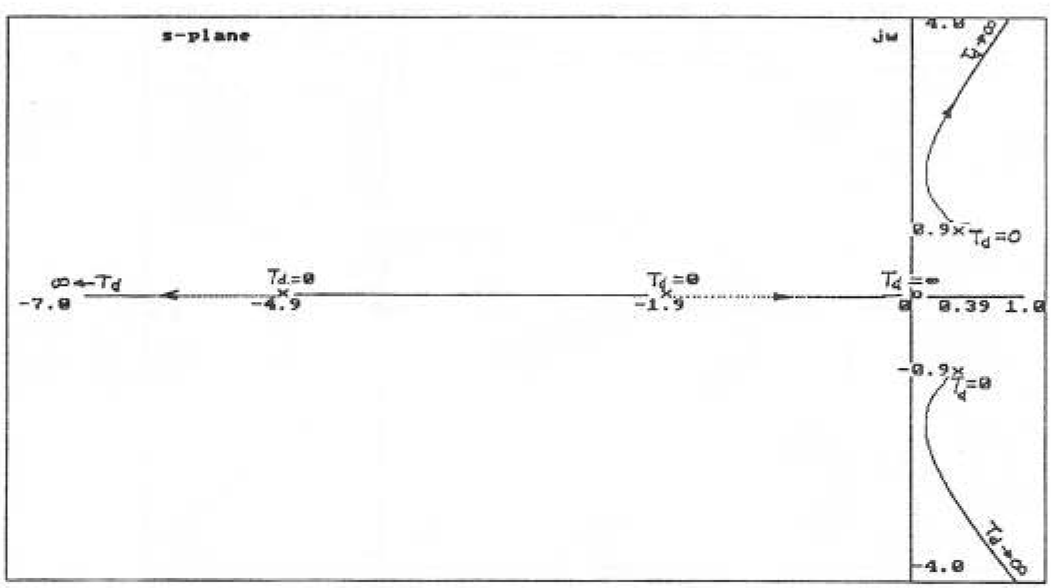
8-17 $P(s) = s^2(s+1)(s+5) + 10 = (s+4.893)(s+1.896)(s-0.394+j0.96)(s-0.394+j0.96)$

$Q(s) = 10s$

Asymptotes: $\quad T_d > 0: \quad 60^o, \quad 180^o, \quad 300^o$

Intersection of Asymptotes: $\quad s_1 = \dfrac{-4.893 - 1.896 + 0.3944 + 0.3944}{4-1} = -2$

There are no breakaway points on the RL.



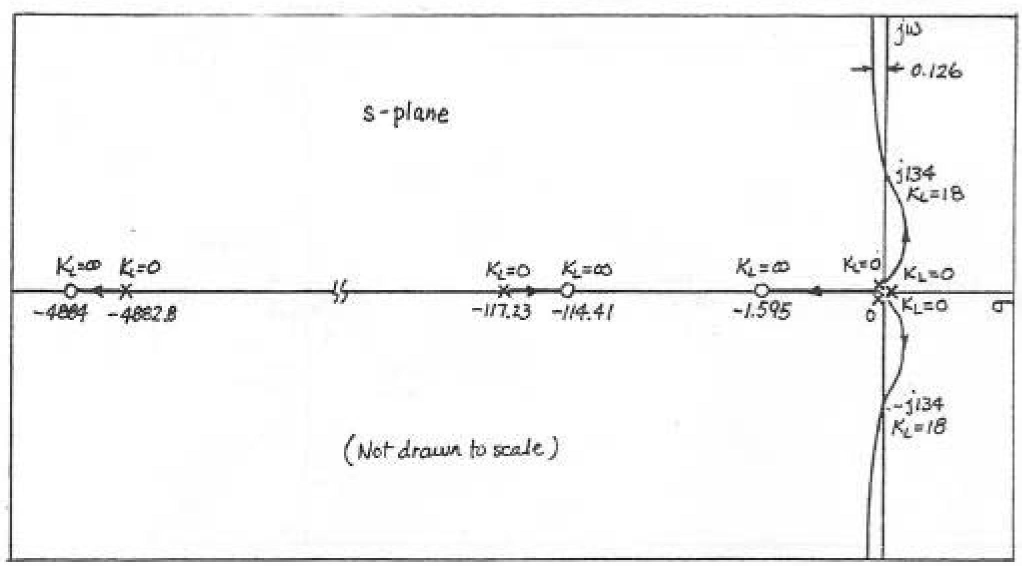
8-18 (a) $K = 1$: $\quad P(s) = s^3(s+117.23)(s+4882.8) \qquad Q(s) = 1010(s+1.5948)(s+114.41)(s+4884)$

Asymptotes: $\quad K_L > 0: \quad 90^o, \quad 270^o$

Intersect of Asymptotes:

$$s_1 = \frac{-117.23 - 4882.8 + 1.5948 + 114.41 + 4884}{5-3} = -0.126$$

Breakaway Point: (RL) $\quad 0$



8-18 (b) $K = 1000$: $\quad P(s) = s^3(s+117.23)(s+4882.8)$

$$Q(s) = 1010(s^3 + 5000s^2 + 5.6673\times10^5 s + 891089110)$$

$$= 1010(s+4921.6)(s+39.18+j423.7)(s+39.18-423.7)$$

Asymptotes: $\quad K_L > 0: \quad 90^o, \quad 270^o$

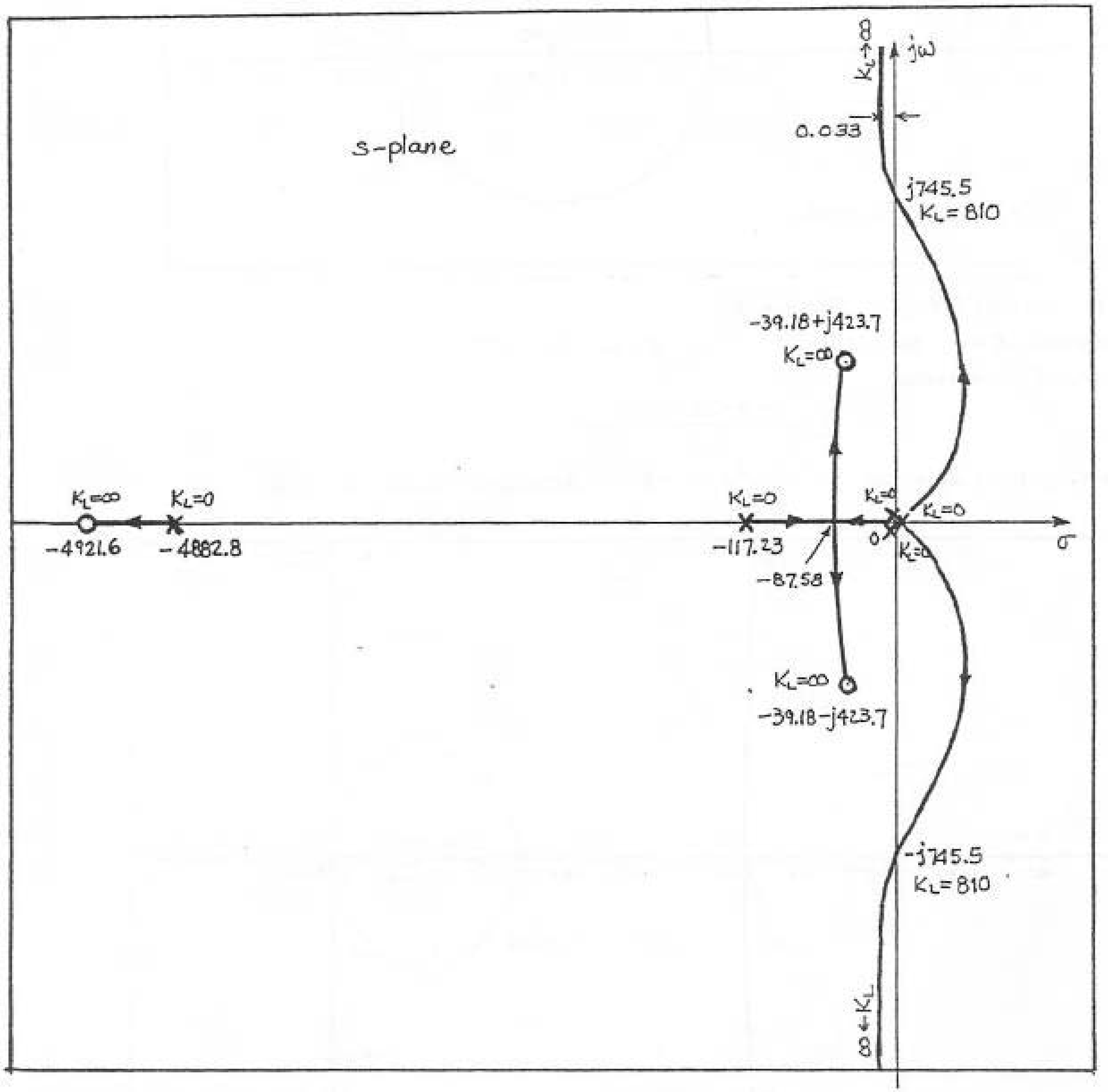
Intersect of Asymptotes:

$$s_1 = \frac{-117.23 - 4882.8 + 4921.6 + 39.18 + 39.18}{5-3} = -0.033$$

Breakaway-point Equation:

$$2020s^7 + 2.02\times10^7 s^6 + 5.279\times10^{10} s^5 + 1.5977\times10^{13} s^4 + 1.8655\times10^{16} s^3 + 1.54455\times10^{18} s^2 = 0$$

Breakaway points: (RL) 0, -87.576



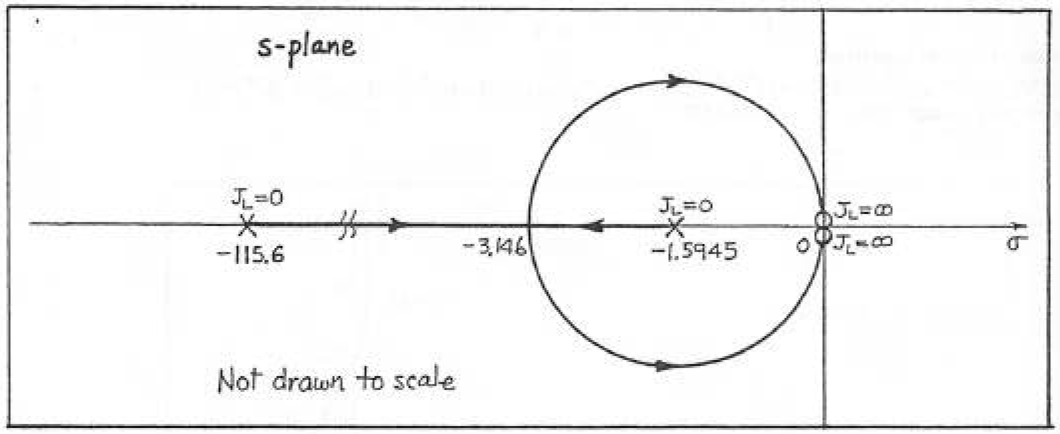
8-19 **Characteristic Equation:** $s^3 + 5000s^2 + 572{,}400s + 900{,}000 + J_L(10s^3 + 50{,}000s^2) = 0$

$P(s) = s^3 + 5000s^2 + 572{,}400s + 900{,}000 = (s+1.5945)(s+115.6)(s+4882.8)$ $\qquad Q(s) = 10s^2(s+5000)$

Since the pole at -5000 is very close to the zero at -4882.8, $P(s)$ and $Q(s)$ can be approximated as:

$P(s) \cong (s+1.5945)(s+115.6)$ $\qquad Q(s) \cong 10.24s^2$

Breakaway-point Equation: $1200s^2 + 3775s = 0$ $\qquad$ **Breakaway Points: (RL):** 0, -3.146



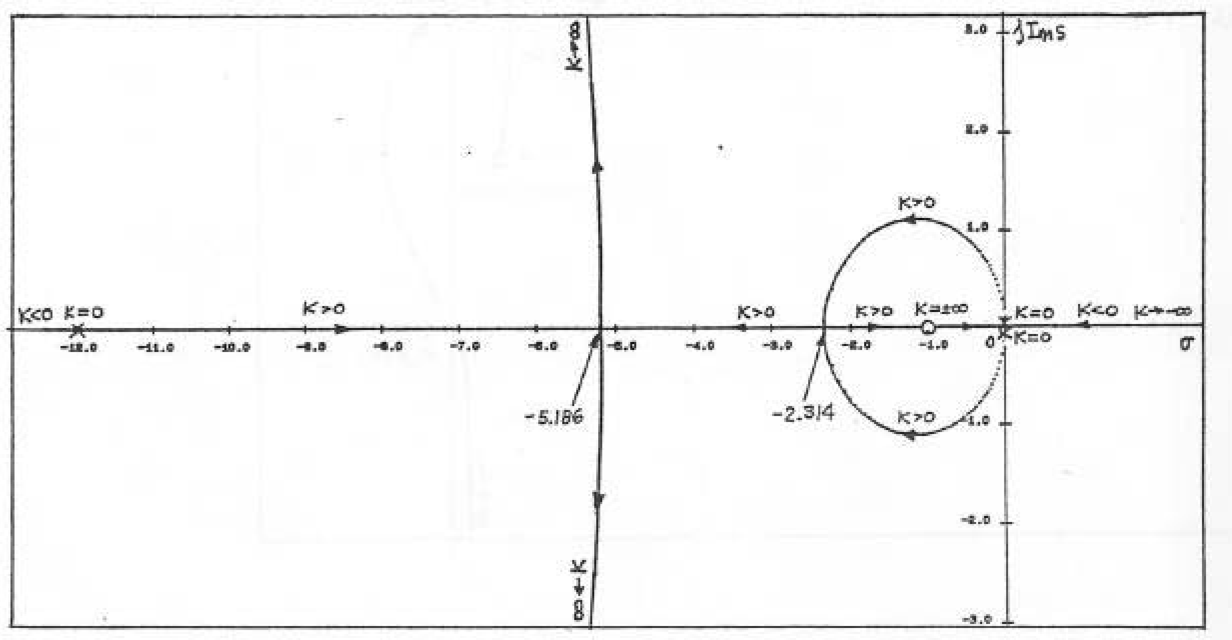
8-20 (a) $a = 12$: $\quad P(s) = s^2(s+12) \qquad Q(s) = s+1$

Asymptotes: $\quad$ ***K* > 0:** $\quad 90^o, \quad 270^o$ $\qquad\qquad$ ***K* < 0:** $\quad 0^o, \quad 180^o$

Intersect of Asymptotes:

$$\sigma_1 = \frac{0+0-12-(-1)}{3-1} = -5.5$$

Breakaway-point Equation: $\quad 2s^3 + 15s^2 + 24s = 0$ $\qquad$ **Breakaway Points:** $\quad 0, \quad -2.314, \quad -5.186$



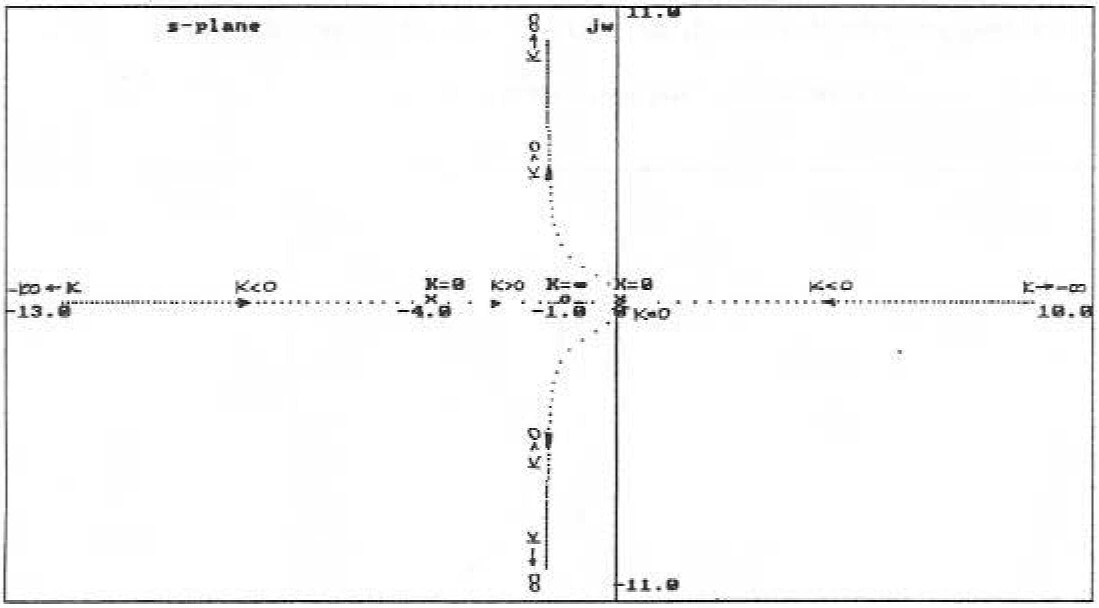
8-20 (b) $a = 4$: $\qquad P(s) = s^2(s+4) \qquad Q(s) = s+1$

Asymptotes: ***K* > 0:** $90^o, \quad 270^o$ $\quad$ ***K* < 0:** $0^o, \quad 180^o$

Intersect of Asymptotes:

$$\sigma_1 = \frac{0+0-4-(-10}{3-1} = -1.5$$

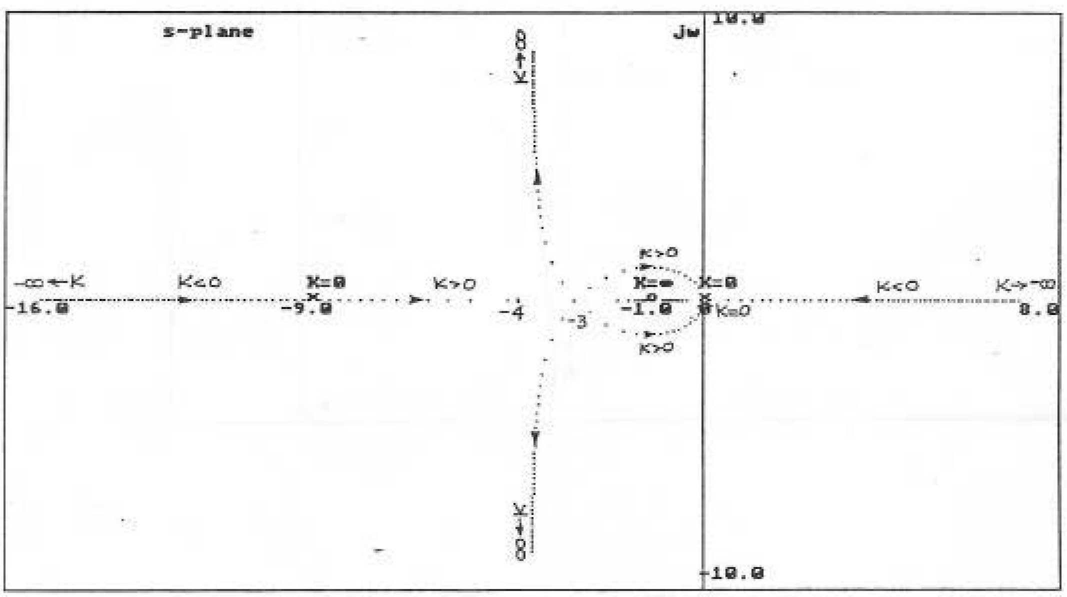
Breakaway-point Equation: $2s^3 + 7s^2 + 8s = 0$ $\qquad$ **Breakaway Points:** ***K* > 0** 0. None for ***K* < 0.**



(c) Breakaway-point Equation: $2s^2 + (a+3)s + 2s = 0$ **Solutions:** $s = -\frac{a+3}{4} \pm \frac{\sqrt{(a+3)^2 - 16a}}{4}$, $s = 0$

For one nonzero breakaway point, the quantity under the square-root sign must equal zero.

Thus, $a^2 - 10a + 9 = 0$, $a = 1$ or $a = 9$. The a nswer is $a = 9$. The $a = 1$ solution represents pole-zero cancellation in the equivalent $G(s)$. When $a = 9$, the nonzero breakaway point is at $s = -3$. $s_1 = -4$.



8-21 (a) $P(s) = s^2(s+3)$ $Q(s) = s + a$

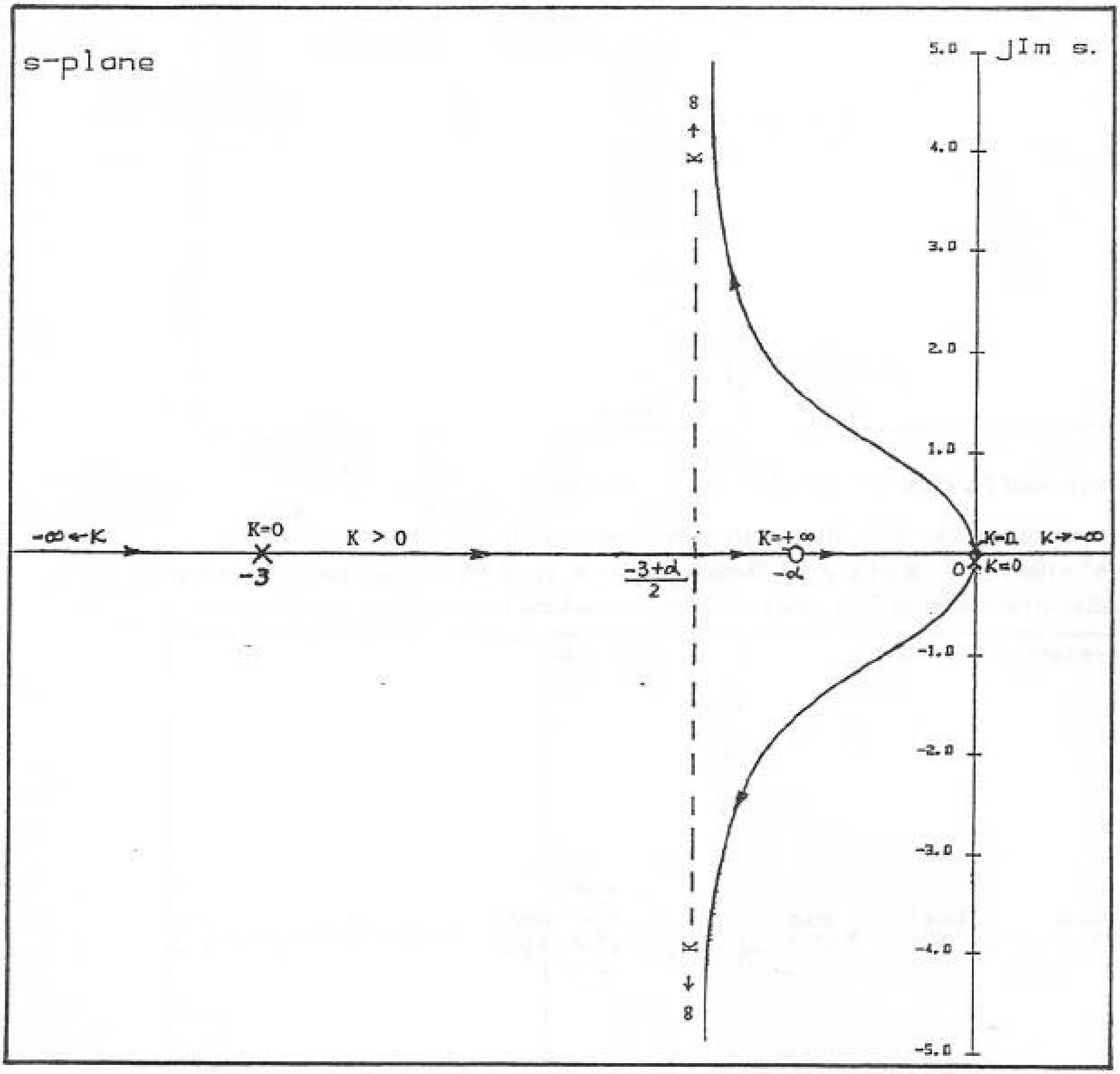
Breakaway-point Equation: $2s^3 + 3(1+a)s + 6a = 0$

The roots of the breakaway-point equation are:

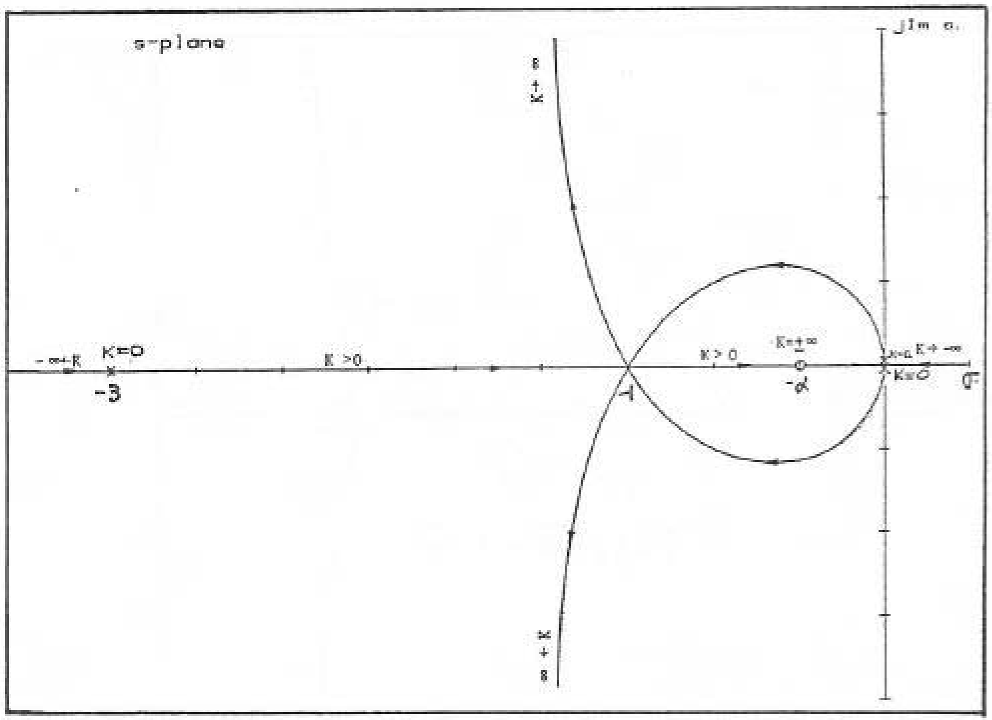
$$s = \frac{-3(1+a)}{4} \pm \frac{\sqrt{9(1+a)^2 - 48a}}{4}$$

For no breakaway point other than at $s = 0$ **, set** $9(1+a)^2 - 48a < 0$ **or** $-0.333 < a < 3$

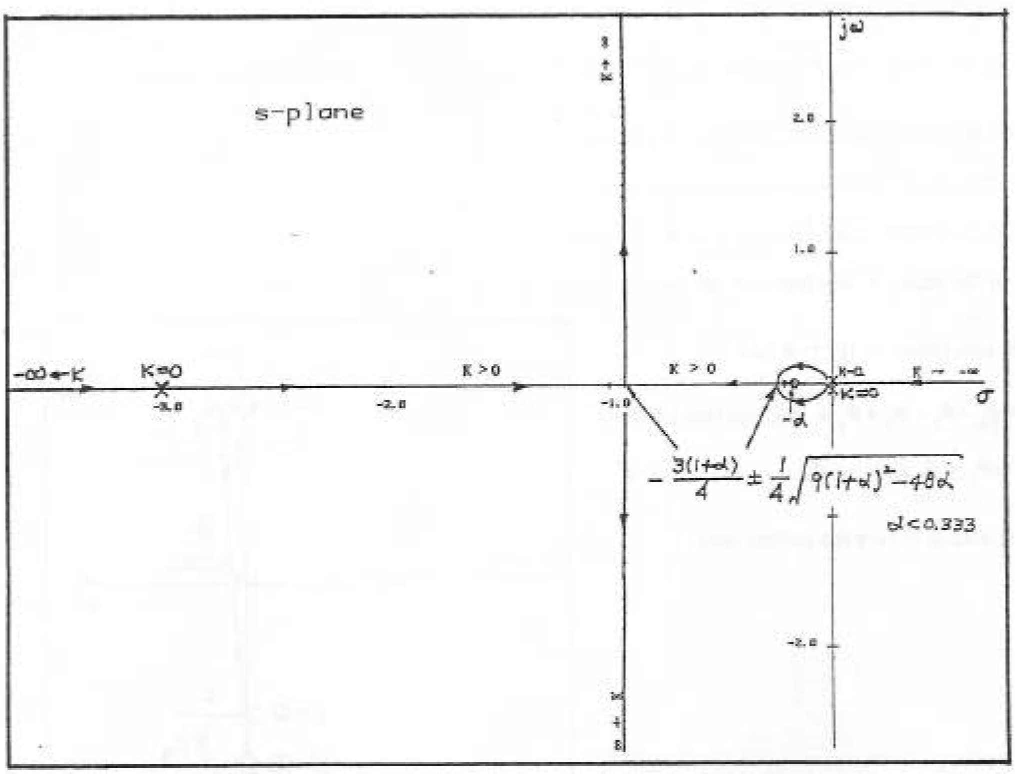
Root Locus Diagram with No Breakaway Point other than at $s = 0$**.**



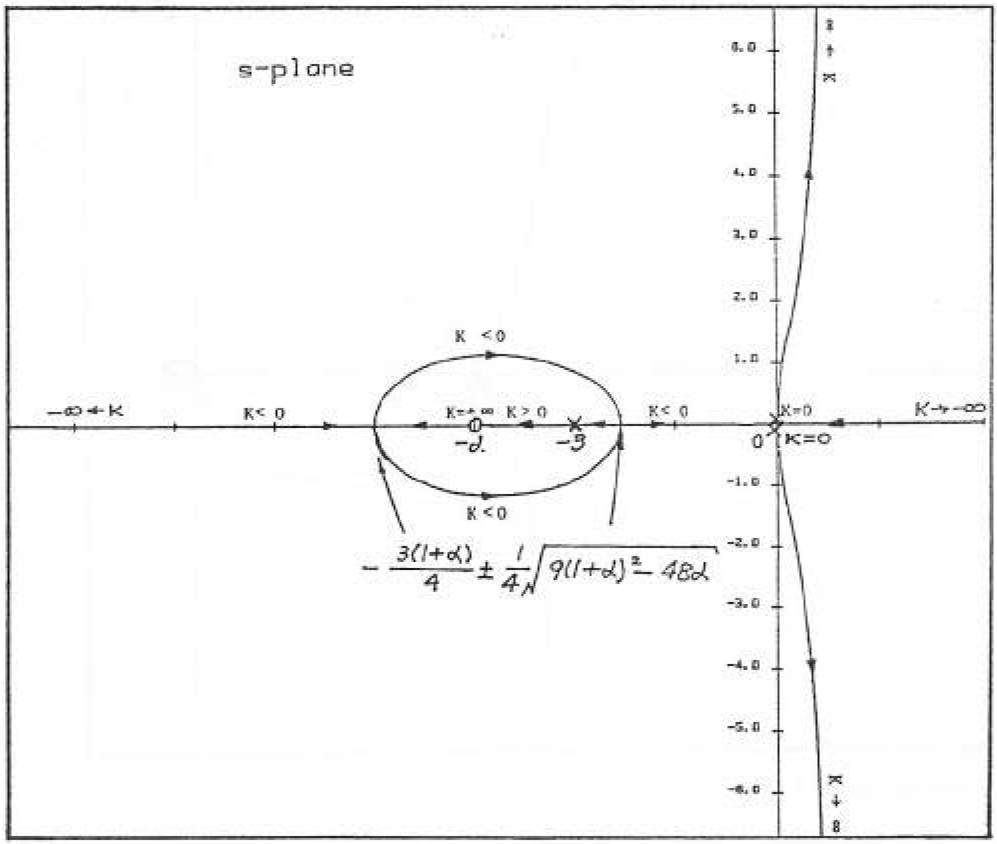
8-21 (b) One breakaway point other than at $s = 0$**:** $a = 0.333$, Breaka way po int at $s = -1$.



(c) Two breakaway points: $\alpha < 0.333$:



8-21 (d) Two breakaway points: $a > 3$:



8-22 Let the angle of the vector drawn from the zero at $s = j12$ to a point s_1 on the root locuss near the zero be q. Let

$q_1 =$ angle of the vector drawn from the pole at $j10$ to s_1.

$q_2 =$ angle of the vector drawn from the pole at 0 to s_1.

$q_3 =$ angle of the vector drawn from the pole at $-j10$ to s_1.

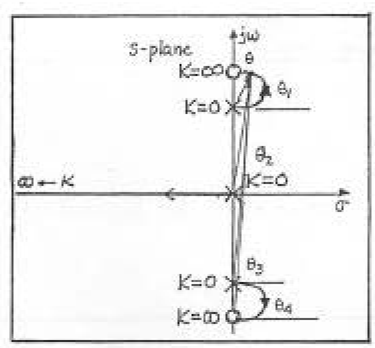
$q_4 =$ angle of the vector drawn from the zero at $-j12$ to s_1.

Then the angle conditions on the root loci are:

$$q = q_1 - q_2 - q_3 + q_4 = \text{odd multiples of } 180^o$$

$$q_1 = q_2 = q_3 = q_4 = 90^o \qquad \text{Thus,} \qquad q = 0^o$$

The root loci shown in (b) are the correct ones.



Chapter 9 FREQUENCY DOMAIN ANALYSIS

9-1 (a) $K = 5$ $w_n = \sqrt{5} = 2.24$ rad / sec $z = \dfrac{6.54}{4.48} = 1.46$ $M_r = 1$ $w_r = 0$ rad / sec

(b) $K = 21.39$ $w_n = \sqrt{21.39} = 4.62$ rad / sec $z = \dfrac{6.54}{9.24} = 0.707$ $M_r = \dfrac{1}{2z\sqrt{1-z^2}} = 1$

$$w_r = w_n\sqrt{1-V^2} = 3.27 \text{ rad / sec}$$

(c) $K = 100$ $w_n = 10$ rad / sec $z = \dfrac{6.54}{20} = 0.327$ $M_r = 1.618$ $w_r = 9.45$ rad / sec

9-2 (a) $M_r = 2.944$ (9.38 dB) $w_r = 3$ rad / sec BW = 4.495 rad / sec

(b) $M_r = 15.34$ (23.71 dB) $w_r = 4$ rad / sec BW = 6.223 rad / sec

(c) $M_r = 4.17$ (12.4 dB) $w_r = 6.25$ rad / sec BW = 9.18 rad / sec

(d) $M_r = 1$ (0 dB) $w_r = 0$ rad / sec BW = 0.46 rad / sec

(e) $M_r = 1.57$ (3.918 dB) $w_r = 0.82$ rad / sec BW = 1.12 rad / sec

(f) $M_r = \infty$ (unstable) $w_r = 1.5$ rad / sec BW = 2.44 rad / sec

(g) $M_r = 3.09$ (9.8 dB) $w_r = 1.25$ rad / sec BW = 2.07 rad / sec

(h) $M_r = 4.12$ (12.3 dB) $w_r = 3.5$ rad / sec BW = 5.16 rad / sec

9-3

Maximum overshoot = 0.1 Thus, $z = 0.59$

$$M_r = \frac{1}{2z\sqrt{1-z^2}} = 1.05 \qquad t_r = \frac{1 - 0.416\,z + 2.917\,z^2}{w_n} = 0.1 \text{ sec}$$

Thus, minimum $w_n = 17.7$ rad / sec Maximum $M_r = 1.05$

$$\text{Minimum BW} = w_n\left(\left(1-2z^2\right) + \sqrt{4z^4 - 4z^2 + 2}\right)^{1/2} = 20.56 \text{ rad/sec}$$

9-4

Maximum overshoot = 0.2 Thus, $0.2 = e^{\frac{-pz}{\sqrt{1-z^2}}}$ $z = 0.456$

$$M_r = \frac{1}{2\zeta\sqrt{1-\zeta^2}} = 1.232 \qquad t_r = \frac{1 - 0.416\zeta + 2.917\zeta^2}{\omega_n} = 0.2 \qquad \text{Thus, minimum } \omega_n = 14.168 \text{ rad/sec}$$

$$\text{Maximum } M_r = 1.232 \qquad \text{Minimum BW} = \left((1-2\zeta^2) + \sqrt{4\zeta^4 - 4\zeta^2 + 2}\right)^{1/2} = 18.7 \text{ rad/sec}$$

9-5

$$\text{Maximum overshoot} = 0.3 \qquad \text{Thus, } 0.3 = e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}} \quad \zeta = 0.358$$

$$M_r = \frac{1}{2\zeta\sqrt{1-\zeta^2}} = 1.496 \qquad t_r = \frac{1 - 0.416\zeta + 2.917\zeta^2}{\omega_n} = 0.2 \qquad \text{Thus, minimum } \omega_n = 6.1246 \text{ rad/sec}$$

$$\text{Maximum } M_r = 1.496 \qquad \text{Minimum BW} = \left((1-2\zeta^2) + \sqrt{4\zeta^4 - 4\zeta^2 + 2}\right)^{1/2} = 1.4106 \text{ rad/sec}$$

9-6

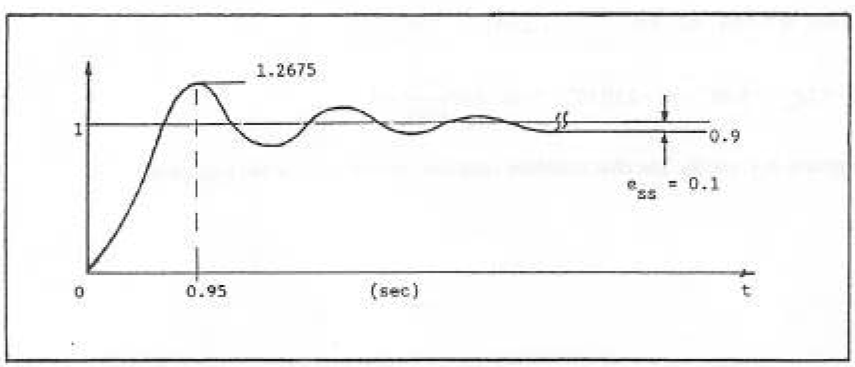
$$M_r = 1.4 = \frac{1}{2\zeta\sqrt{1-\zeta^2}} \qquad \text{Thus, } \zeta = 0.387 \qquad \text{Maximum overshoot} = e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}} = 0.2675 \quad (26.75\%)$$

$$\omega_r = 3 \text{ rad/sec} = \omega_n\sqrt{1-2\zeta^2} = 0.8367\,\omega_n \text{ rad/sec} \qquad \omega_n = \frac{3}{0.8367} = 3.586 \text{ rad/sec}$$

$$t_{max} = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{\pi}{3.586\sqrt{1-(0.387)^2}} = 0.95 \text{ sec} \qquad \text{At } \omega = 0, \quad |M| = 0.9.$$

This indicates that the steady-state value of the unit-step response is 0.9.

Unit-step Response:



9-7

T	**BW (rad/sec)**	M_r
0	1.14	1.54
0.5	1.17	1.09
1.0	1.26	1.00
2.0	1.63	1.09

3.0	1.96	1.29
4.0	2.26	1.46
5.0	2.52	1.63

9-8

T	**BW (rad/sec)**	M_r
0	1.14	1.54
0.5	1.00	2.32
1.0	0.90	2.65
2.0	0.74	2.91
3.0	0.63	3.18
4.0	0.55	3.37
5.0	0.50	3.62

9-9 (a)

$$L(s) = \frac{20}{s(1+0.1s)(1+0.5s)} \qquad P_w = 1, \quad P = 0$$

When $w = 0$: $\angle L(jw) = -90^o$ $\quad |L(jw)| = \infty$ $\qquad$ When $w = \infty$: $\angle L(jw) = -270^o$ $\quad |L(jw)| = 0$

$$L(jw) = \frac{20}{-0.6w^2 + jw\left(1-0.05w^2\right)} = \frac{20\left[-0.6w^2 - jw\left(1-0.05w^2\right)\right]}{0.36w^4 + w^2\left(1-0.05w^2\right)^2}$$

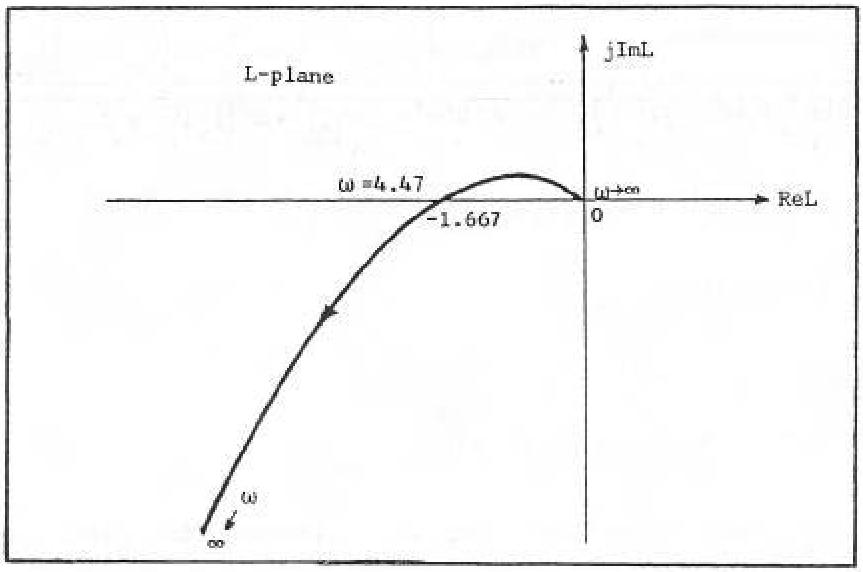
Setting $\text{Im}\left[L(jw)\right] = 0$

$1 - 0.05w^2 = 0$ $\quad$ Thus, $w = \pm 4.47$ rad / sec $\qquad L(j4.47) = -1.667$

$$\Phi_{11} = 270^o = \left(Z - 0.5P_w - P\right)180^o = \left(Z - 0.5\right)180^o \quad \text{Thus, } Z = \frac{360^o}{180^o} = 2$$

The closed-loop system is unstable. The characteristic equation has two roots in the right-half *s*-plane.

Nyquist Plot of $L(jw)$:



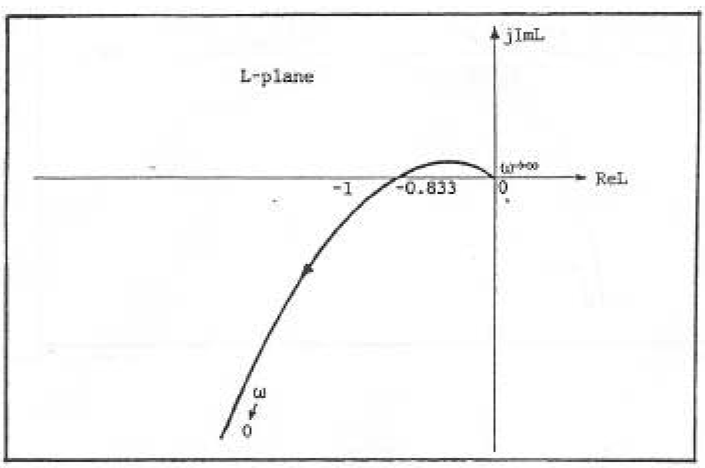
(b)

$$L(s) = \frac{10}{s(1+0.1s)(1+0.5s)}$$

Based on the analysis conducted in part (a), the intersect of the negative real axis by the $L(j\omega)$ plot is at -0.8333, and the corresponding ω is 4.47 rad/sec.

$\Phi_{11} = -90^o = (Z - 0.5P_\omega - P)180^o = 180Z - 90^o$ Thus, $Z = 0$. **The closed-loop system is stable.**

Nyquist Plot of $L(j\omega)$:



(c)

$$L(s) = \frac{100(1+s)}{s(1+0.1s)(1+0.2s)(1+0.5s)}$$

$P_\omega = 1, \quad P = 0.$

When $\omega = 0$: $\angle L(j0) = -90^o$ $|L(j0)| = \infty$ When $\omega = \infty$: $\angle L(j\infty) = -270^o$ $|L(j\infty)| = 0$

When $\omega = \infty$: $\angle L(j\omega) = -270^o$ $|L(j\omega)| = 0$ When $\omega = \infty$: $\angle L(j\omega) = -270^o$ $|L(j\omega)| = 0$

$$L(jw) = \frac{100(1+jw)}{\left(0.01w^4 - 0.8w^2\right) + jw\left(1-0.17w^2\right)} = \frac{100(1+jw)\left[\left(0.01w^4 - 0.8w^2\right) - jw\left(1-0.17w^2\right)\right]}{\left(0.01w^4 - 0.8w^2\right)^2 + w^2\left(1-0.17w^2\right)^2}$$

Setting $\text{Im}\left[L(jw)\right] = 0$ $\quad 0.01w^4 - 0.8w^2 - 1 + 0.17w^2 = 0 \quad\quad w^4 - 63w^2 - 100 = 0$

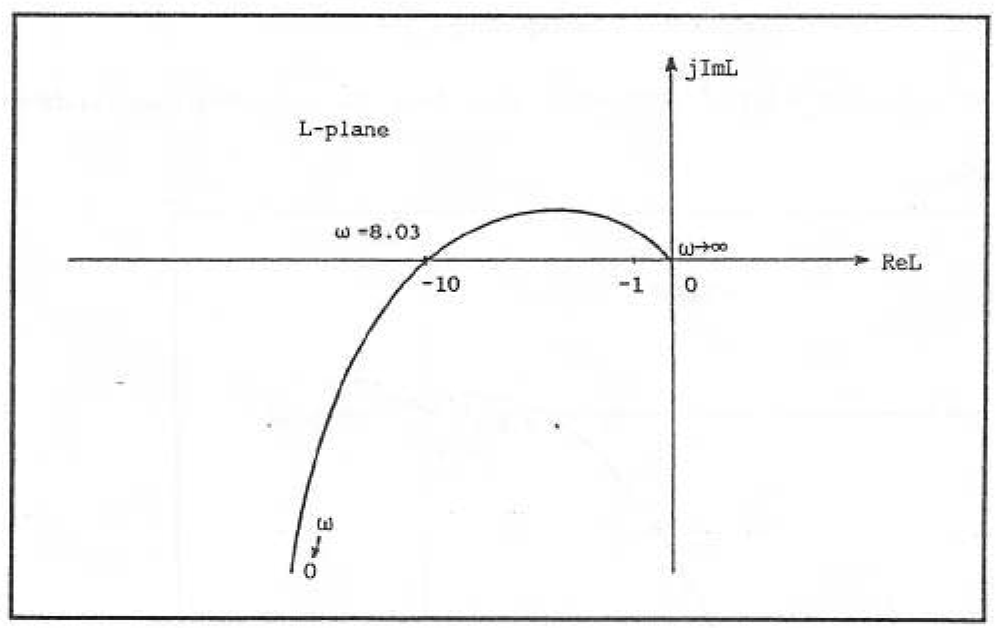
Thus, $w^2 = 64.55 \quad w = \pm 8.03$ rad/sec

$$L(j8.03) = \left|\left(\frac{100\left[\left(0.01w^4 - 0.8w^2\right) + w^2\left(1-0.17w^2\right)\right]}{\left(0.01w^2 - 0.8w^2\right)^2 + w^2\left(1-0.17w^2\right)^2}\right)\right|_{w=8.03} = -10$$

$\Phi_{11} = 270^o = \left(Z - 0.5P_w - P\right)180^o = \left(Z - 0.5\right)180^o$ Thus, $Z = 2$ **The closed-loop system is unstable.**

The characteristic equation has two roots in the right-half s-plane.

Nyquist Plot of $L(jw)$:



(d)

$$L(s) = \frac{10}{s^2(1+0.2s)(1+0.5s)} \qquad P_w = 2 \qquad P = 0$$

When $w = 0$: $\angle L(jw) = -180^o \quad |L(jw)| = \infty$ $\quad$ When $w = \infty$: $\angle L(jw) = -360^o \quad |L(jw)| = 0$

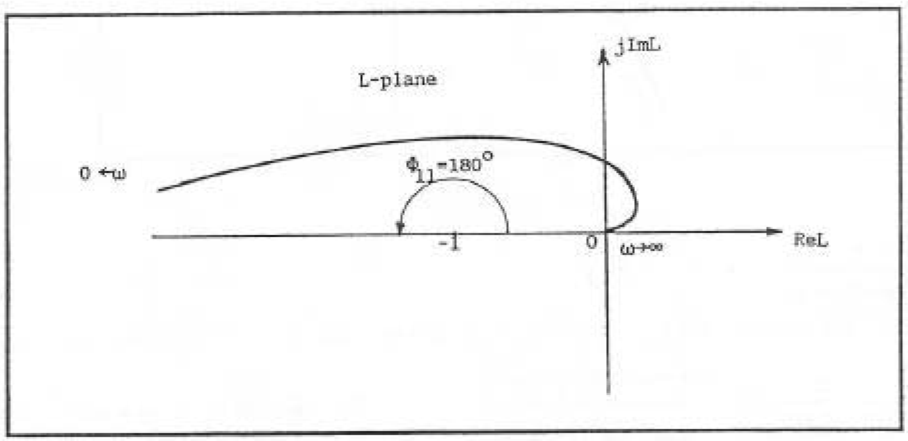
$$L(jw) = \frac{10}{\left(0.1w^4 - w^2\right) - j0.7w^3} = \frac{10\left(0.1w^4 - w^2 + j0.7w^3\right)}{\left(0.1w^4 - w^2\right)^2 + 0.49w^6}$$

Setting $\text{Im}\left[L(jw)\right] = 0$, $w = \infty$. The Nyquist plot of $L(jw)$ does not intersect the real axis except at the origin where $w = \infty$.

$$\Phi_{11} = \left(Z - 0.5P_w - P\right)180^\circ = \left(Z - 1\right)180^\circ \quad \text{Thus, } Z = 2.$$

The closed-loop system is unstable. The characteristic equation has two roots in the right-half *s*-plane.

Nyquist Plot of $L(jw)$:



9-9 (e)

$$L(s) = \frac{3(s+2)}{s\left(s^3 + 3s + 1\right)} \qquad P_w = 1 \qquad P = 2$$

When $w = 0$: $\angle L(j0) = -90^o$ $\quad \left|L(j0)\right| = \infty$ $\qquad$ When $w = \infty$: $\angle L(j\infty) = -270^o$ $\quad \left|L(j\infty)\right| = 0$

$$L(jw) = \frac{3(jw+2)}{\left(w^4 - 3w^2\right) + jw} = \frac{3(jw+2)\left[\left(w^4 - 3w^2\right) - jw\right]}{\left(4^4 - 3w^2\right)^2 + w^2}$$

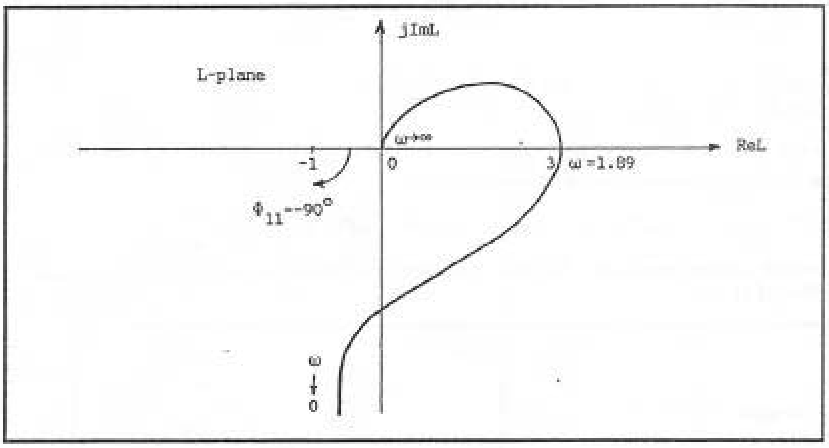
Setting $\text{Im}\left[L(jw)\right] = 0$,

$w^4 - 3w^2 - 2 = 0$ or $w^2 = 3.56$ $\quad w = \pm 1.89$ rad/sec. $L(j1.89) = 3$

$$\Phi_{11} = \left(Z - 0.5P_w - P\right)180^\circ = \left(Z - 2.5\right)180^\circ = -90^\circ \qquad \text{Thus, } Z = 2$$

The closed-loop system is unstable. The characteristic equation has two roots in the right-half *s*-plane.

Nyquist Plot of $L(jw)$:



9-9 (f)

$$L(s)=\frac{0.1}{s\left(s+1\right)\left(s^2+s+1\right)} \qquad P_w=1 \qquad P=0$$

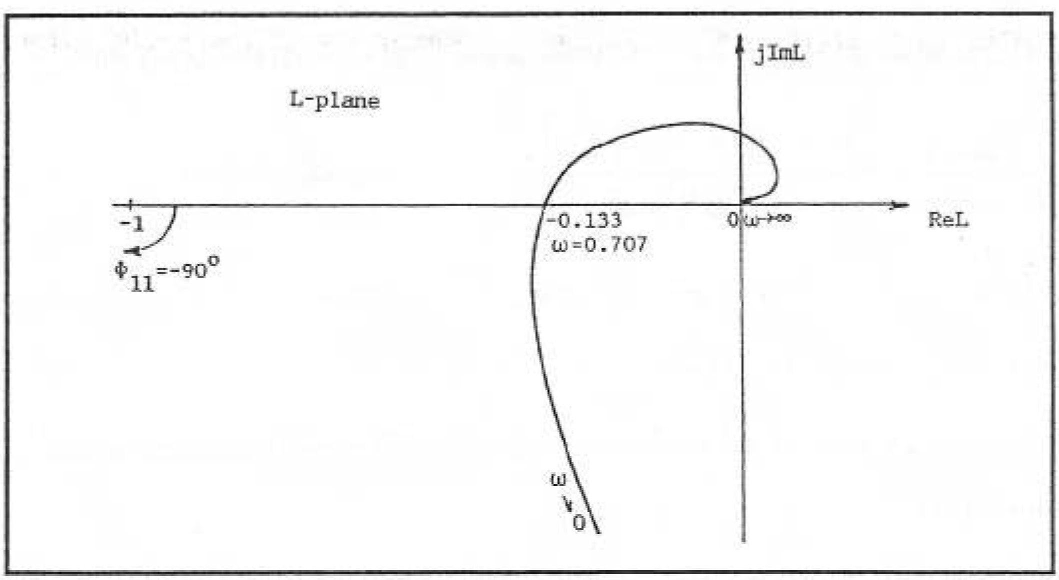
When $w=0$: $\angle L(j0)=-90^o$ $\left|L(j0)\right|=\infty$ When $w=\infty$: $\angle L(j\infty)=-360^o$ $\left|L(j\infty)\right|=0$

$$L(jw)=\frac{0.1}{\left(w^4-2w^2\right)+jw\left(1-2w^2\right)}=\frac{0.1\left[\left(w^4-2w^2\right)-jw\left(1-2w^2\right)\right]}{\left(w^4-2w^2\right)^2+w^2\left(1-2w^2\right)^2}$$ Settiing $\text{Im}\left[L(jw)\right]=0$

$w=\infty$ or $w^2=0.5$ $w=\pm 0.707$ rad / sec $L(j0.707)=-0.1333$

$\Phi_{11}=\left(Z-0.5P_w-P\right)180^\circ=\left(Z-0.5\right)180^\circ=-90^\circ$ Thus, $Z=0$ **The closed-loop system is stable.**

Nyquist Plot of $L(jw)$:



9-9 (g)

$$L(s)=\frac{100}{s\left(s+1\right)\left(s^2+2\right)} \qquad P_w=3 \qquad P=0$$

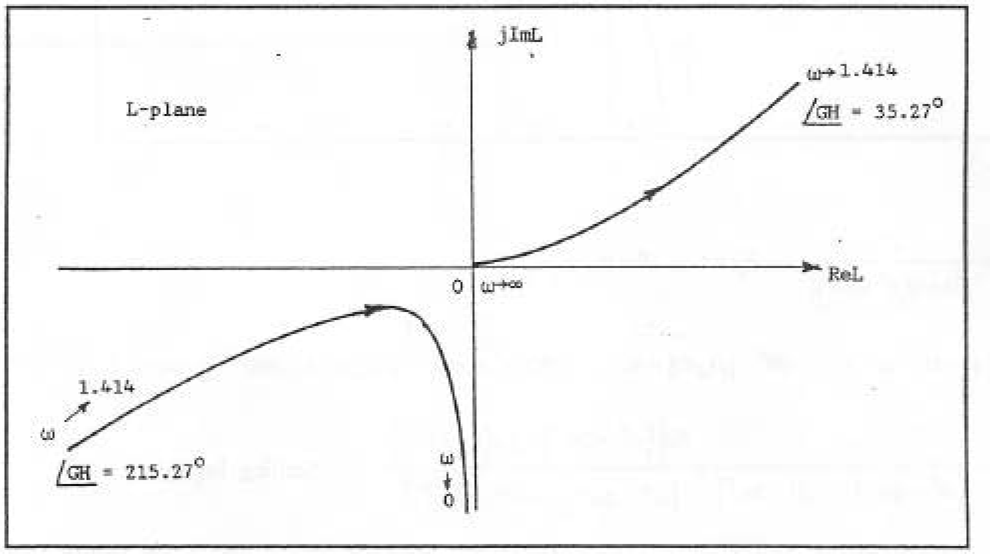
When $\omega = 0$: $\angle L(j0) = -90^o$ $|L(j0)| = \infty$ When $\omega = \infty$: $\angle L(j\infty) = -360^o$ $|L(j\infty)| = 0$

The phase of $L(j\omega)$ is discontinuous at $\omega = 1.414$ rad/sec.

$$\Phi_{11} = 35.27^o + \left(270^o - 215.27^o\right) = 90^o \quad \Phi_{11} = (Z - 1.5)180^o = 90^o \qquad \text{Thus, } P_{11} = \frac{360^o}{180^o} = 2$$

The closed-loop system is unstable. The characteristic equation has two roots in the right-half *s*-plane.

Nyquist Plot of $L(j\omega)$:



9-9 (h)

$$L(s) = \frac{10(s+10)}{s(s+1)(s+100)} \qquad P_\omega = 1 \qquad P = 0$$

When $\omega = 0$: $\angle L(j0) = -90^o$ $|L(j0)| = \infty$ When $\omega = \infty$: $\angle L(j\infty) = -180^o$ $|L(j\infty)| = 0$

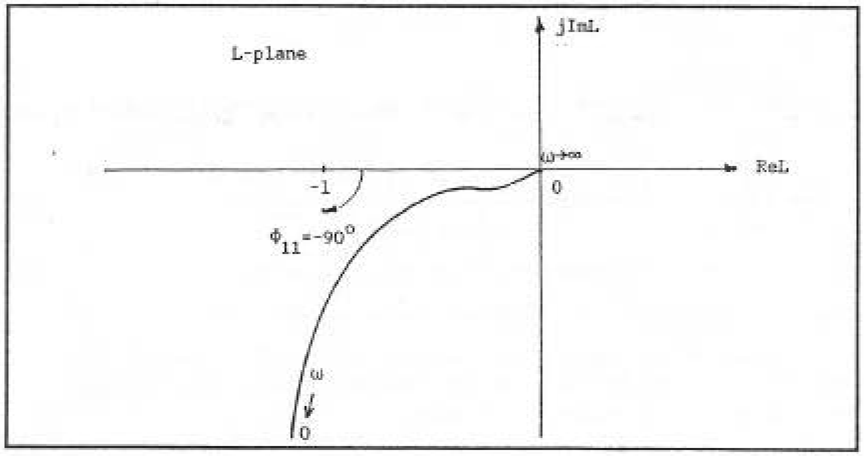
$$L(j\omega) = \frac{10(j\omega + 10)}{-101\omega^2 + j\omega(100 - \omega^2)} = \frac{10(j\omega + 10)\left[-101\omega^2 - j\omega\left(100 - \omega^2\right)\right]}{10201\omega^4 + \omega^2\left(100 - \omega^2\right)^2}$$

Setting $\text{Im}\left[L(j\omega)\right] = 0$, $\omega = 0$ is the only solution. Thus, the Nyquist plot of $L(j\omega)$ does not intersect the real axis, except at the origin.

$$\Phi_{11} = \left(Z - 0.5P_\omega - P\right)180^o = \left(Z - 0.5\right)180^o = -90^o \qquad \text{Thus, } Z = 0.$$

The closed-loop system is stable.

Nyquist Plot of $L(j\omega)$:



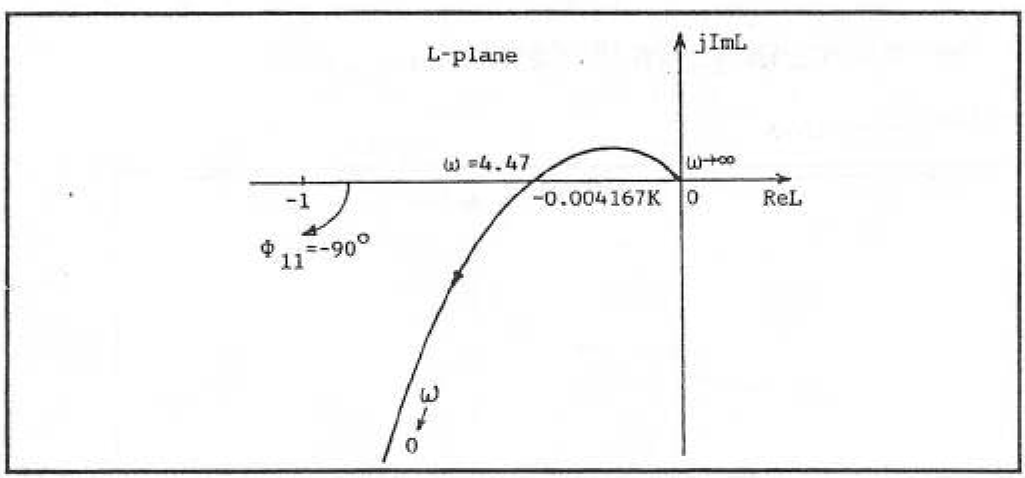
9-10 (a)

$$L(s)=\frac{K}{s(s+2)(s+10)} \qquad P_w=1 \qquad P=0$$

For stability, $Z = 0$. $\Phi_{11}=-0.5P_w\times 180^o=-90^o$ This means that the $(-1, j0)$ point must **not** be enclosed by the Nyquist plot, or

$$0<0.004167K<1. \qquad \text{Thus,} \qquad 0<K<240$$

Nyquist Plot of $L(jW)$:



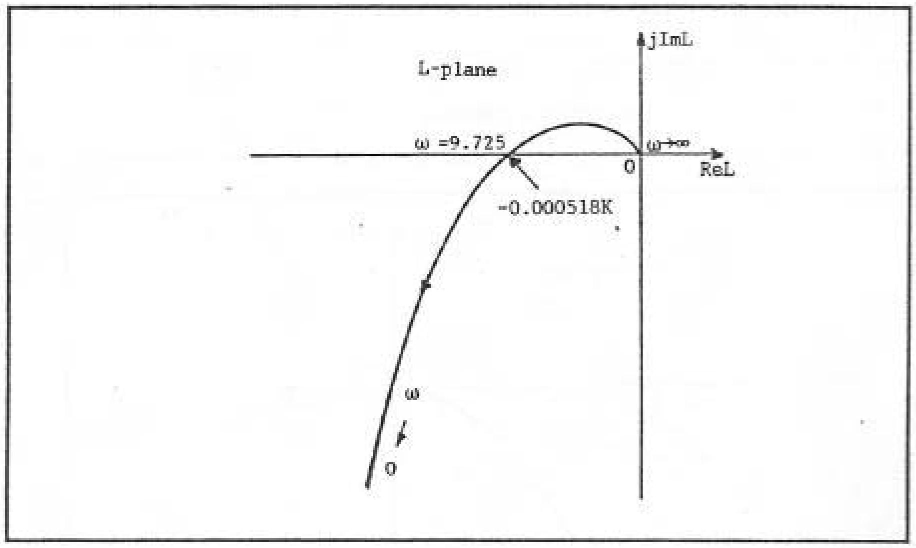
9-10 (b)

$$L(s)=\frac{K(s+1)}{s(s+2)(s+5)(s+15)} \qquad P_w=1 \qquad P=0$$

For stability, $Z = 0$. $\Phi_{11}=-0.5\times 180^o=-90^o$ This means that the $(-1, j0)$ must not be enclosed by the Nyquist plot, or

$$0<0.000517\ 9K<1 \qquad \text{Thus,} \qquad \mathbf{0<K<1930.9}$$

Nyquist Plot of $L(jW)$:



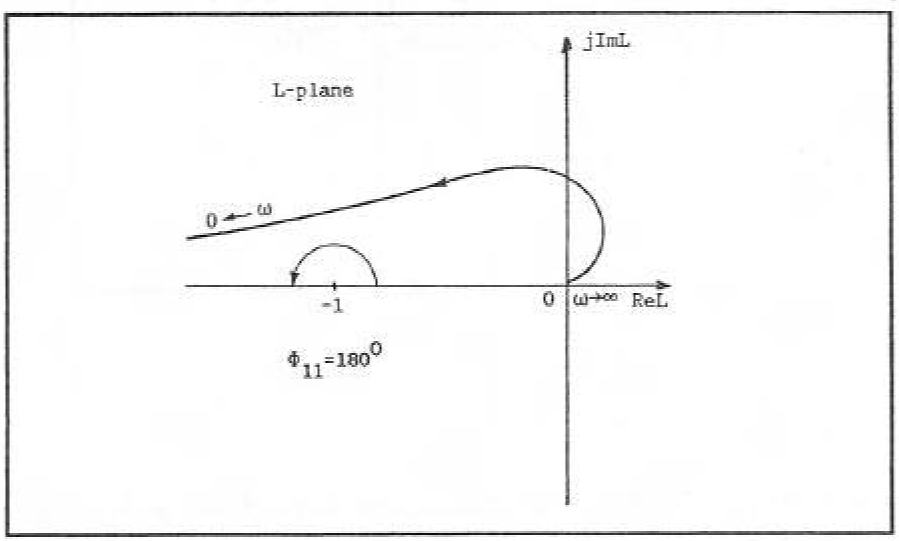
9-10 (c)

$$L(j\omega)=\frac{K}{s^2(s+2)(s+10)} \qquad P_\omega=2 \qquad P=0$$

For stability, $Z = 0$. $\Phi_{11}=-0.5P_\omega\times 180^o=-180^o$ For all $K > 0$ $\Phi_{11}=180^o$, not -180^o.

Thus, **the system is unstable for all $K > 0$.** For $K < 0$, the critical point is $(1, j0)$, $\Phi_{11}=0^o$ for all $K < 0$. **Thus, the system is unstable for all values of K.**

Nyquist Plot of $L(j\omega)$:



9-11 (a)

$$G(s)=\frac{K}{(s+5)^2} \qquad P_\omega=0 \qquad P=0$$

$\angle G(j0) = 0^o \quad (K > 0) \qquad \angle G(0) = 180^o \quad (K < 0) \qquad |G(j0)| = \frac{K}{25}$

$G(j\infty) = -180^o \quad (K > 0) \qquad \angle G(j\infty) = 0^o \quad (K < 0) \qquad |G(j\infty)| = 0$

For stability, $Z = 0$.

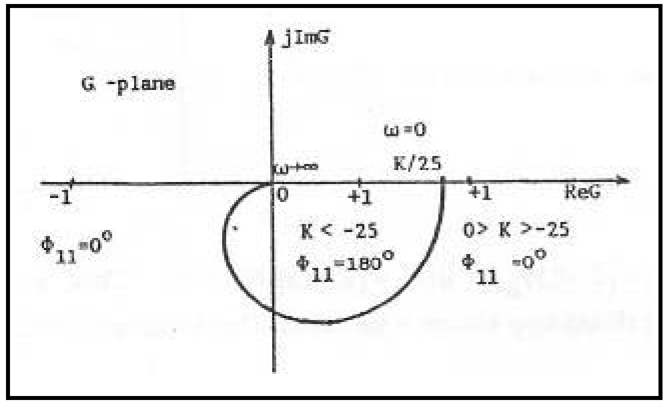
$\Phi_{11} = -(0.5P_w + P)180^o = 0^o$

$0 < K < \infty \quad \Phi_{11} = 0^o$ **Stable**

$K < -25 \quad \Phi_{11} = 180^o$ **Unstable**

$-25 < K < 0 \quad \Phi_{11} = 0^o$ **Stable**

The system is stable for $-25 < K < \infty$.



9-11 (b)

$G(s) = \frac{K}{(s+5)^3} \qquad P_w = 0 \qquad P = 0$

$\angle G(j0) = 0^o \quad (K > 0) \qquad \angle G(0) = 180^o \quad (K < 0) \qquad |G(j0)| = \frac{K}{125}$

$G(j\infty) = -270^o \quad (K > 0) \qquad \angle G(j\infty) = 270^o \quad (K < 0) \qquad |G(j\infty)| = 0$

For stability, $Z = 0$.

$\Phi_{11} = -(0.5P_w + P)180^o = 0^o$

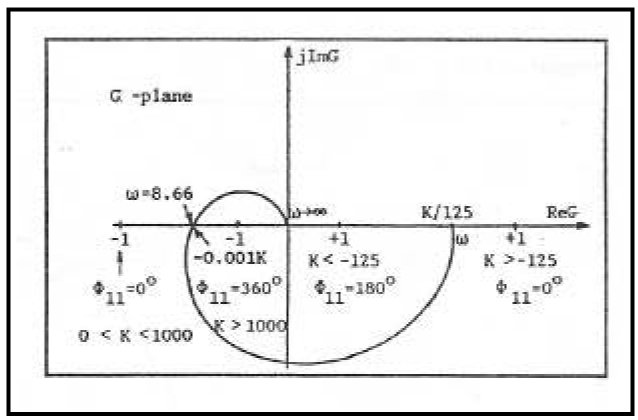
$0 < K < 1000 \quad \Phi_{11} = 0^o$ **Stable**

$K > 1000 \quad \Phi_{11} = 360^o$ **Unstable**

$K < -125 \quad \Phi_{11} = 180^o$ **Unstable**

$-125 < K < 0 \quad \Phi_{11} = 0^o$ **Stable**

The system is stable for $-125 < K < 0$.



9-11 (c)

$G(s) = \frac{K}{(s+5)^4} \qquad P_w = P = 0$

$\angle G(j0) = 0^o \quad (K > 0) \qquad \angle G(0) = 180^o \quad (K < 0) \qquad |G(j0)| = \frac{K}{625}$

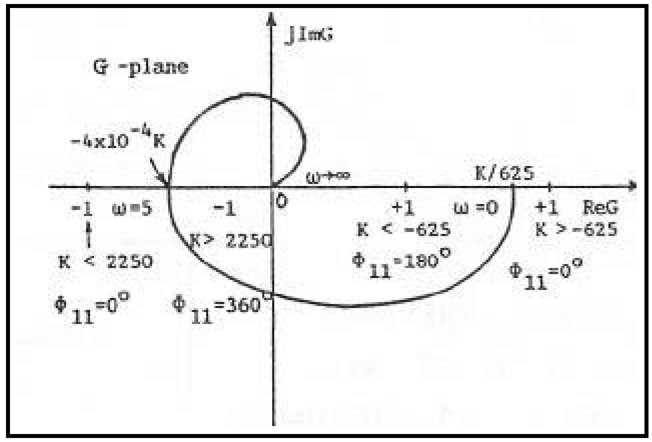
$G(j\infty) = 0^o \quad (K > 0) \qquad \angle G(j\infty) = 180^o \quad (K < 0) \qquad |G(j\infty)| = 0$

For stability, $Z = 0$.

$$\Phi_{11} = -\left(0.5P_w + P\right)180^\circ = 0^\circ$$

$0 < K < 2500$	$\Phi_{11} = 0^o$	**Stable**
$K > 2500$	$\Phi_{11} = 360^o$	**Unstable**
$K < -625$	$\Phi_{11} = 180^o$	**Unstable**
$-625 < K < 0$	$\Phi_{11} = 0^o$	**Stable**

The system is stable for $-625 < K < 2500$.



9-12

$$s\left(s^3 + 2s^2 + s + 1\right) + K\left(s^2 + s + 1\right) = 0$$

$$L_{eq}(s) = \frac{K\left(s^2 + s + 1\right)}{s\left(s^3 + 2s^2 + s + 1\right)} \qquad P_w = 1 \quad P = 0 \quad L_{eq}(j0) = \infty\angle -90^o \quad L_{eq}(j\infty) = 0\angle 180^o$$

$$L_{eq}(jw) = \frac{K\left[\left(1 - w^2\right) + jw\right]}{\left(w^4 - w^2\right) + jw\left(1 - 2w^2\right)} = \frac{K\left[-\left(w^6 + w^4\right) - jw\left(w^4 - 2w^2 + 1\right)\right]}{\left(w^4 - w^2\right)^2 + w^2\left(1 - 2w^2\right)^2}$$

Setting $\text{Im}\left[L_{eq}(jw)\right] = 0$

$$w^4 - 2w^2 + 1 = 0$$

Thus, $w = \pm 1$ rad/sec are the real solutions.

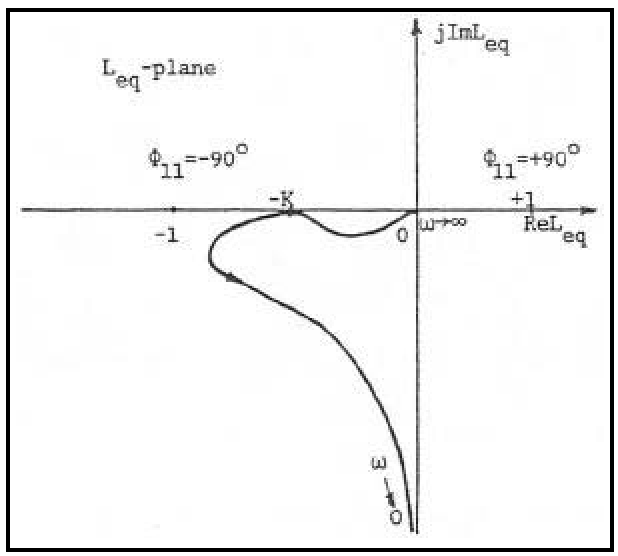
$L_{eq}(j1) = -K$

For stability,

$$\Phi_{11} = -(0.5P_w + P)180^o = -90^o$$

When $K = 1$ the system is marginally stable.

$K > 0$	$\Phi_{11} = -90^o$	**Stable**
$K < 0$	$\Phi_{11} = +90^o$	**Unstable**



Routh Tabulation

s^4	1	$K+1$	K
s^3	2	$K+1$	
s^2	$\dfrac{K+1}{2}$	K	$K > -1$
s^1	$\dfrac{K^2-2K+1}{K+1} = \dfrac{(K-1)^2}{K+1}$		
s^0	K		$K > 0$

When $K = 1$ the coefficients of the s^1 row are all zero. The auxiliary equation is $s^2 + 1 = 0$ The solutions are $w = \pm 1$ rad/sec. Thus the Nyquist plot of $L_{eq}(jw)$ intersects the -1 point when $K = 1$, when $w = \pm 1$ rad/sec. **The system is stable for 0 < K < ∞, except at K = 1.**

9-13

Parabolic error constant $K_a = \lim_{s\to 0} s^2 G(s) = \lim_{s\to 0} 10\left(K_P + K_D s\right) = 10K_P = 100$ Thus $K_P = 10$

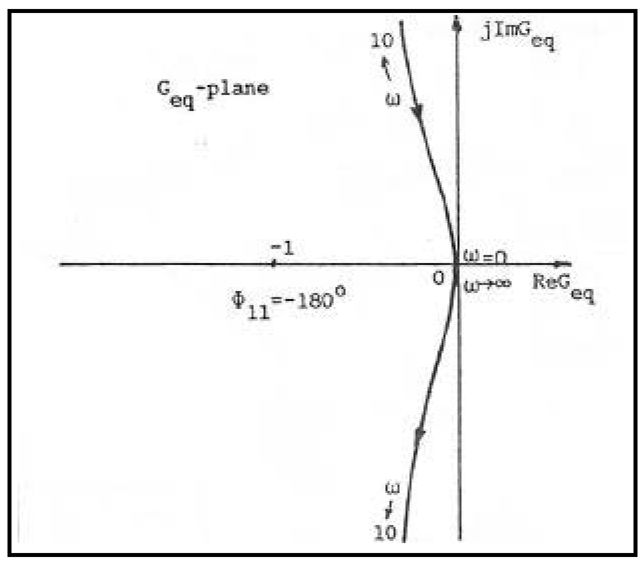
Characteristic Equation: $s^2 + 10K_D s + 100 = 0$

$G_{eq}(s) = \dfrac{10K_D s}{s^2 + 100}$ $\quad P_w = 2 \quad P = 0$

For stability,

$\Phi_{11} = -(0.5P_w + P)180^o = -180^o$

The system is stable for $0 < K_D < \infty$**.**



9-14 (a)

The characteristic equation is

$$1 + G(s) - G(s) - 2\left[G(s)\right]^2 = 1 - 2\left[G(s)\right]^2 = 0$$

$$G_{eq}(s) = -2\left[G(s)\right]^2 = \frac{-2K^2}{(s+4)^2(s+5)^2} \qquad P_w = 0 \qquad P = 0$$

$$G_{eq}(jw) = \frac{-2K^2}{\left(400 - 120w^2 + w^4\right) + jw\left(360 - 18w^2\right)} = \frac{-2K^2\left[\left(400 - 120w^2 + w^2\right) - jw\left(360 - 18w^2\right)\right]}{\left(400 - 120w^2 + w^2\right) + w^2\left(360 - 18w^2\right)^2}$$

$$G_{eq}(j0) = \frac{K^2}{200}\angle 180^o \qquad G_{eq}(j\infty) = 0\angle 180^o \qquad \text{Setting } \operatorname{Im}\left[G_{eq}(jw)\right] = 0$$

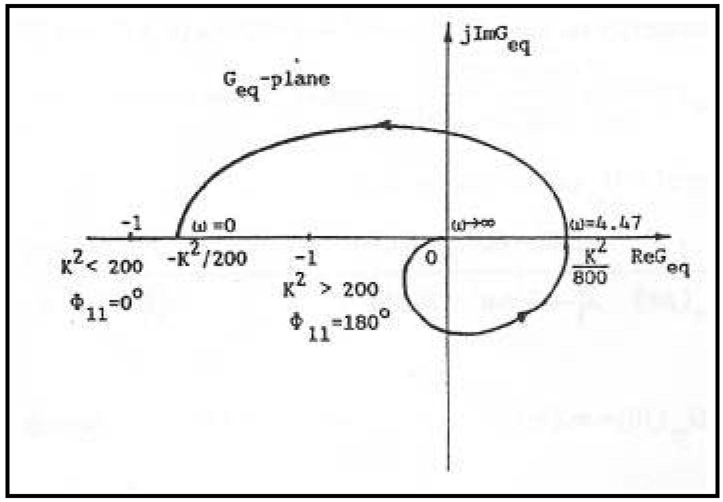
$\omega = 0$ and $\omega = \pm 4.47$ rad / sec $\quad G_{eq}(j4.47) = \dfrac{K^2}{800}$

For stability,

$$\Phi_{11} = -\left(0.5P_\omega + P\right)180^o = 0^o$$

The system is stable for $K^2 < 200$

or $|K| < \sqrt{200}$



Characteristic Equation:
$s^4 + 18s^3 + 121s^2 + 360s + 400 - 2K^2 = 0$

Routh Tabulation

s^4	1	121	$400 - 2K^2$
s^3	18	360	
s^2	101	$400 - 2K^2$	
s^1	$\dfrac{29160 - 36K^2}{101}$		$29160 + 36K^2 > 0$
s^0	$400 - 2K^2$		$K^2 < 200$

Thus for stability, $|K| < \sqrt{200}$

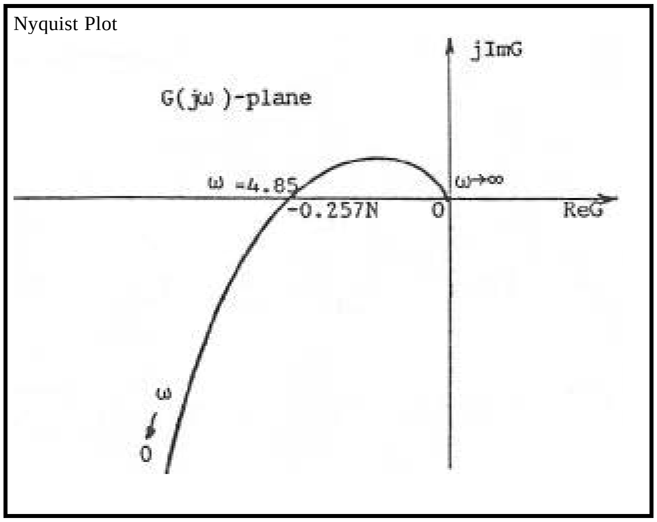
9-15 (a)

$$G(s) = \frac{83.33N}{s(s+2)(s+11.767)}$$

For stability, $N < 3.89$

Thus $N < 3$ since N must be an integer.

Nyquist Plot



(b)

$$G(s) = \frac{2500}{s(0.06s + 0.706)(As + 100)}$$

Characteristic Equation: $0.06As^3 + (6 + 0.706A)s^2 + 70.6s + 2500 = 0$

$G_{eq}(s) = \dfrac{As^2(0.06s + 0.706)}{6s^2 + 70.6s + 2500}$ Since $G_{eq}(s)$ has more zeros than poles, we should sketch the Nyquist plot of $1/G_{eq}(s)$ for stability study.

$$\frac{1}{G_{eq}(jw)} = \frac{\left(2500 - 6w^2\right) + j70.6w}{A\left(-0.706w^2 - j0.06w^3\right)} = \frac{\left[\left(2500 - 6w^2\right) + j70.6w\right]\left(-0.706w^2 + j0.06w^3\right)}{A\left(0.498w^4 + 0.0036w^6\right)}$$

$1/G_{eq}(j0) = \infty\angle -180^o$ $\quad 1/G_{eq}(j\infty) = 0\angle -90^o$ $\quad$ Setting $\mathrm{Im}\left[\dfrac{1}{G_{eq}(jw)}\right] = 0$

$100.156 - 0.36w^2 = 0 \quad w = \pm 16.68$ rad/sec $\quad \dfrac{1}{G_{eq}(j16.68)} = \dfrac{-4.23}{A}$

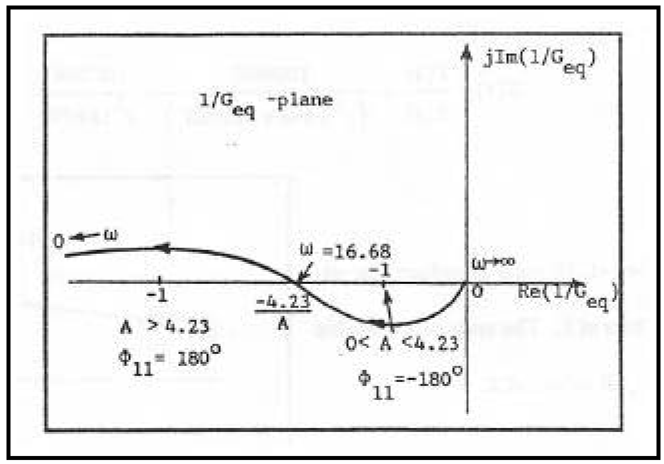
For stability,

$$\Phi_{11} = -\left(0.5P_w + P\right)180^o = -180^o$$

For $A > 4.23$ $\quad \Phi_{11} = 180^o$ $\quad$ **Unstable**

For $0 < A < 4.23$ $\quad \Phi_{11} = -180^o$ $\quad$ **Stable**

The system is stable for $0 < A < 4.23$.



(c)

$$G(s) = \frac{2500}{s(0.06s + 0.706)(50s + K_o)}$$

Characteristic Equation: $\quad s(0.06s + 0.706)(50s + K_o) + 2500 = 0$

$G_{eq}(s) = \dfrac{K_o s(0.06s + 0.706)}{3s^3 + 35.3s^2 + 2500}$ $\quad P_w = 0 \quad P = 0 \quad G_{eq}(j0) = 0\angle 90^o \quad G_{eq}(j\infty) = 0\angle -90^o$

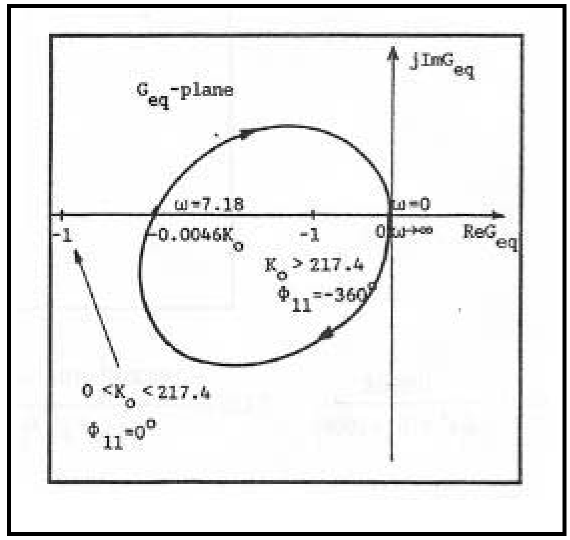
$$G_{eq}(jw) = \frac{K_o(-0.06w^3 + 0.706jw)}{\left(2500 - 35.3w^2\right) - j3w^3} = \frac{K_o\left(-0.06w^2 + 0.706jw\right)\left[\left(2500 - 35.5w^2\right) + j3w^3\right]}{\left(2500 - 35.3w^2\right)^2 + 9w^6}$$

Setting $\mathrm{Im}\left[G_{eq}(jw)\right] = 0 \quad w^4 + 138.45w^2 - 9805.55 = 0 \quad w^2 = 51.6 \quad w = \pm 7.18$ rad/sec

$G_{eq}(j7.18) = -0.004K_o$

For stability, $\Phi_{11} = -\left(0.5P_w + P\right)180^o = 0^o$

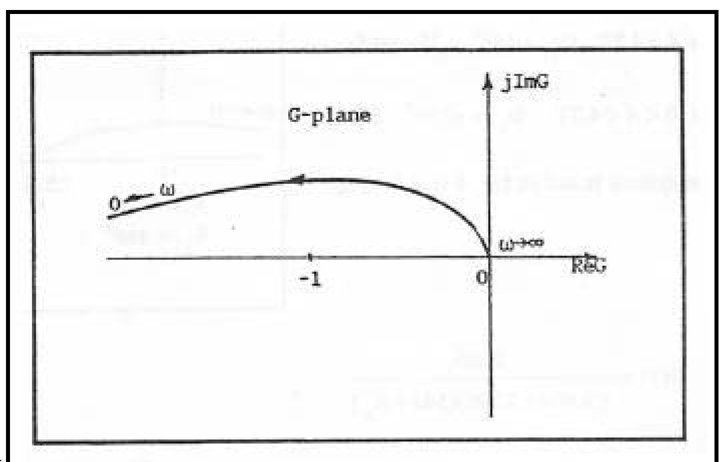
For stability, $0 < K_o < 217.4$



9-16 (a) $K_t = 0$:

$$G(s) = \frac{Y(s)}{E(s)} = \frac{10000K}{s\left(s^2 + 10s + 10000K_t\right)} = \frac{10000K}{s^2(s+10)}$$

The $(-1, j0)$ point is enclosed for all values of K. **The system is unstable for all values of *K*.**



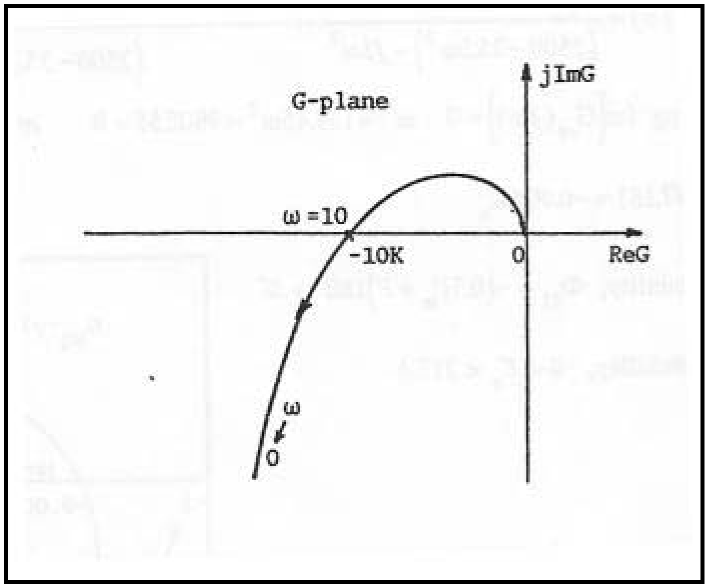
(b) $K_t = 0.01$:

$$G(s) = \frac{10000K}{s\left(s^2 + 10s + 100\right)} \qquad G(jw) = \frac{10000K\left[-10w^2 - jw\left(100 - w^2\right)\right]}{100w^4 + w^2\left(100 - w^2\right)^2}$$

Setting $\text{Im}\left[G(jw)\right] = 0 \quad w^2 = 100$

$w = \pm 10$ rad / sec $\quad G(j10) = -10K$

The system is stable for $0 < K < 0.1$

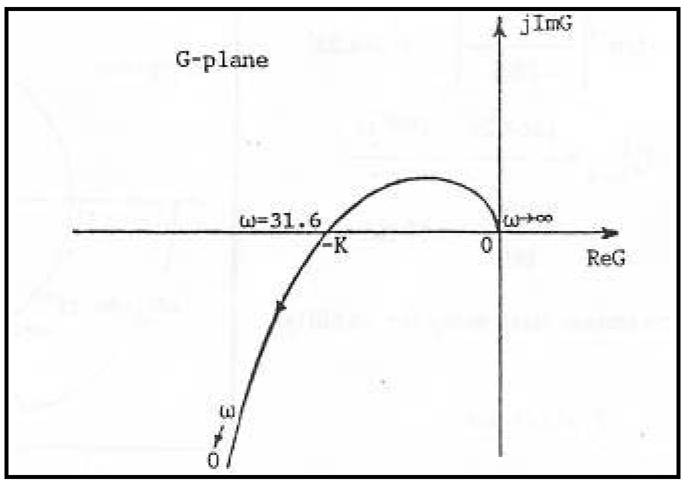


(c) $K_t = 0.1$:

$$G(s) = \frac{10000K}{s\left(s^2 + 10s + 1000\right)} \qquad G(jw) = \frac{10000K\left[-10w^2 - jw\left(1000 - w^2\right)\right]}{100w^4 + w^2\left(1000 - w^2\right)^2}$$

Setting $\text{Im}\left[G(jw)\right] = 0 \quad w^2 = 100 \quad w = \pm 31.6$ rad/sec $\quad G(j31.6) = -K$

For stability, $0 < K < 1$



9-17 The characteristic equation for $K = 10$ is:

$$s^3 + 10s^2 + 10{,}000K_t s + 100{,}000 = 0$$

$$G_{eq}(s) = \frac{10{,}000K_t s}{s^3 + 10s^2 + 100{,}000} \qquad P_w = 0 \qquad P = 2$$

$$G_{eq}(jw) = \frac{10{,}000K_t jw}{100{,}000 - 10w^2 - jw^3} = \frac{10{,}000K_t\left[-w^4 + jw\left(10{,}000 - 10w^2\right)\right]}{\left(10{,}000 - 10w^2\right)^2 + w^6} \qquad \text{Setting } \text{Im}\left[G_{eq}(jw)\right] = 0$$

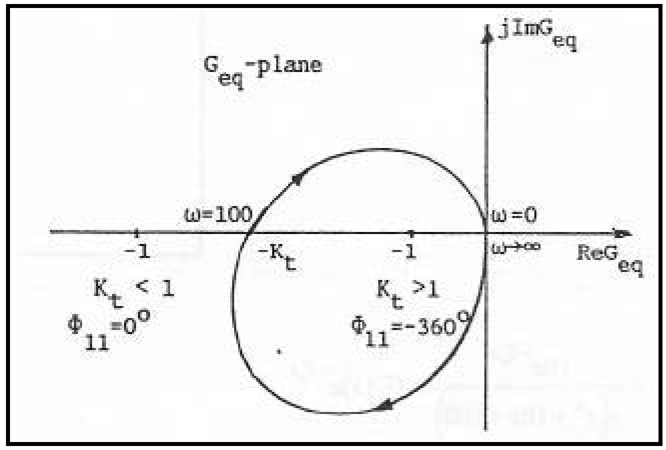
$w = 0, \quad w^2 = 10{,}000$

$w = \pm 100 \ \text{rad / sec} \qquad G_{eq}(j100) = -K_t$

For stability,

$$\Phi_{11} = -\left(0.5P_w + P\right)180^o = -360^o$$

The system is stable for $K_t > 0$.



9-18 (a)

Let $G(s) = G_1(s)e^{-T_d s}$ Then

$$G_1(s) = \frac{100}{s\left(s^2 + 10s + 100\right)}$$

Let $$\left|\frac{100}{-10w^2 + jw\left(100 - w^2\right)}\right| = 1 \quad \text{or} \quad \frac{100}{\left[100w^4 + w^2\left(100 - w^2\right)^2\right]^{1/2}} = 1$$

Thus $$100w^4 + w^2\left(100 - w^2\right)^2 = 10{,}000 \qquad w^6 - 100w^4 + 10{,}000w^2 - 10{,}000 = 0$$

The real solution for w are $w = \pm 1$ rad/sec.

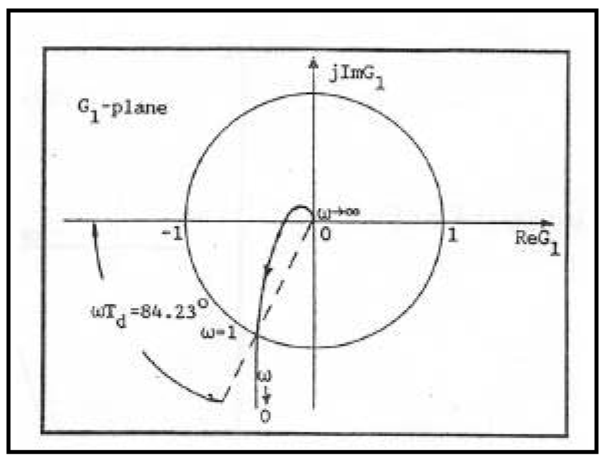
$$\angle G_1(j1) = -\tan^{-1}\left[\frac{100 - w^2}{-10w}\right]\Bigg|_{w=1} = 264.23^o$$

Equating $$wT_d\big|_{w=1} = \frac{\left(264.23^o - 180^o\right)p}{180}$$

$$= \frac{84.23\ p}{180} = 1.47 \ \text{rad}$$

Thus the maximum time delay for stability is

$T_d = 1.47$ **sec.**



(b) $T_d = 1$ sec.

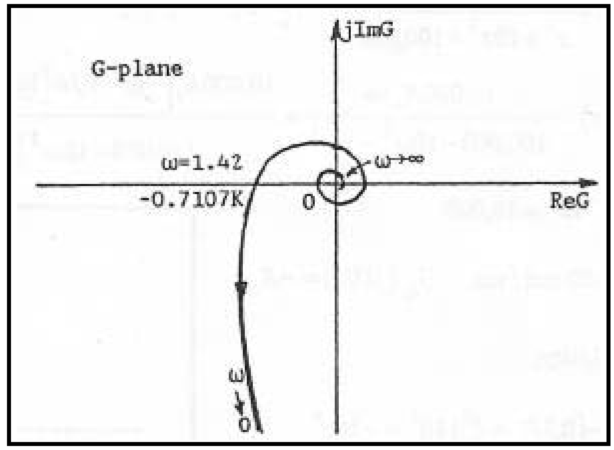
$$G(s) = \frac{100Ke^{-s}}{s\left(s^2 + 10s + 100\right)} \qquad G(jw) = \frac{100Ke^{-jw}}{-10w^2 + jw\left(100 - w^2\right)}$$

At the intersect on the negative real axis, $w = 1.42$ rad/sec.

$G(j1.42) = -0.7107\ K.$

The system is stable for

$\mathbf{0 < K < 1.407}$



9-19 (a) $K = 0.1$

$$G(s) = \frac{10e^{-T_d s}}{s\left(s^2 + 10s + 100\right)} = G_1(s)e^{-T_d s}$$

Let $$\left|\frac{10}{-10w^2 + jw\left(100 - w^2\right)}\right| = 1 \qquad \text{or} \qquad \frac{10}{\left[100w^4 + w^2\left(100 - w^2\right)^2\right]^{1/2}} = 1$$

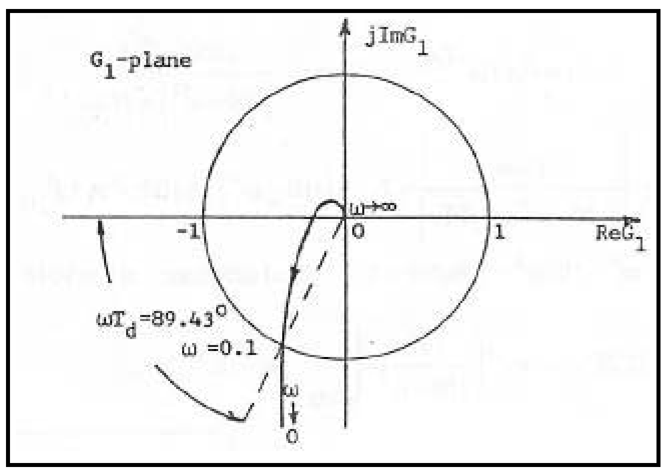
Thus $\quad w^6 - 100w^4 + 10{,}000w^2 - 100 = 0 \quad$ The real solutions for w is $w = \pm 0.1$ rad/sec.

$$\angle G_1(j0.1) = -\tan^{-1}\left[\frac{100 - w^2}{-10w}\right]\Bigg|_{w=0.1} = 269.43^o$$

Equate $\quad wT_d\big|_{w=0.1} = \dfrac{\left(269.43^o - 180^o\right)p}{180^o} = 1.56$ rad $\qquad$ We have $\quad T_d = 15.6$ sec.

We have the maximum time delay

for stability is 15.6 sec.



9-19 (b) $T_d = 0.1$ sec.

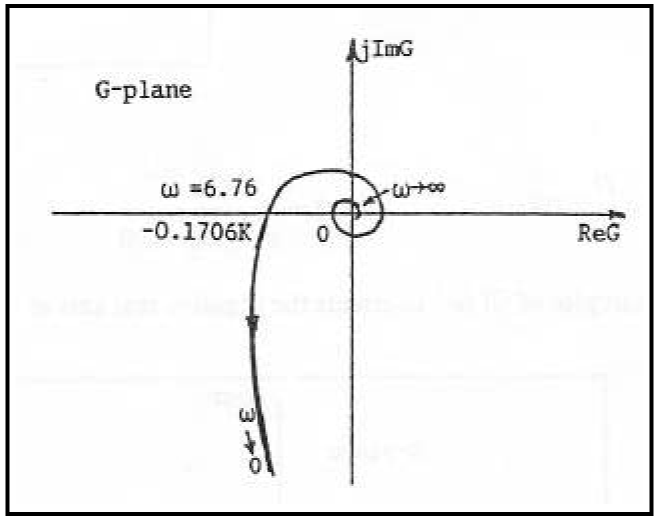
$$G(s) = \frac{100Ke^{-0.1s}}{s\left(s^2 + 10s + 100\right)} \qquad G(jw) = \frac{100Ke^{-0.1jw}}{-10w^2 + jw\left(100 - w^2\right)}$$

At the intersect on the negative real axis,

$w = 6.76$ rad/sec. $\quad G(j6.76) = -0.1706\ K$

The system is stable for

$$0 < K < 5.86$$



9-20 (a) The transfer function (gain) for the sensor-amplifier combination is 10 V/0.1 in = 100 V/in. The velocity of flow of the solution is

$$v = \frac{10 \text{ in}^3/\text{sec}}{0.1 \text{ in}} = 100 \quad \text{in/sec}$$

The time delay between the valve and the sensor is $T_d = D/v$ sec. The loop transfer function is

$$G(s) = \frac{100Ke^{-T_d s}}{s^2 + 10s + 100}$$

(b) $K = 10$:

$$G(s) = G_1(s)e^{-T_d s} \qquad G(j\omega) = \frac{1000e^{-j\omega T_d}}{\left(100-\omega^2\right)+j10\omega}$$

Setting $\left|\dfrac{1000}{\left(100-\omega^2\right)+j10\omega}\right| = 1 \qquad \left(100-\omega^2\right)^2 + 100\omega^2 = 10^6$

Thus, $\omega^4 - 100\omega^2 - 990{,}000 = 0$ Real solutions: $\omega^2 = 1046.2 \qquad \omega = 32.35$ rad/sec

$$\angle G_1(j32.25) = -\tan^{-1}\left(\frac{10\omega}{100-\omega^2}\right)\Bigg|_{\omega=32.25} = -161^o$$

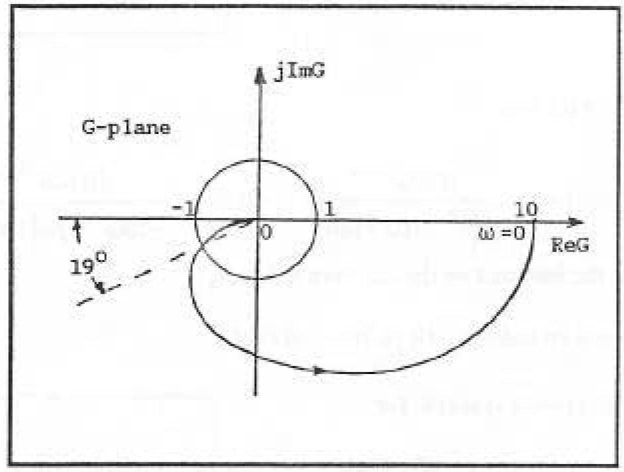
Thus,

$$32.35T_d = \frac{19^o \pi}{180^o} = 0.33 \text{ rad}$$

Thus,

$$T_d = 0.0103 \text{ sec}$$

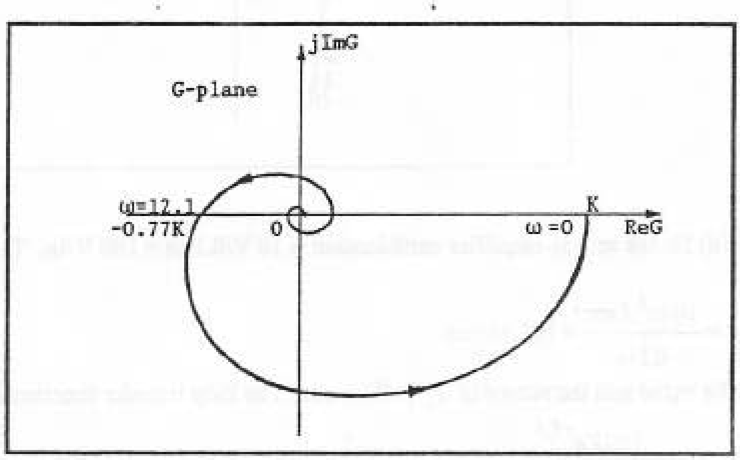
Maximum $D = vT_d = 100 \times 0.0103 = 1.03$ in



(c) $D = 10$ **in.**

$$T_d = \frac{D}{v} = 0.1 \text{ sec} \qquad G(s) = \frac{100Ke^{-0.1s}}{s^2+10s+100}$$

The Nyquist plot of $G(j\omega)$ intersects the negative real axis at $\omega = 12.1$ rad/sec.



9-21 (a) The transfer function (gain) for the sensor-amplifier combination is 1 V/0.1 in = 10 V/in. The velocity of flow of the solutions is

$$v = \frac{10 \ \text{in}^3/\text{sec}}{0.1 \ \text{in}} = 100 \ \text{in}/\text{sec}$$

The time delay between the valve and sensor is $T_d = D/v$ sec. The loop transfer function is

$$G(s) = \frac{10Ke^{-T_d s}}{s^2 + 10s + 100}$$

(b) ***K* = 10:**

$$G(s) = G_1(s)e^{-T_d s} \qquad G(jw) = \frac{100e^{-jwT_d}}{\left(100 - w^2\right) + j10w}$$

Setting $\left|\frac{100}{\left(100 - w^2\right) + j10w}\right| = 1 \qquad \left(100 - w^2\right)^2 + 100w^2 = 10{,}000$

Thus, $w^4 - 100w^2 = 0$ Real solutions: $w = 0$, $w = \pm 10$ rad/sec

$$\angle G_1(j10) = -\tan^{-1}\left(\frac{10w}{100 - w^2}\right)\Bigg|_{w=10} = -90^o$$

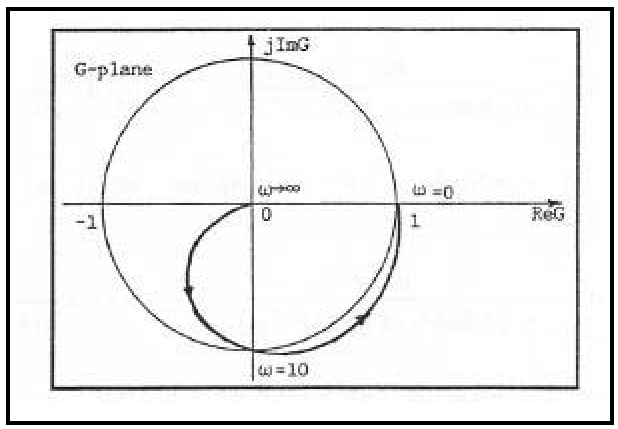
Thus,

$$10T_d = \frac{90^o p}{180^o} = \frac{p}{2} \text{ rad}$$

Thus,

$$T_d = \frac{p}{20} = 0.157 \text{ sec}$$

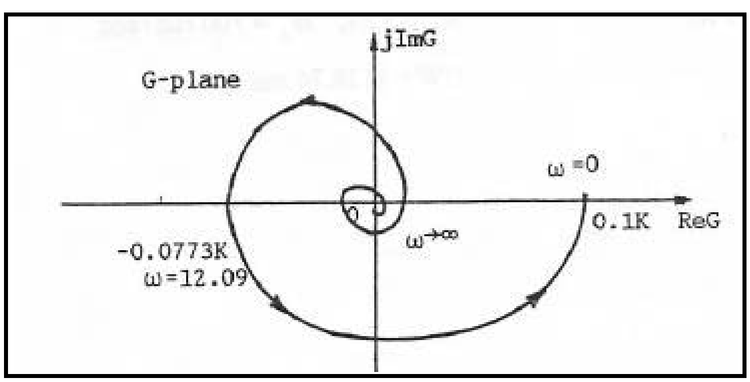
Maximum $D = vT_d = 100 \times 0.157 = 15.7$ in



(c) ***D* = 10 in.**

$$T_d = \frac{D}{v} = \frac{10}{100} = 0.1 \text{ sec} \qquad G(s) = \frac{10Ke^{-0.1s}}{s^2 + 10s + 100}$$

The Nyquist plot of $G(jw)$ intersects the negative real axis at $w = 12.09$ rad/sec. $G(j) = -0.0773K$
For stability, the maximum value of K is 12.94 .



9-22 (a)

$$G_H(s) = \frac{8}{s\left(s^2 + 6s + 12\right)} \qquad G_L(s) = \frac{2.31}{s(s + 2.936)}$$

$M_r = 1$, $\omega_r = 0$ rad / sec, BW = 1.02 rad / sec $\qquad$ $M_r = 1$, $\omega_r = 0$ rad / sec, BW = 1.03 rad / sec

(b)

$$G_H(s) = \frac{0.909}{s\left(1 + 0.5455s + 0.0455s^2\right)} \qquad G_L(s) = \frac{0.995}{s(1 + 0.4975s)}$$

$M_r = 1$, $\omega_r = 0$ rad / sec, BW = 1.4 rad / sec $\qquad$ $M_r = 1$, $\omega_r = 0$ rad / sec, BW = 1.41 rad / sec

(c)

$$G_H(s) = \frac{0.5}{s\left(1 + 0.75s + 0.25s^2\right)} \qquad G_L(s) = \frac{0.707}{s(1 + 0.3536s)}$$

$M_r = 1$, $\omega_r = 0$ rad / sec, BW = 0.87 rad / sec $\qquad$ $M_r = 1$, $\omega_r = 0$ rad / sec, BW = 0.91 rad / sec

(d)

$$G_H(s) = \frac{90.3}{s\left(1 + 0.00283s + 8.3056\times10^{-7}s^2\right)} \qquad G_L(s) = \frac{92.94}{s(1 + 0.002594s)}$$

$M_r = 1$, $\omega_r = 0$ rad / sec, BW = 119.74 rad / sec $\qquad$ $M_r = 1$, $\omega_r = 0$ rad / sec BW = 118.76 rad / sec

(e)

$$G_H(s) = \frac{180.6}{s\left(1 + 0.00283s + 8.3056\times10^{-7}s^2\right)} \qquad G_L(s) = \frac{189.54}{s(1 + 0.002644s)}$$

$M_r = 1$, $\omega_r = 0$ rad / sec, BW = 270.55 rad / sec $\qquad$ $M_r = 1$, $\omega_r = 0$ rad / sec, BW = 268.11 rad / sec

(f)

$$G_H(s) = \frac{1245.52}{s\left(1 + 0.00283s + 8.3056\times10^{-7}s^2\right)} \qquad G_L(s) = \frac{2617.56}{s(1 + 0.0053s)}$$

$M_r = 2.96$, $\omega_r = 666.67$ rad / sec $\qquad$ $M_r = 3.74$, $\omega_r = 700$ rad / sec

BW = 1054.41 rad / sec $\qquad$ BW = 1128.74 rad / sec

9-23 (a) $M_r = 2.06$, $\omega_r = 9.33$ rad / sec, BW = 15.2 rad / sec

(b)

$$M_r = \frac{1}{2z\sqrt{1-z^2}} = 2.06 \qquad z^4 - z^2 + 0.0589 = 0 \quad \text{The solution for } z < 0.707 \text{ is } z = 0.25.$$

$$w_r\sqrt{1-2z^2} = 9.33 \text{ rad / sec} \qquad \text{Thus} \quad w_n = \frac{9.33}{0.9354} = 9.974 \text{ rad / sec}$$

$$G_L(s) = \frac{w_n^2}{s\left(s+2zw_n\right)} = \frac{99.48}{s(s+4.987)} = \frac{19.94}{s(1+0.2005s)}$$

BW = 15.21 rad/sec

9-24 (a) $M_r = 2.96$, $w_r = 666.67$ rad / sec, BW $= 1054.41$ rad / sec

(b) $$M_r = \frac{1}{2z\sqrt{1-z^2}} = 2.96 \qquad z^4 - z^2 + 0.0285 = 0 \quad \text{The solution for } z < 0.707 \text{ is } z = 0.1715$$

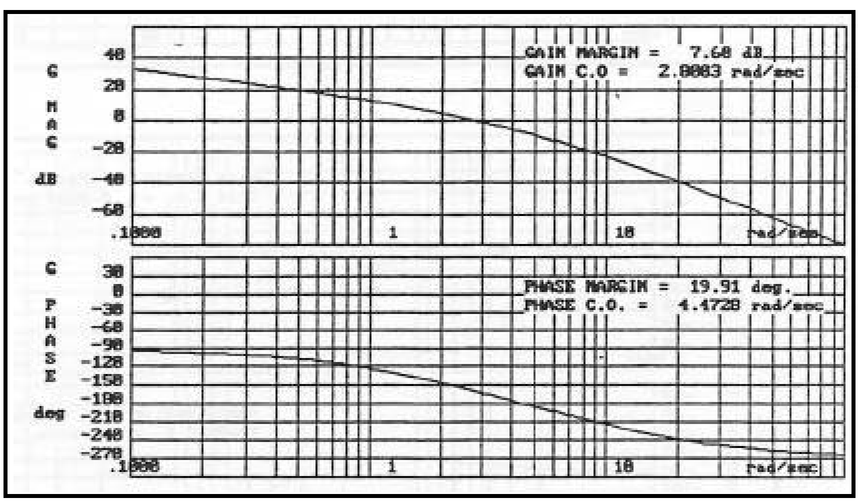
$$w_r\sqrt{1-2z^2} = 666.67 \text{ rad / sec} \qquad \text{Thus} \quad w_n = \frac{666.67}{0.97} = 687.19 \text{ rad / sec}$$

$$G_L(s) = \frac{w_n^2}{s\left(s+2zw_n\right)} = \frac{472227.43}{s(s+235.7)} = \frac{2003.5}{s(1+0.00424s)}$$

BW = 1079.28 rad/sec

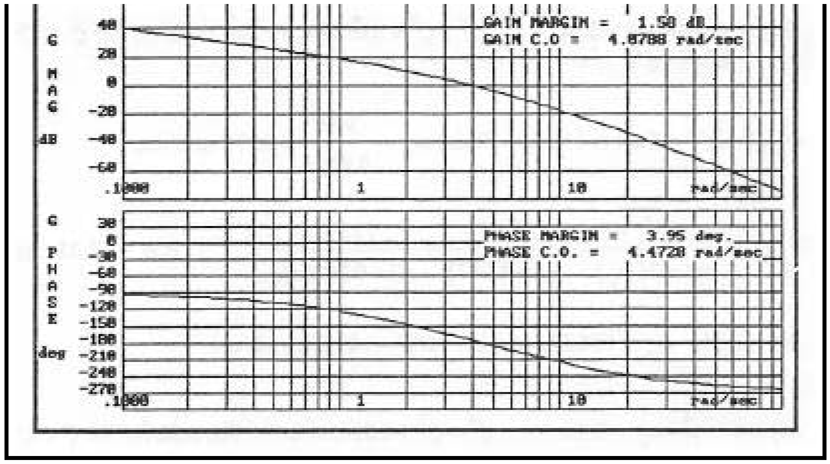
9-25 (a)

$$G(s) = \frac{5}{s(1+0.5s)(1+0.1s)}$$



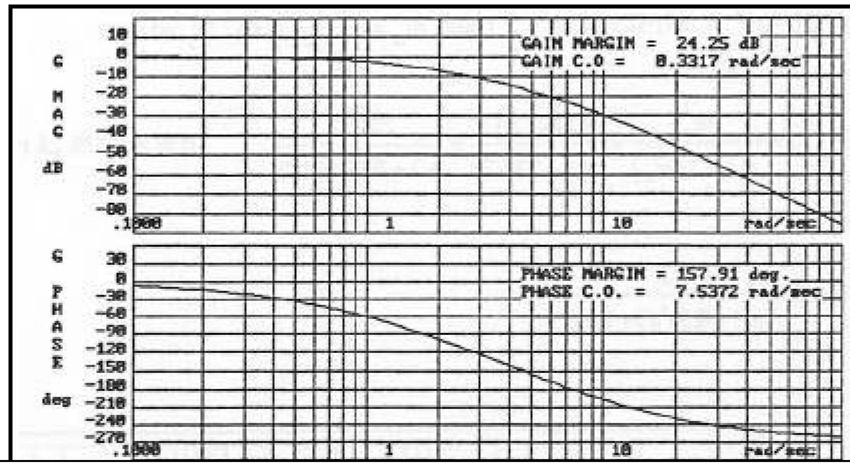
9-25 (b)

$$G(s) = \frac{10}{s(1+0.5s)(1+0.1s)}$$



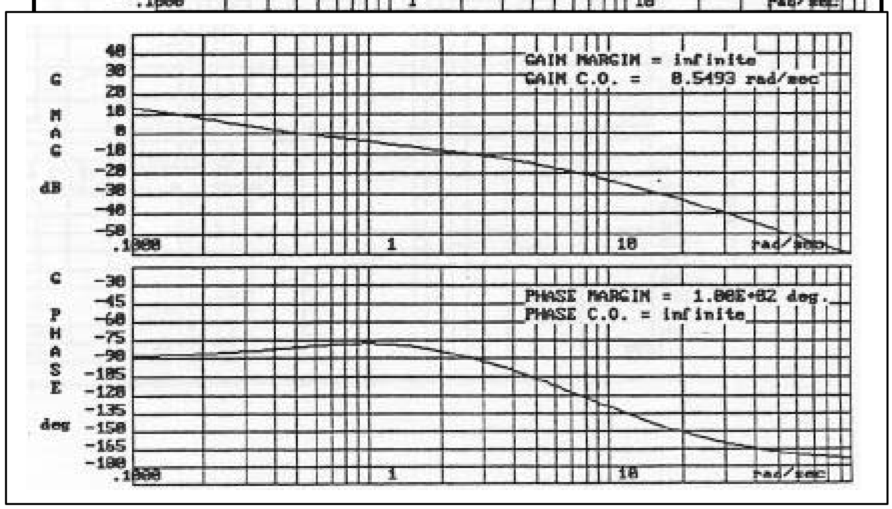
(c)

$$G(s) = \frac{500}{(s+1.2)(s+4)(s+10)}$$



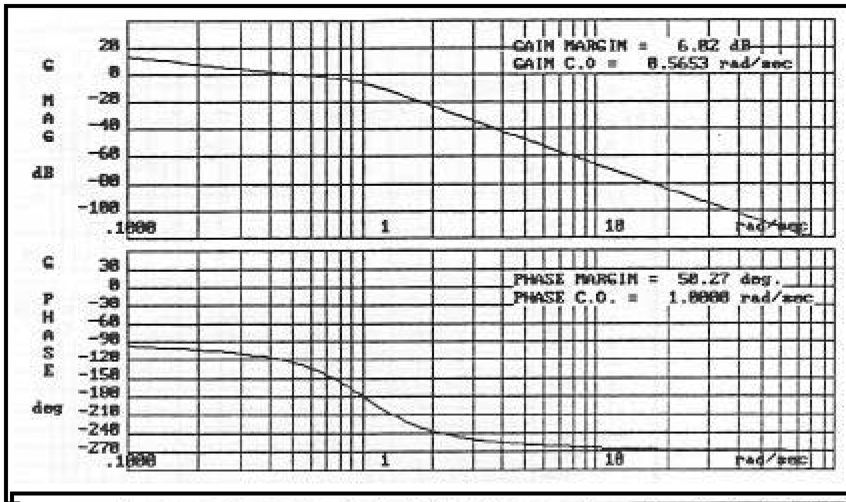
(d)

$$G(s) = \frac{10(s+1)}{s(s+2)(s+10)}$$



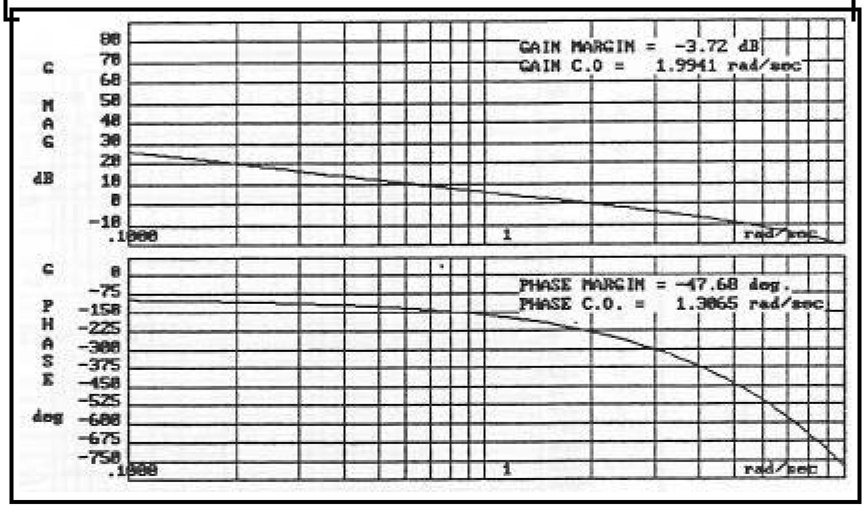
9-25 (e)

$$G(s) = \frac{0.5}{s\left(s^2 + s + 1\right)}$$



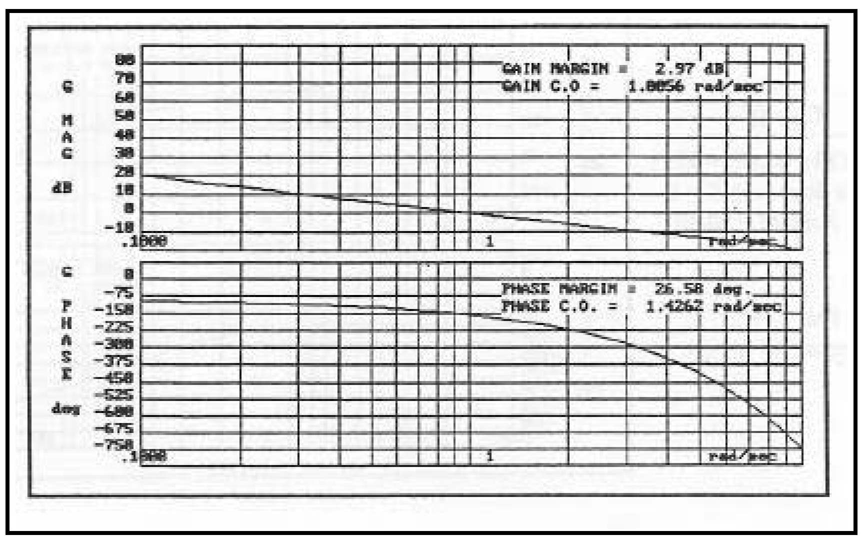
(f)

$$G(s) = \frac{100e^{-s}}{s\left(s^2 + 10s + 50\right)}$$



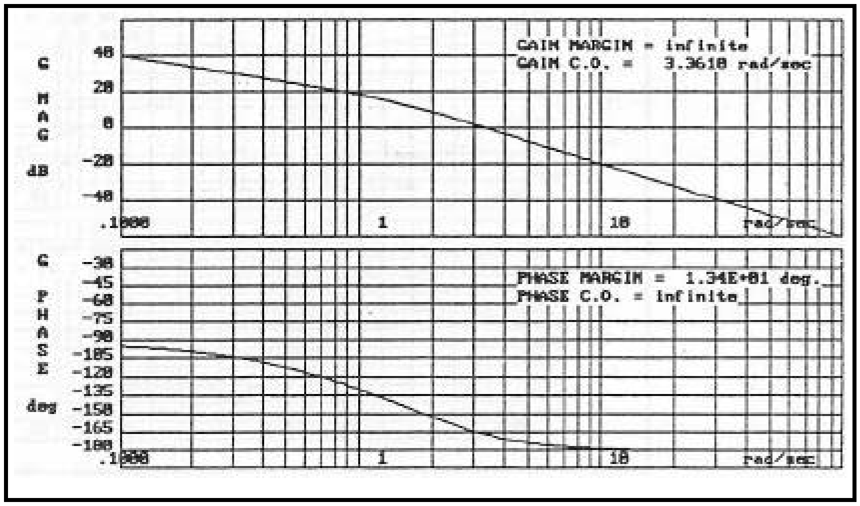
(g)

$$G(s) = \frac{100e^{-s}}{s\left(s^2 + 10s + 100\right)}$$



9-25 (h)

$$G(s) = \frac{10(s+5)}{s\left(s^2+5s+5\right)}$$



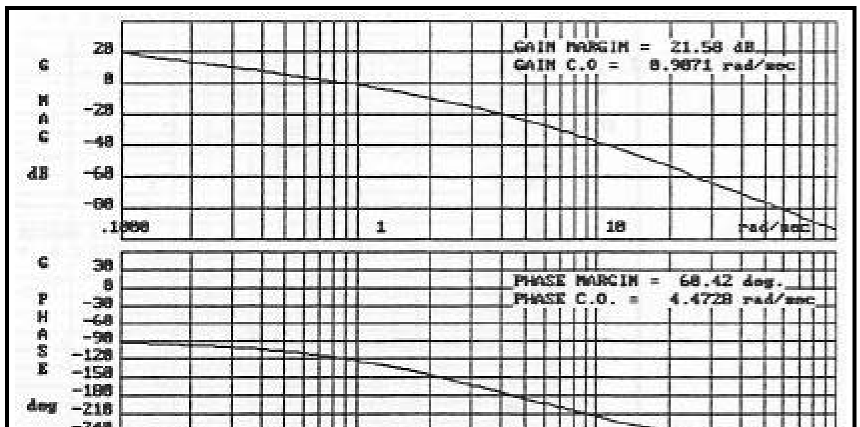
9-26 (a)

$$G(s) = \frac{K}{s(1+0.1s)(1+0.5s)}$$

The Bode plot is done with $K = 1$.

GM = 21.58 dB For GM = 20 dB,
K must be reduced by -1.58 dB.
Thus $K = 0.8337$

PM = 60.42^o. For PM = 45^o
K should be increased by 5.6 dB.
Or, $K = 1.91$

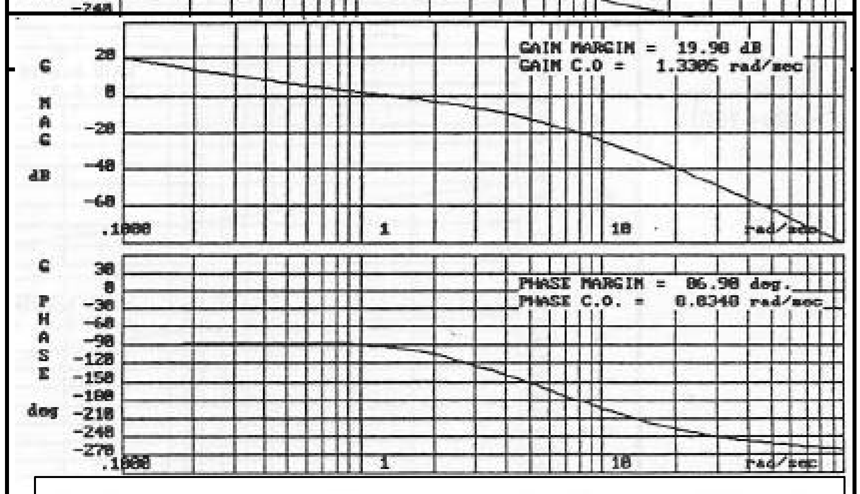


(b)

$$G(s) = \frac{K(s+1)}{s(1+0.1s)(1+0.2s)(1+0.5s)}$$

The Bode plot is done with $K = 1$.
GM = 19.98 dB. For GM = 20 dB,
$K \cong 1$.

PM = 86.9^o. For PM = 45^o
K should be increased by 8.9 dB.
Or, $K = 2.79$.

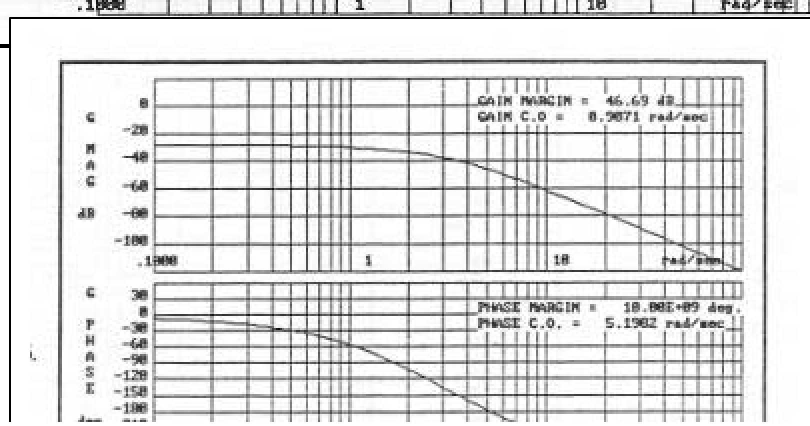


9-26 (c)

$$G(s) = \frac{K}{(s+3)^3}$$

The Bode plot is done with $K = 1$.

GM = 46.69 dB
PM = infinity.



For GM = 20 dB K can be increased by 26.69 dB or $K = 21.6$.
For PM = 45 deg. K can be increased by 28.71 dB, or $K = 27.26$.

(d)

$$G(s) = \frac{K}{(s+3)^4}$$

The Bode plot is done with $K = 1$.

GM = 50.21 dB
PM = infinity.
For GM = 20 dB K can be increased by 30.21 dB or $K = 32.4$
For PM = 45 deg. K can be increased by 38.24 dB, or $K = 81.66$

(e)

$$G(s) = \frac{Ke^{-s}}{s\left(1 + 0.1s + 0.01s^2\right)}$$

The Bode plot is done with $K = 1$.

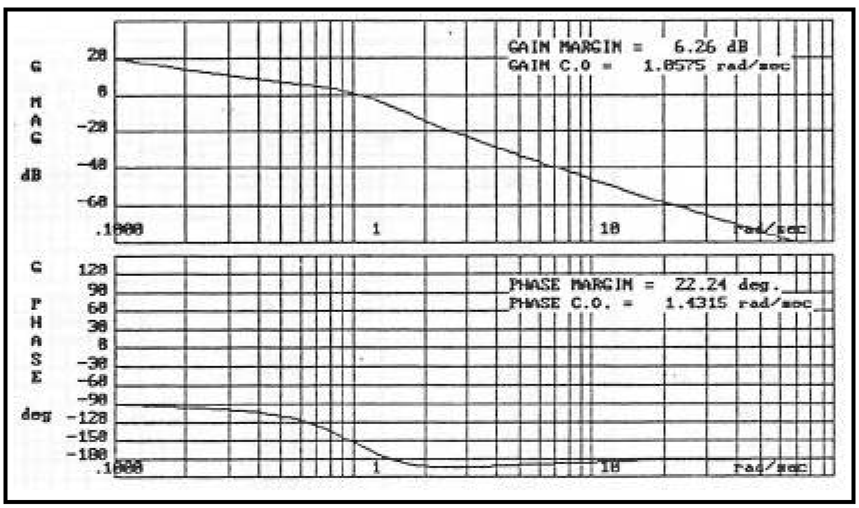
GM=2.97 dB
PM = 26.58 deg
For GM = 20 dB K must be decreased by -17.03 dB or $K = 0.141$.
For PM = 45 deg. K must be decreased by -2.92 dB or $K = 0.71$.

9-26 (f)

$$G(s) = \frac{K(1+0.5s)}{s\left(s^2 + s + 1\right)}$$

The Bode plot is done with $K = 1$.

GM = 6.26 dB
PM = 22.24 deg
For GM = 20 dB K must be decreased by -13.74 dB or $K = 0.2055$.
For PM = 45 deg K must be decreased by -3.55 dB or $K = 0.665$.



9-27 (a)

$$G(s) = \frac{10K}{s(1+0.1s)(1+0.5s)}$$

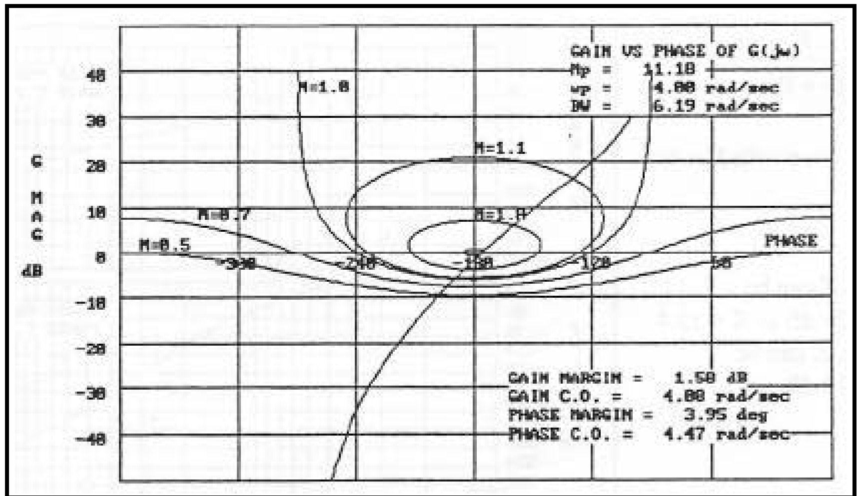
The gain-phase plot is done with
$K = 1$.

GM = 1.58 dB
PM = 3.95 deg.

For GM = 10 dB, K must be decreased by −8.42 dB or $K = 0.38$.

For PM = 45 deg, K must be decreased by −14 dB, or $K = 0.2$.

For $M_r = 1.2$, K must be decreased to 0.16.



(b)

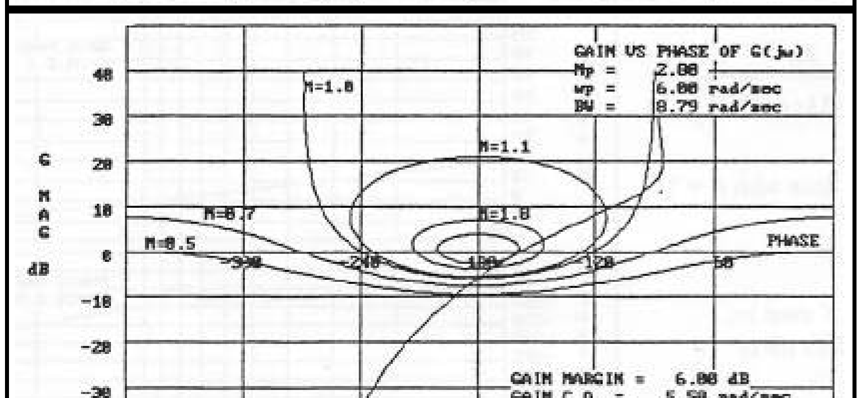
$$G(s) = \frac{5K(s+1)}{s(1+0.1s)(1+0.2s)(1+0.5s)}$$

The Gain-phase plot is done with
$K = 1$.

GM = 6 dB
PM = 22.31 deg.

For GM = 10 dB, K must be decreased by −4 dB or $K = 0.631$.
For PM = 45 deg, K must be
decrease by −5 dB.

For $M_r = 1.2$, K must be decreased to 0.48.



9-27 (c)

$$G(s) = \frac{10K}{s\left(1+0.1s+0.01s^2\right)}$$

The gain-phase plot is done for
$K = 1$.

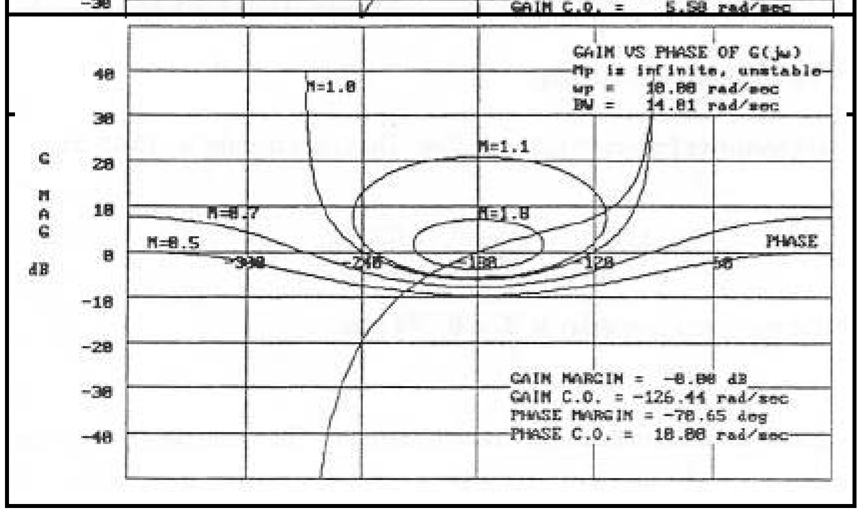
GM = 0 dB $\quad M_r = \infty$

PM = 0 deg

For GM = 10 dB, K must be decreased by −10 dB or $K = 0.316$.

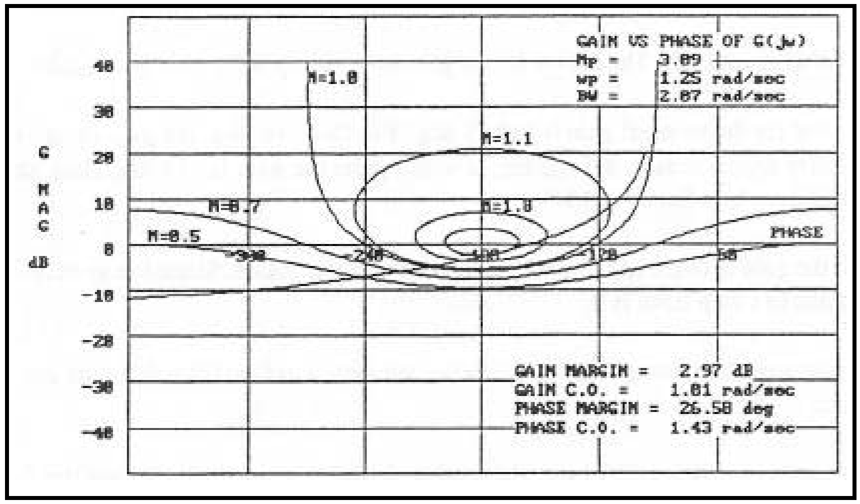
For PM = 45 deg, K must be decreased by −5.3 dB, or
$K = 0.543$.

For $M_r = 1.2$, K must be decreased to 0.2213.



(d)

$$G(s) = \frac{Ke^{-s}}{s\left(1 + 0.1s + 0.01s^2\right)}$$



The gain-phase plot is done for $K = 1$.

GM = 2.97 dB $\quad M_r = 3.09$

PM = 26.58 deg

For GM = 10 dB, K must be decreased by -7.03 dB, $K = 0.445$.

For PM = 45 deg, K must be decreased by -2.92 dB, or $K = 0.71$.

For $M_r = 1.2$, $K = 0.61$.

9-28 **(a)** Gain crossover frequency = 2.09 rad/sec PM = 115.85 deg
Phase crossover frequency = 20.31 rad/sec GM = 21.13 dB

(b) Gain crossover frequency = 6.63 rad/sec PM = 72.08 deg
Phase crossover frequency = 20.31 rad/sec GM = 15.11 dB

(c) Gain crossover frequency = 19.1 rad/sec PM = 4.07 deg
Phase crossover frequency = 20.31 rad/sec GM = 1.13 dB

(d) For GM = 40 dB, reduce gain by (40 − 21.13) dB = 18.7 dB, or gain = 0.116 × nominal value.

(e) For PM = 45 deg, the magntude curve reads −10 dB. This means that the loop gain can be increased by 10 dB from the nominal value. Or gain = 3.16 × nominal value.

(f) The system is type 1, since the slope of $|G(j\omega)|$ is −20 dB/decade as $\omega \to 0$.

(g) GM = 12.7 dB. PM = 109.85 deg.

(h) The gain crossover frequency is 2.09 rad/sec. The phase margin is 115.85 deg.
Set

$$\omega T_d = 2.09\,T_d = \frac{115.85^o \pi}{180^o} = 2.022 \text{ rad}$$

Thus, the maximum time delay is $T_d = 0.9674$ sec.

9-29 **(a)** The gain is increased to four times its nominal value. The magnitude curve is raised by 12.04 dB.

Gain crossover frequency = 10 rad/sec PM = 46 deg
Phase crossover frequency = 20.31 rad/sec GM = 9.09 dB

(b) The GM that corresponds to the nominal gain is 21.13 dB. To change the GM to 20 dB we need to increase the gain by 1.13 dB, or 1.139 times the nominal gain.

(c) The GM is 21.13 dB. The forward-path gain for stability is 21.13 dB, or 11.39.

(d) The PM for the nominal gain is 115.85 deg. For PM = 60 deg, the gain crossover frequency must be moved to approximately 8.5 rad/sec, at which point the gain is −10 dB. Thus, the gain must be increased by 10 dB, or by a factor of 3.162.

(e) With the gain at twice its nominal value, the system is stable. Since the system is type 1, the steady-state error due to a step input is 0.

(f) With the gain at 20 times its nominal value, the system is unstable. Thus the steady-state error would be infinite.

(g) With a pure time delay of 0.1 sec, the magnitude curve is not changed, but the the phase curve is subject to a negative phase of -0.1ω rad. The PM is

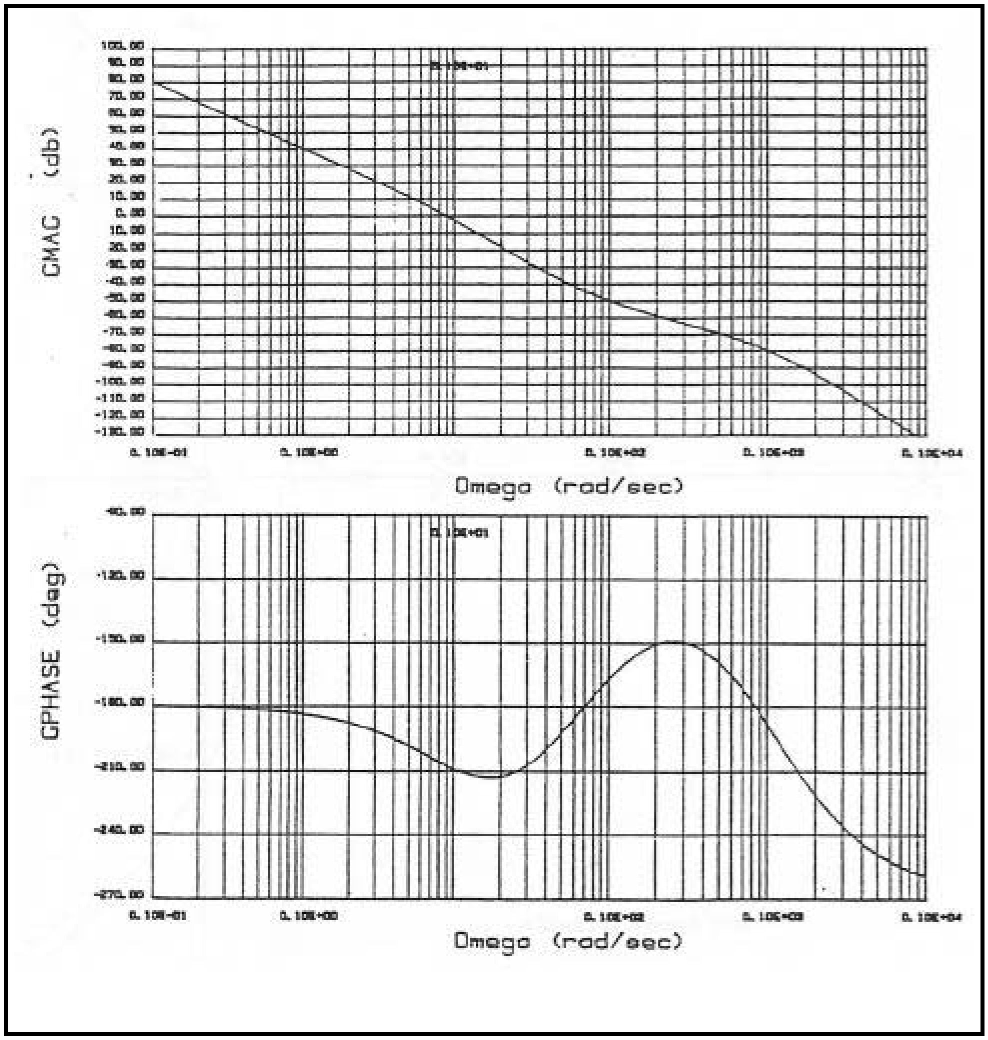
$$PM = 115.85 - 0.1 \times \text{gain crossover frequency} = 115.85 - 0.209 = 115.64 \text{ deg}$$

The new phase crossover frequency is approximately 9 rad/sec, where the original phase curve is reduced by -0.9 rad or -51.5 deg. The magnitude of the gain curve at this frequency is -10 dB. Thus, the gain margin is 10 dB.

(h) When the gain is set at 10 times its nominal value, the magnitude curve is raised by 20 dB. The new gain crossover frequency is approximately 17 rad/sec. The phase at this frequency is -30 deg. Thus, setting

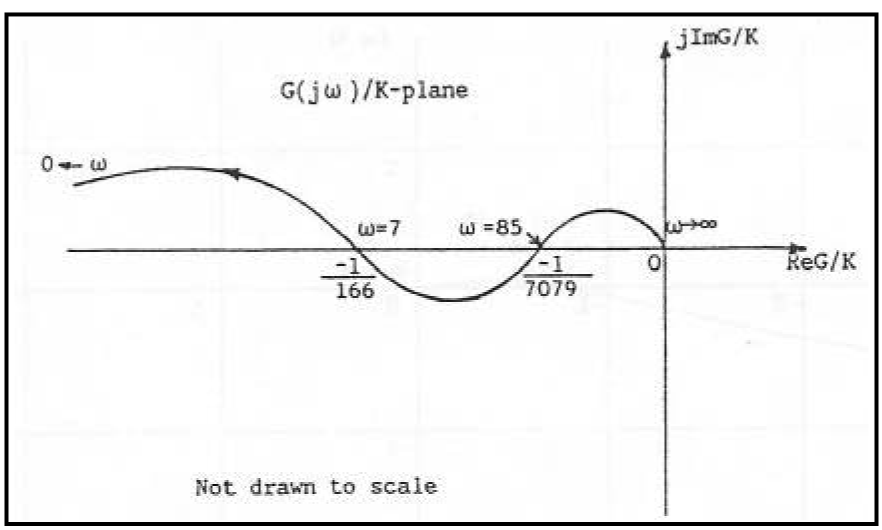
$$\omega T_d = 17 T_d = \frac{30^o \pi}{180^o} = 0.5236 \qquad \text{Thus} \qquad T_d = 0.0308 \text{ sec.}$$

9-30 (a) Bode Plot:

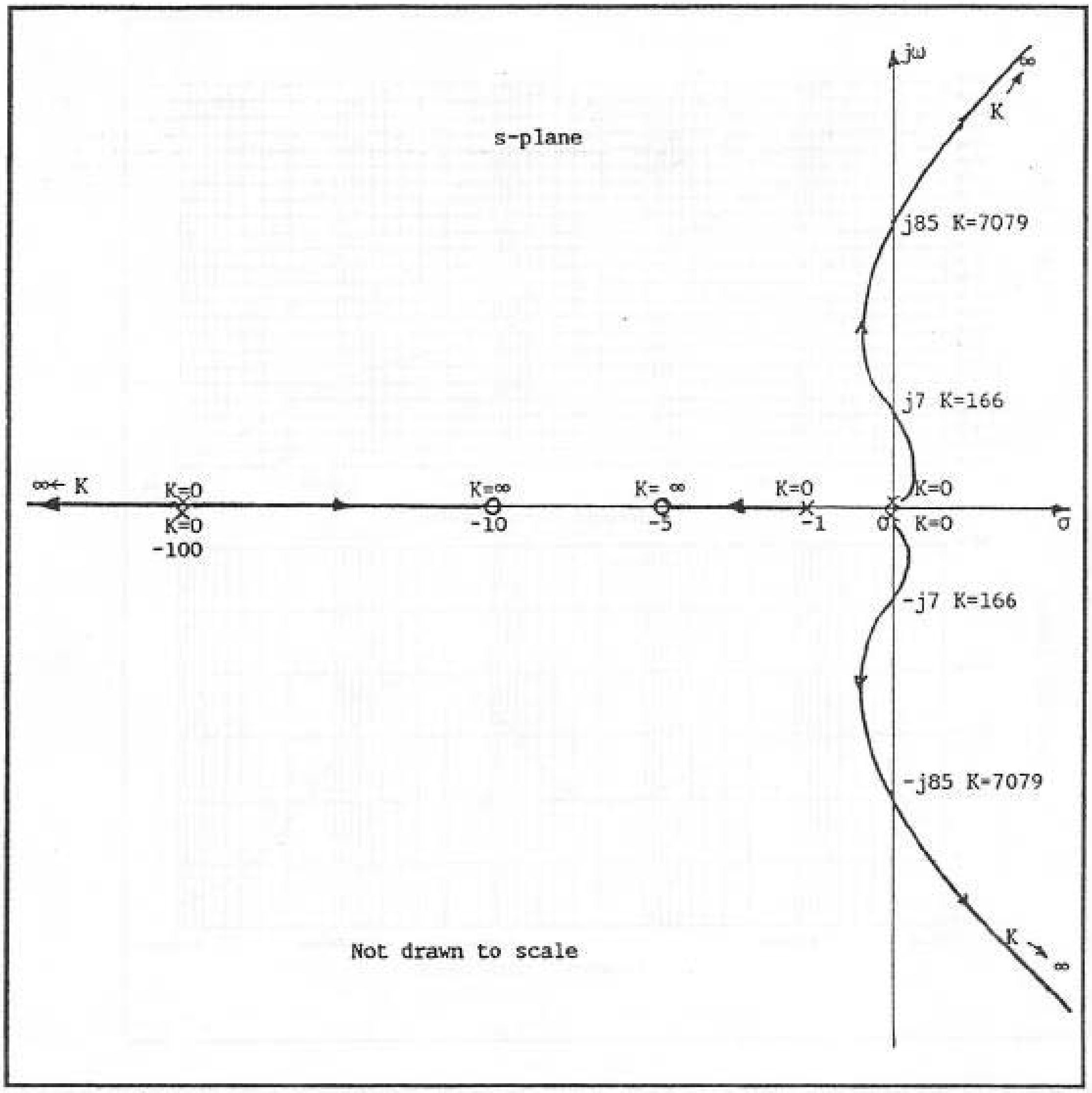


For stability: 166 (44.4 dB) < K < 7079 (77 dB) Phase crossover frequencies: 7 rad/sec and 85 rad/sec

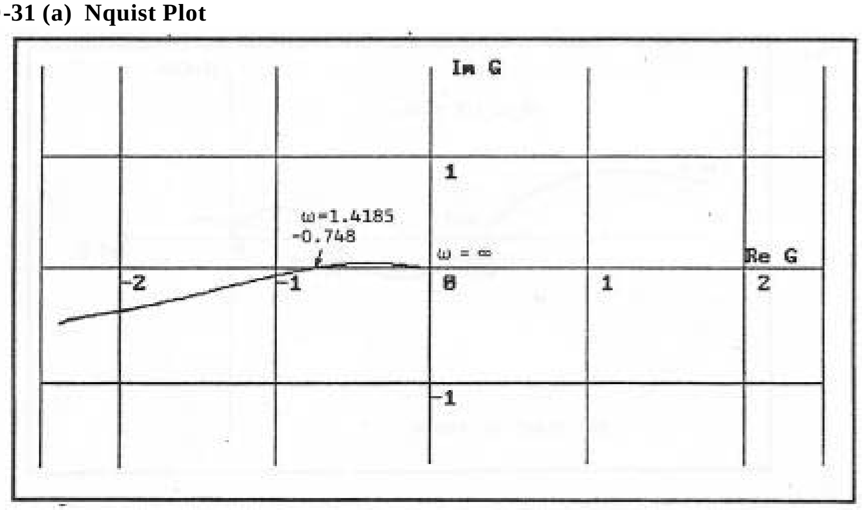
Nyquist Plot:



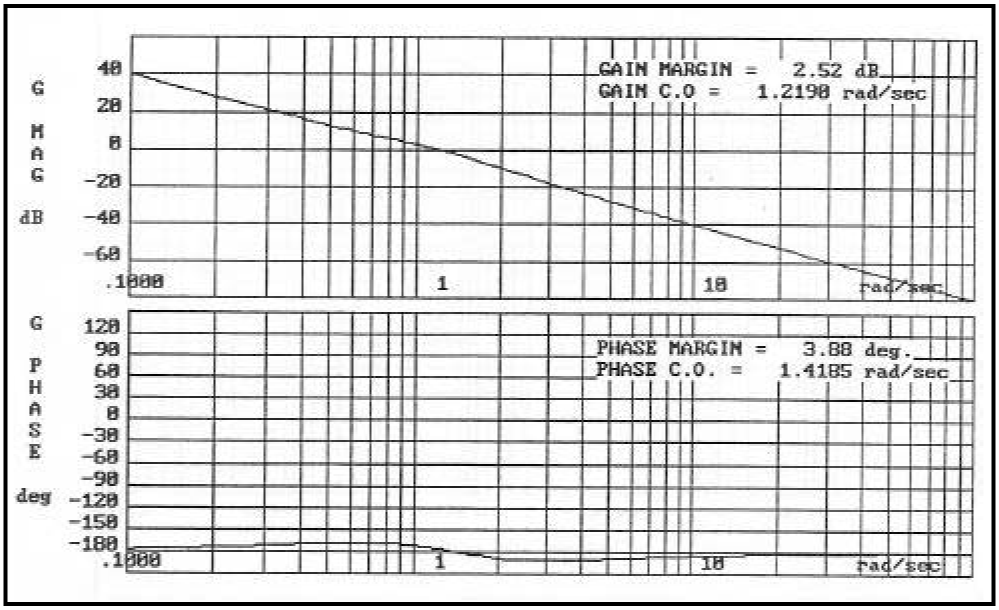
9-30 (b) Root Loci.



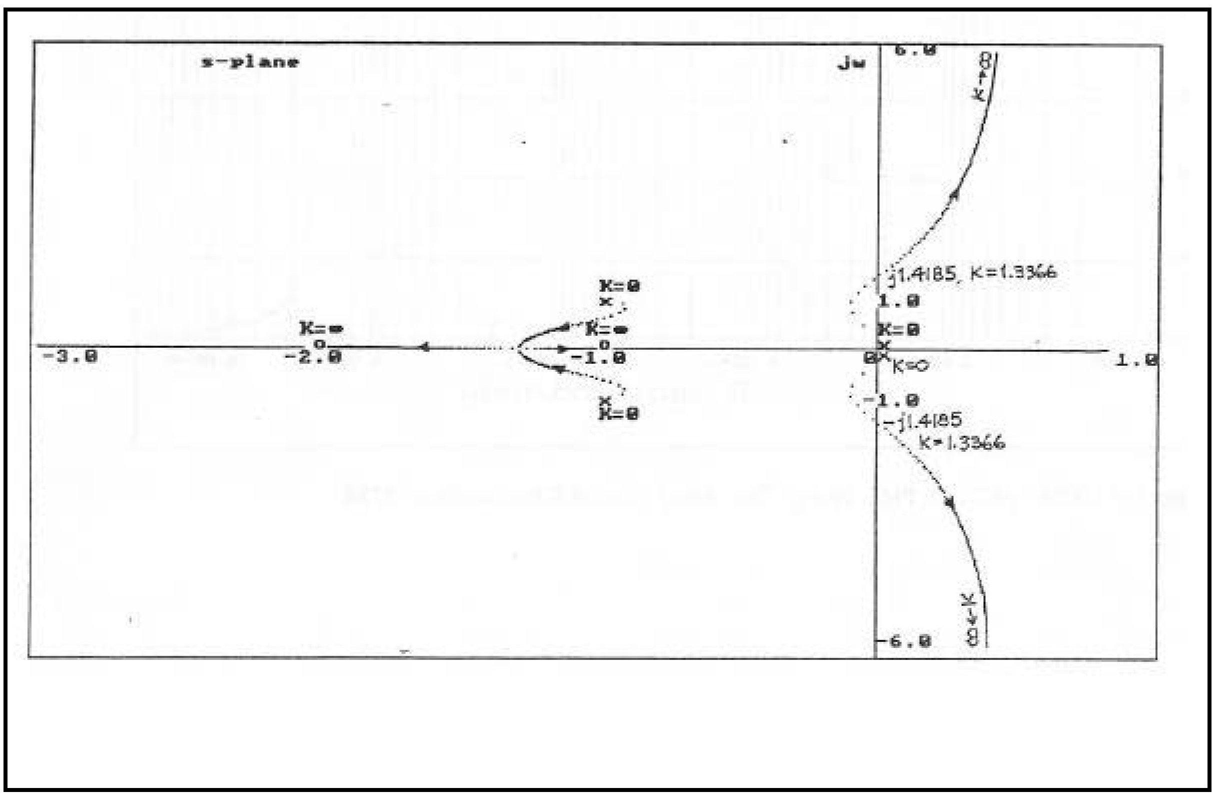
9-31 (a) Nquist Plot



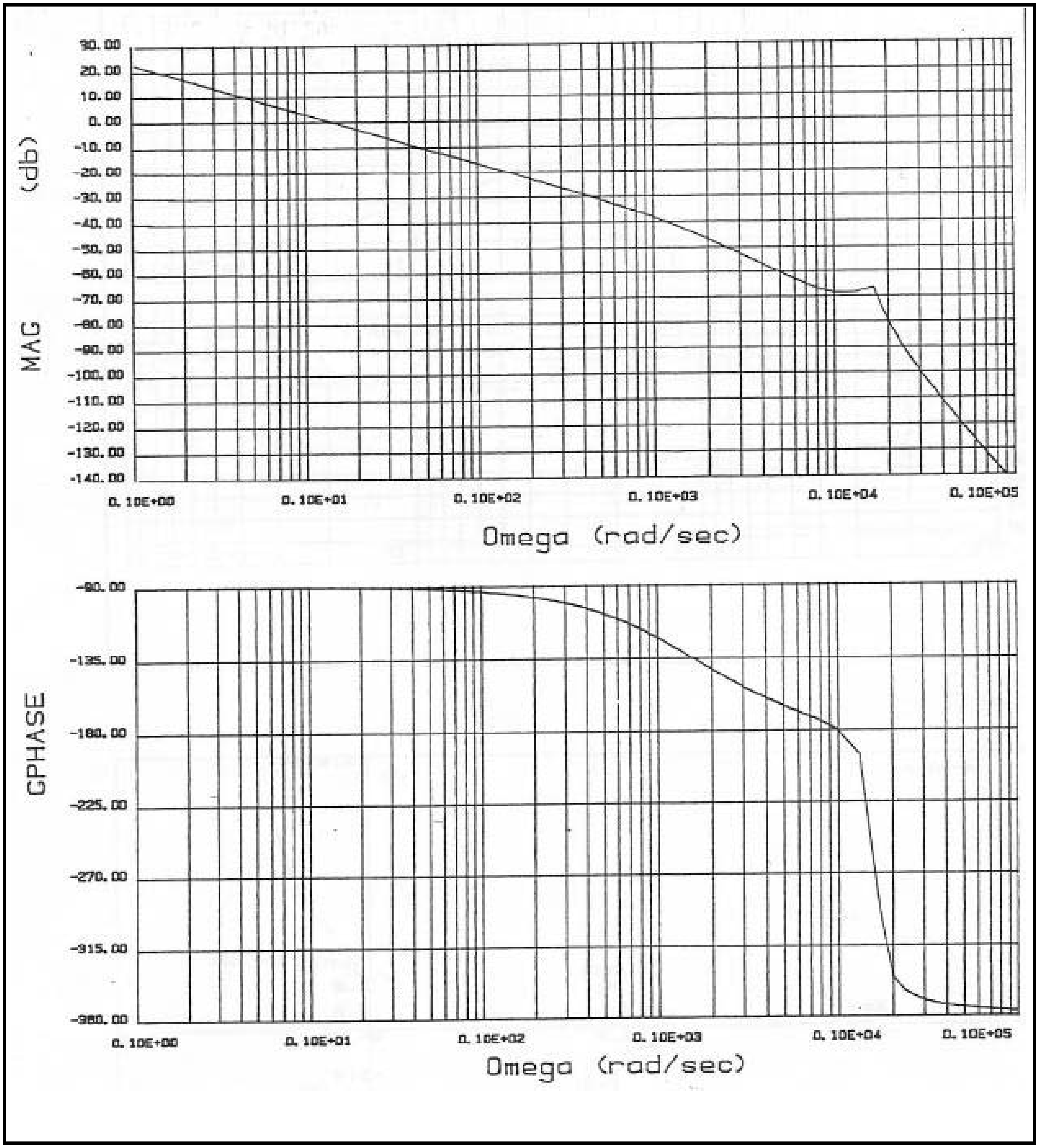
9-31 **Bode Plot**



9-31 (b) Root Loci



9-32 **Bode Diagram**



When $K = 1$, GM = 68.75 dB, PM = 90 deg. The critical value of K for stability is 2738.

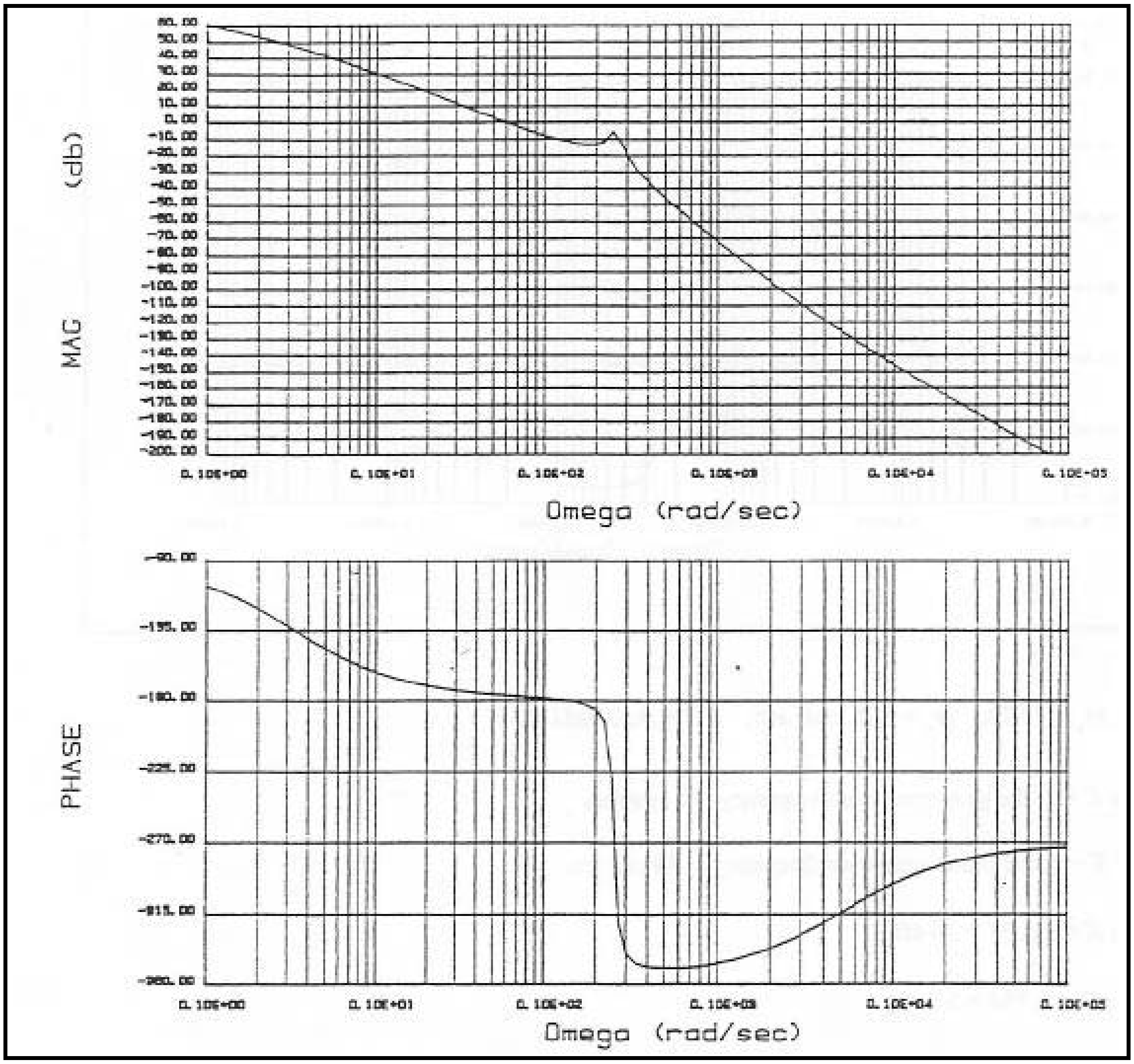
9-33 (a) Forward-path transfer function:

$$G(s) = \frac{\Theta_L(s)}{E(s)} = K_a G_p(s) = \frac{K_a K_i (Bs + K)}{\Delta_o}$$

where

$$\Delta_o = 0.12s\left(s + 0.0325\right)\left(s^2 + 2.5675s + 6667\right)$$
$$= s\left(0.12s^3 + 0.312s^2 + 80.05s + 26\right)$$
$$G(s) = \frac{43.33(s+500)}{s\left(s^3 + 2.6s^2 + 667.12s + 216.67\right)}$$

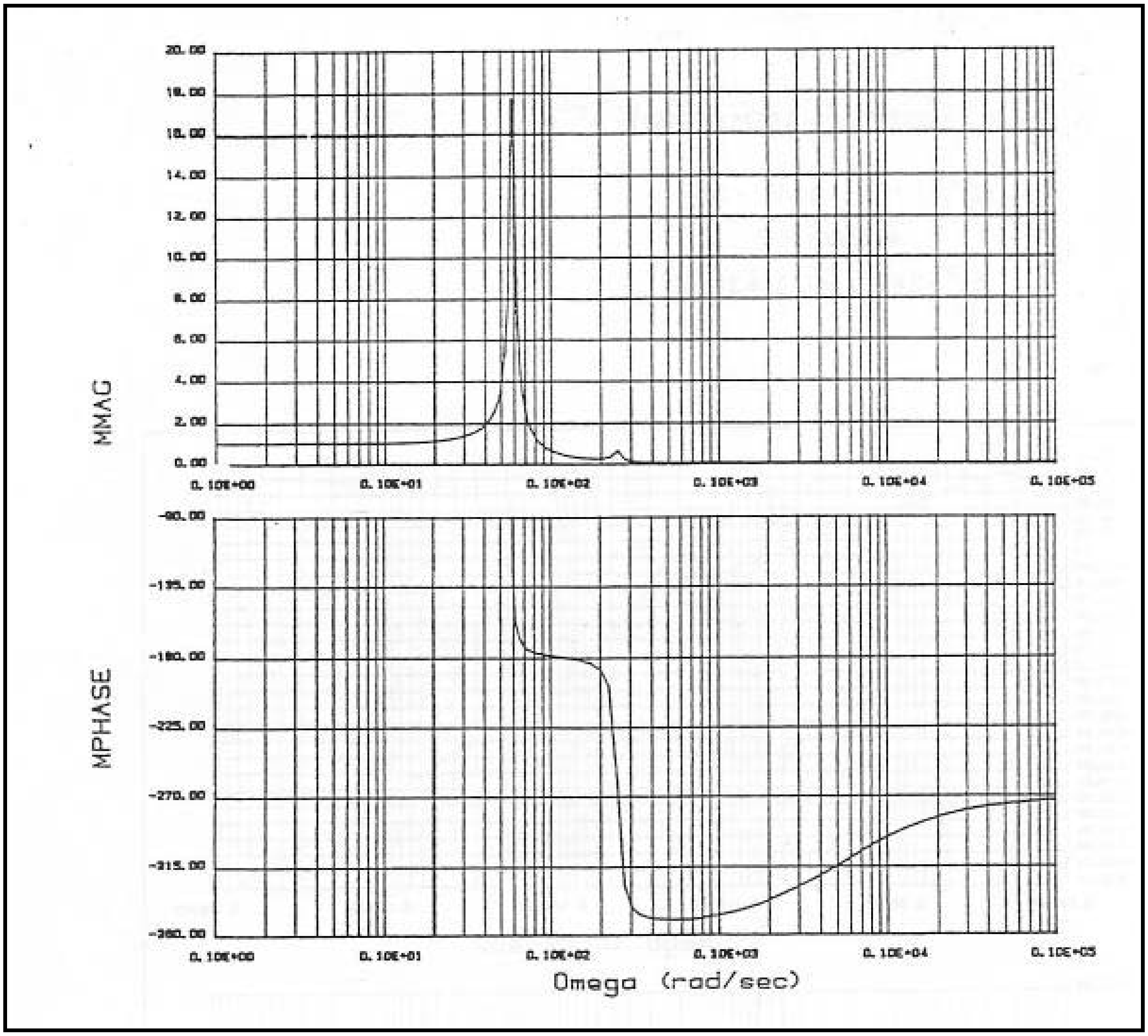
(b) Bode Diagram:



Gain crossover frequency = 5.85 rad/sec PM = 2.65 deg.

Phase crossover frequency = 11.81 rad/sec GM = 10.51 dB

9-33 (c) Closed-loop Frequency Response:



$M_r = 17.72$, $\omega_r = 5.75$ rad / sec, BW = 9.53 rad / sec

9-34 **(a)** When $K = 1$, the gain crossover frequency is 8 rad/sec.

(b) When $K = 1$, the phase crossover frequency is 20 rad/sec.

(c) When $K = 1$, GM = 10 dB.

(d) When $K = 1$, PM = 57 deg.

(e) When $K = 1$, $M_r = 1.2$.

(f) When $K = 1$, $\omega_r = 3$ rad/sec.

(g) When $K = 1$, BW = 15 rad/sec.

(h) When $K = -10$ dB (0.316), GM = 20 dB

(i) When $K = 10$ dB (3.16), the system is marginally stable. The frequency of oscillation is 20 rad/sec.

(j) The system is type 1, since the gain-phase plot of $G(j\omega)$ approaches infinity at -90 deg. Thus, the steady-state error due to a unit-step input is zero.

9-35 When K = 5 dB, the gain-phase plot of $G(j\omega)$ is raised by 5 dB.

(a) The gain crossover frequency is 10 rad/sec.

(b) The phase crossover frequency is 20 rad/sec.

(c) GM = 5 dB.

(d) PM = 34.5 deg.

(e) $M_r = 2$

(f) $\omega_r = 15$ rad/sec

(g) BW = 30 rad/sec

(h) When $K = -30$ dB, the GM is 40 dB.

9-36 (a) The phase margin with K = 1 and $T_d = 0$ sec is approximately 57 deg. For a PM of 40 deg, the time delay produces a phase lag of -17 deg. The gain crossover frequency is 8 rad/sec.
Thus,

$$\omega T_d = 17^o = \frac{17^o \pi}{180^o} = 0.2967 \text{ rad / sec} \qquad \text{Thus} \quad \omega = 8 \text{ rad / sec}$$

$$T_d = \frac{0.2967}{8} = 0.0371 \text{ sec}$$

(b) With K = 1, for marginal stability, the time delay must produce a phase lag of -57 deg.
Thus, at $\omega = 8$ rad/sec,

$$\omega T_d = 57^o = \frac{57^o \pi}{180^o} = 0.9948 \text{ rad} \qquad T_d = \frac{0.9948}{8} = 0.1244 \text{ sec}$$

9-37 (a) The phase margin with K = 5 dB and $T_d = 0$ is approximately 34.5 deg. For a PM of 30 deg, the time delay must produce a phase lag of -4.5 deg. The gain crossover frequency is 10 rad/sec. Thus,

$$\omega T_d = 4.5^o = \frac{4.5^o \pi}{180^o} = 0.0785 \text{ rad} \qquad \text{Thus} \qquad T_d = \frac{0.0785}{10} = 0.00785 \text{ sec}$$

(b) With K = 5 dB, for marginal stability, the time delay must produce a phase lag of -34.5 deg.

Thus at $\omega = 10$ rad/sec,

$$\omega T_d = 34.5^o = \frac{34.5^o \pi}{180^o} = 0.602 \text{ rad} \qquad \text{Thus} \qquad T_d = \frac{0.602}{10} = 0.0602 \text{ sec}$$

9-38 For a GM of 5 dB, the time delay must produce a phase lag of -34.5 deg at $\omega = 10$ rad/sec. Thus,

$$\omega T_d = 34.5^o = \frac{34.5^o \pi}{180^o} = 0.602 \text{ rad} \qquad \text{Thus} \qquad T_d = \frac{0.602}{10} = 0.0602 \text{ sec}$$

9-39 **(a)** **Forward-path Transfer Function:**

$$G(s) = \frac{Y(s)}{E(s)} = \frac{e^{-2s}}{(1+10s)(1+25s)}$$

From the Bode diagram, phase crossover frequency = 0.21 rad/sec GM = 21.55 dB

gain crossover frequency = 0 rad/sec PM = infinite

(b)

$$G(s) = \frac{1}{(1+10s)(1+25s)\left(1+2s+2s^2\right)}$$

From the Bode diagram, phase crossover frequency = 0.26 rad/sec GM = 25 dB

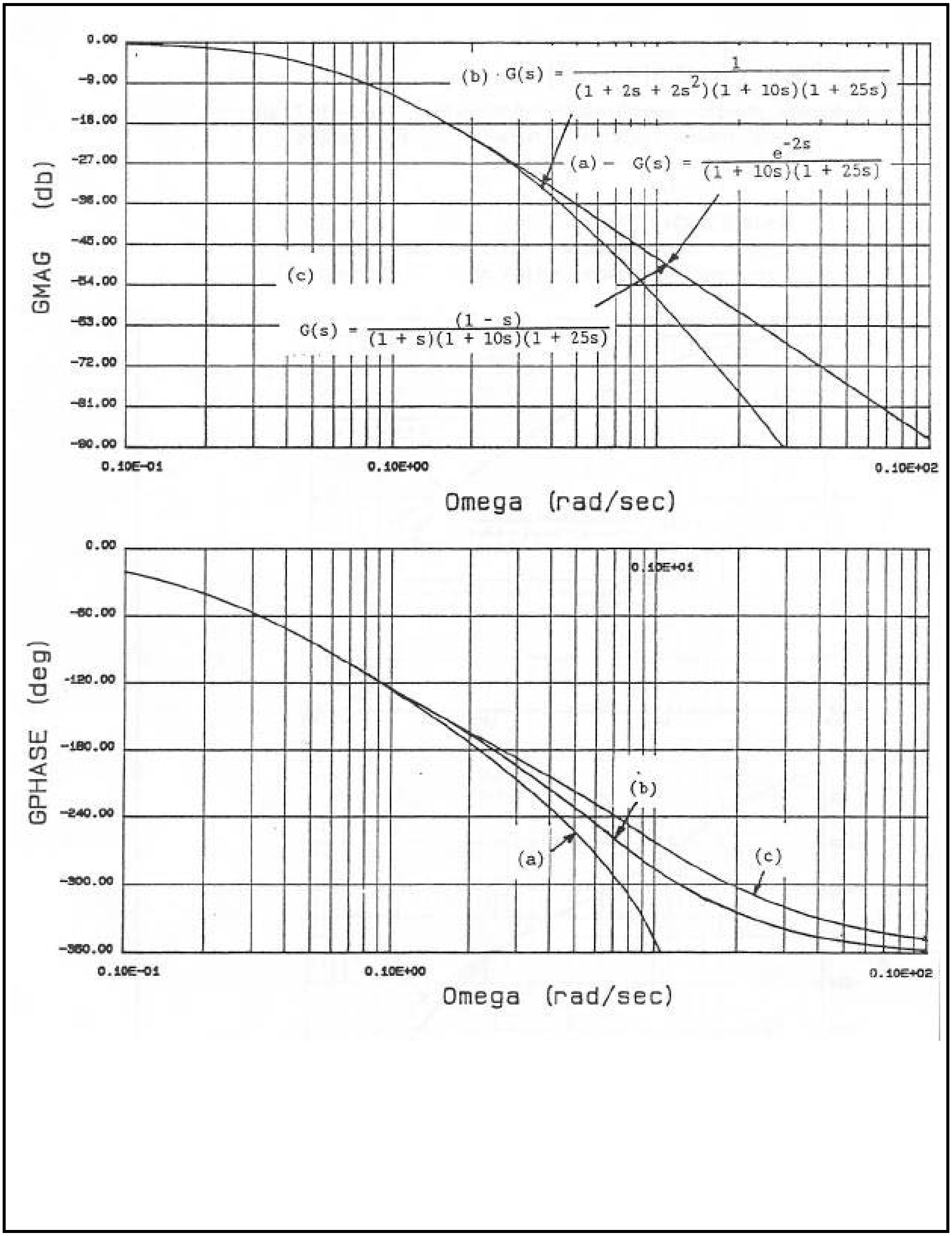
gain crossover frequency = 0 rad/sec PM = infinite

(c)

$$G(s) = \frac{1-s}{(1+s)(1+10s)(1+2s)}$$

From the Bode diagram, phase crossover frequency = 0.26 rad/sec GM = 25.44 dB
gain crossover frequency = 0 rad/sec PM = infinite

9-39 **(continued)** **Bode diagrams for all three parts.**



9-40 (a) **Forward-path Transfer Function:**

$$G(s) = \frac{e^{-s}}{(1+10s)(1+25s)}$$

From the Bode diagram, phase crossover frequency = 0.37 rad/sec GM = 31.08 dB
gain crossover frequency = 0 rad/sec PM = infinite

(b)

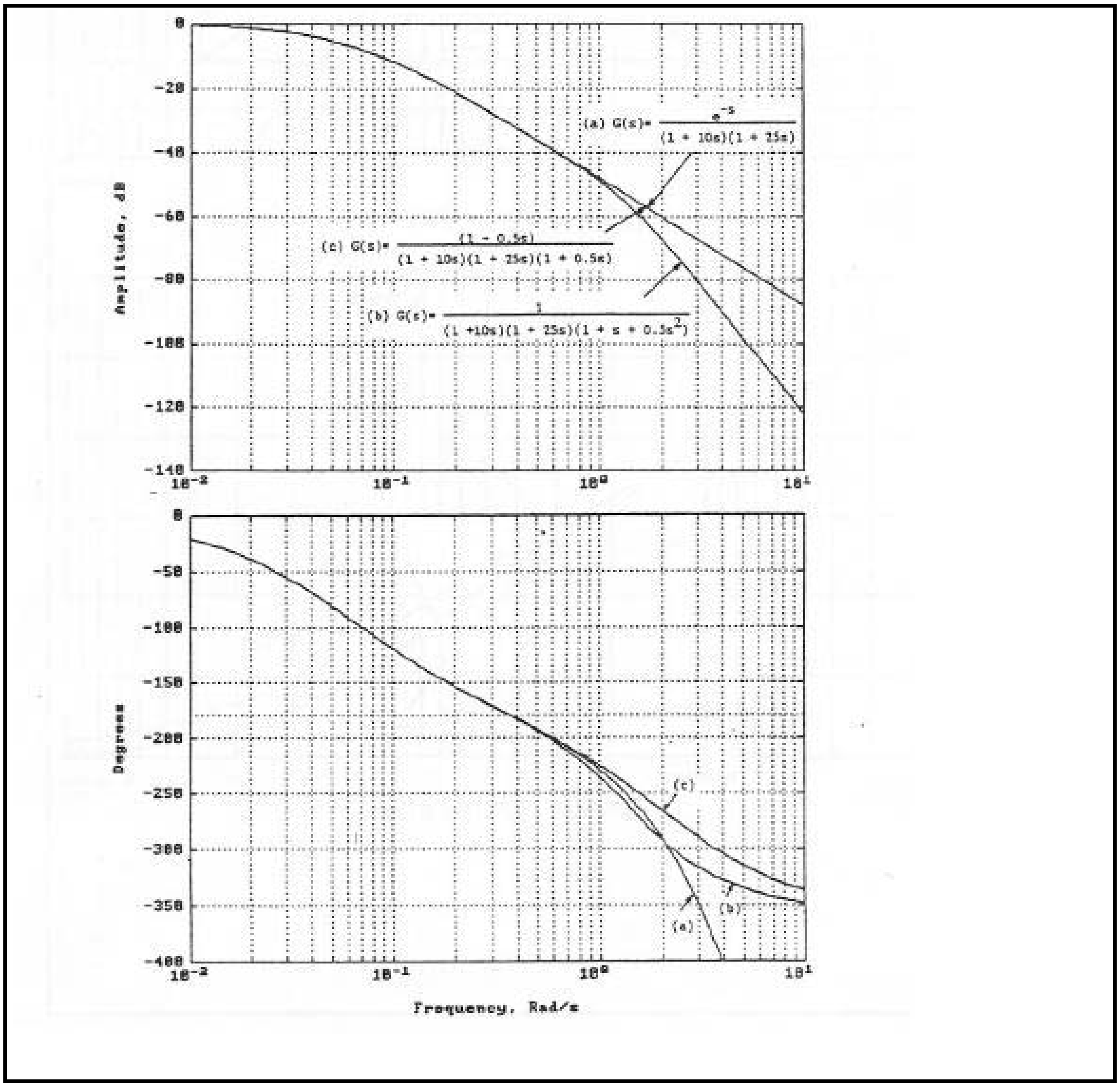
$$G(s) = \frac{1}{(1+10s)(1+25s)(1+s+0.5s^2)}$$

From the Bode diagram, phase crossover frequency = 0.367 rad/sec GM = 30.72 dB
gain crossover frequency = 0 rad/sec PM = infinite

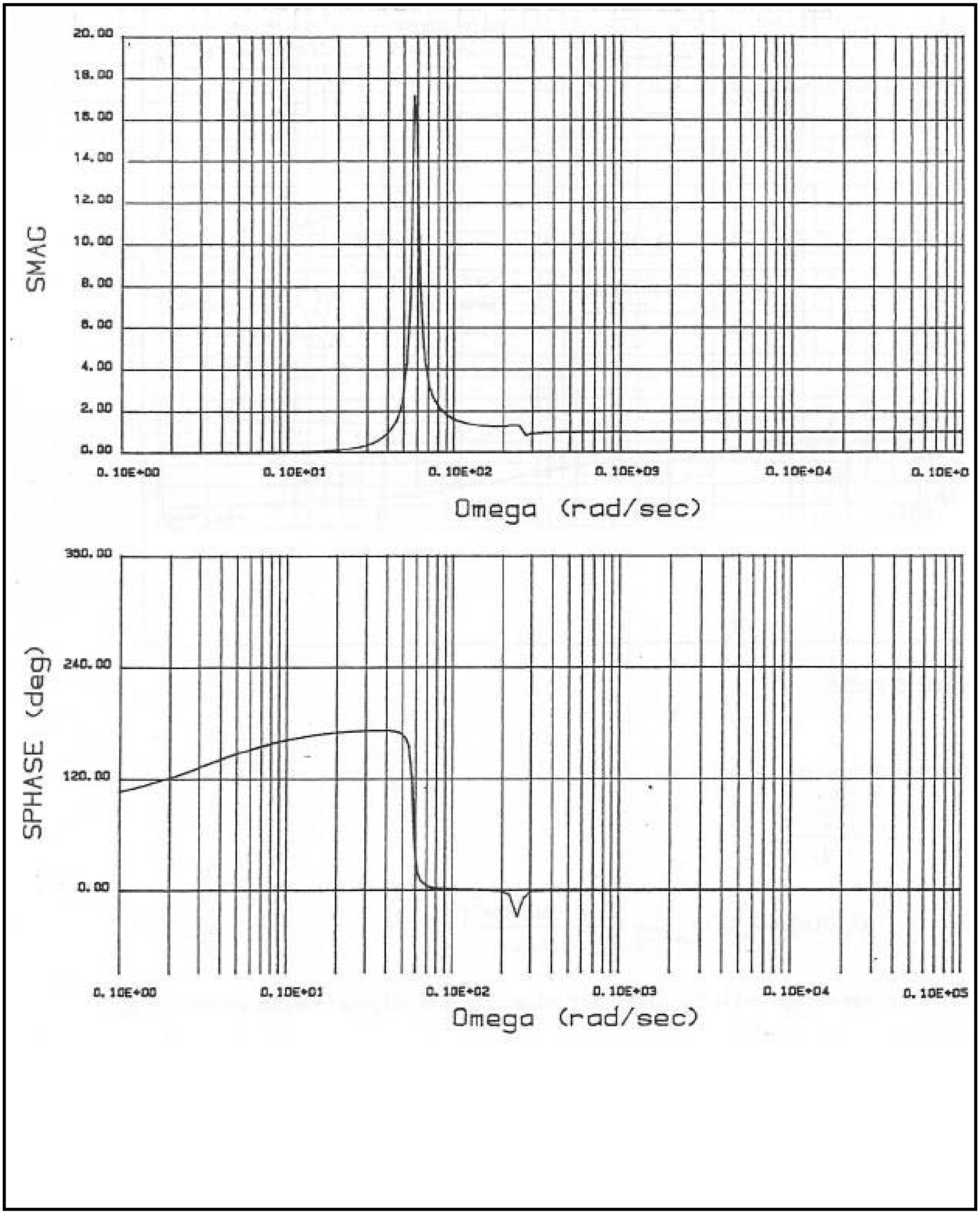
(c)

$$G(s) = \frac{(1-0.5s)}{(1+10s)(1+25s)(1+0.5s)}$$

From the Bode diagram, phase crossover frequency = 0.3731 rad/sec GM = 31.18 dB
gain crossover frequency = 0 rad/sec PM = infinite



9-41 Sensitivity Plot:



$\left|S_G^M\right|_{max} = 17.15 \qquad w_{max} = 5.75$ rad/sec

Chapter 10 DESIGN OF CONTROL SYSTEMS

10-1 Forward-path Transfer Function:

$$G(s)=\frac{M(s)}{1-M(s)}=\frac{K}{s^3+(20+a)s^2+(200+20a)s+200a-K}$$

For type 1 system, $200a-K=0$ Thus $K = 200a$
Ramp-error constant:

$$K_v=\lim_{s\to 0} sG(s)=\frac{K}{200+20a}=\frac{200a}{200+20a}=5 \qquad \text{Thus} \quad a=10 \qquad K=2000$$

The forward-path transfer function is

$$G(s)=\frac{2000}{s\left(s^2+30s+400\right)}$$

The controller transfer function is

$$G_c(s)=\frac{G(s)}{G_p(s)}=\frac{20\left(s^2+10s+100\right)}{\left(s^2+30s+400\right)}$$

The maximum overshoot of the unit-step response is 0 percent.

10-2 Forward-path Transfer Function:

$$G(s)=\frac{M(s)}{1-M(s)}=\frac{K}{s^3+(20+a)s^2+(200+20a)s+200a-K}$$

For type 1 system, $200a-K=0$ Thus $K = 200a$
Ramp-error constant:

$$K_v=\lim_{s\to 0} sG(s)=\frac{K}{200+20a}=\frac{200a}{200+20a}=9 \qquad \text{Thus} \quad a=90 \qquad K=18000$$

The forward-path transfer function is

$$G(s)=\frac{18000}{s\left(s^2+110s+2000\right)}$$

The controller transfer function is

$$G_c(s)=\frac{G(s)}{G_p(s)}=\frac{180\left(s^2+10s+100\right)}{\left(s^2+110s+2000\right)}$$

The maximum overshoot of the unit-step response is 4.3 percent.

From the expression for the ramp-error constant, we see that as a or K goes to infinity, K_v approaches 10. Thus the maximum value of K_v that can be realized is 10. The difficulties with very large values of K and a are that a high-gain amplifier is needed and unrealistic circuit parameters are needed for the controller.

10-3 (a) Ramp-error Constant:

$$K_v=\lim_{s\to 0} s\frac{1000\left(K_P+K_Ds\right)}{s(s+10)}=\frac{1000K_P}{10}=100K_P=1000 \qquad \text{Thus} \qquad K_P=10$$

Characteristic Equation: $s^2+\left(10+1000K_D\right)s+1000K_P=0$

$$\omega_n=\sqrt{1000K_P}=\sqrt{10000}=100 \text{ rad/sec} \qquad 2\zeta\omega_n=10+1000K_D=2\times 0.5\times 100=100$$

Thus $$K_D=\frac{90}{1000}=0.09$$

10-3 (b) For $K_v = 1000$ and $z = 0.707$, and from part (a), $w_n = 100$ rad/sec,

$$2zw_n = 10 + 1000 K_D = 2 \times 0.707 \times 100 = 141.4 \qquad \text{Thus} \qquad K_D = \frac{131.4}{1000} = 0.1314$$

(c) For $K_v = 1000$ and $z = 1.0$, and from part (a), $w_n = 100$ rad/sec,

$$2zw_n = 10 + 1000 K_D = 2 \times 1 \times 100 = 200 \qquad \text{Thus} \qquad K_D = \frac{190}{1000} = 0.19$$

10-4 The ramp-error constant:

$$K_v = \lim_{s \to 0} s \frac{1000\left(K_P + K_D s\right)}{s(s+10)} = 100K_P = 10{,}000 \qquad \text{Thus} \quad K_P = 100$$

The forward-path transfer function is: $G(s) = \dfrac{1000\left(100 + K_D s\right)}{s(s+10)}$

K_D	PM (deg)	GM	M_r	BW (rad/sec)	Max overshoot (%)
0	1.814	$\propto$	13.5	493	46.6
0.2	36.58	$\propto$	1.817	525	41.1
0.4	62.52	$\propto$	1.291	615	22
0.6	75.9	$\propto$	1.226	753	13.3
0.8	81.92	$\propto$	1.092	916	8.8
1.0	84.88	$\propto$	1.06	1090	6.2

The phase margin increases and the maximum overshoot decreases monotonically as K_D increases.

10-5 (a) Forward-path Transfer Function:

$$G(s) = G_c(s)G_p(s) = \frac{4500K\left(K_D + K_P s\right)}{s(s+361.2)}$$

Ramp Error Constant: $K_v = \lim_{s \to 0} sG(s) = \dfrac{4500 KK_P}{361.2} = 12.458 KK_P$

$$e_{ss} = \frac{1}{K_v} = \frac{0.0802}{KK_P} \le 0.001 \qquad \text{Thus} \qquad KK_P \ge 80.2 \qquad \text{Let} \quad K_P = 1 \text{ and } K = 80.2$$

Attributes of Unit-step Response:

K_D	t_r (sec)	t_s (sec)	Max Overshoot (%)
0	0.00221	0.0166	37.1
0.0005	0.00242	0.00812	21.5
0.0010	0.00245	0.00775	12.2
0.0015	0.0024	0.0065	6.4
0.0016	0.00239	0.00597	5.6
0.0017	0.00238	0.00287	4.8
0.0018	0.00236	0.0029	4.0
0.0020	0.00233	0.00283	2.8

Select $K_D \ge 0.0017$

(b) BW must be less than 850 rad/sec.

K_D	GM	PM (deg)	M_r	BW (rad/sec)
0.0005	∞	48.45	1.276	827
0.0010	∞	62.04	1.105	812
0.0015	∞	73.5	1.033	827
0.0016	∞	75.46	1.025	834
0.0017	∞	77.33	1.018	842
0.00175	∞	78.22	1.015	847
0.0018	∞	79.07	1.012	852

Select $K_D \cong 0.00175$. A larger K_D would yield a BW larger than 850 rad/sec.

10-6 **The forward-path Transfer Function: N = 20**

$$G(s) = \frac{200\left(K_P + K_D s\right)}{s(s+1)(s+10)}$$

To stabilize the system, we can reduce the forward-path gain. Since the system is type 1, reducing the gain does not affect the steady-state liquid level to a step input. Let $K_P = 0.05$

$$G(s) = \frac{200\left(0.05 + K_D s\right)}{s(s+1)(s+10)}$$

Unit-step Response Attributes:

K_D	t_s (sec)	Max Overshoot (%)
0.01	5.159	12.7
0.02	4.57	7.1
0.03	2.35	3.2
0.04	2.526	0.8
0.05	2.721	0
0.06	3.039	0
0.10	4.317	0

When $K_D = 0.05$ the rise time is 2.721 sec, and the step response has no overshoot.

10-7 (a) For $e_{ss} = 1$,

$$K_v = \lim_{s\to 0} sG(s) = \lim_{s\to 0} s\frac{200\left(K_P + K_D s\right)}{s(s+1)(s+10)} = 20K_P = 1 \qquad \text{Thus} \quad K_P = 0.05$$

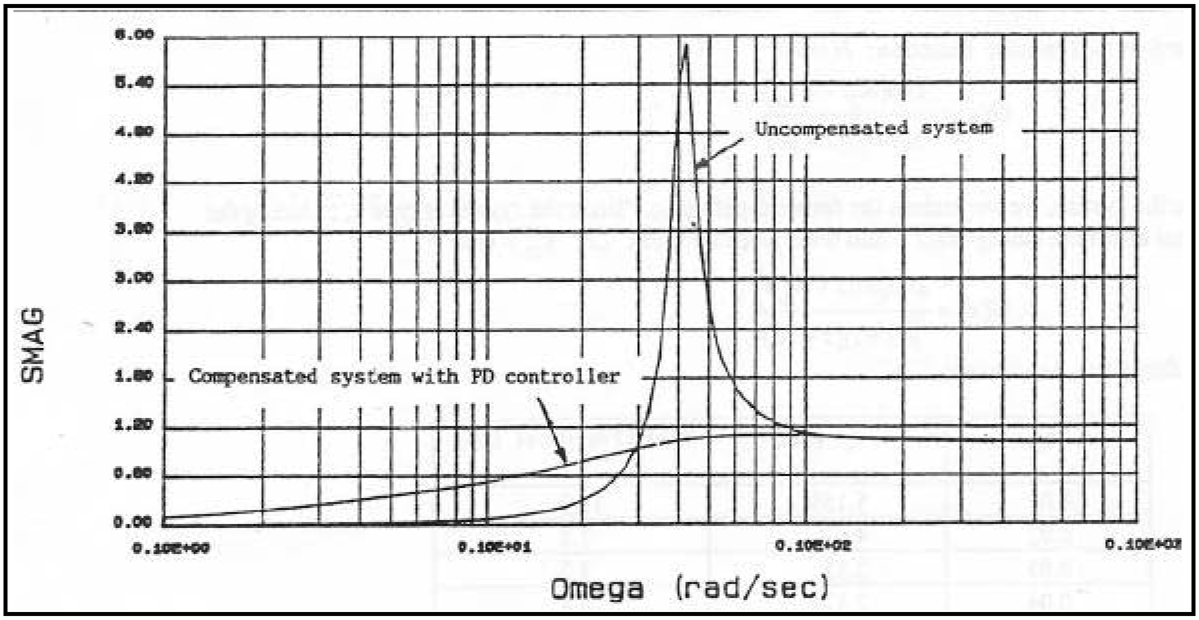
Forward-path Transfer Function:

$$G(s) = \frac{200\left(0.05 + K_D s\right)}{s(s+1)(s+10)}$$

Attributes of Frequency Response:

K_D	PM (deg)	GM (deg)	M_r	BW (rad/sec)
0	47.4	20.83	1.24	1.32
0.01	56.11	∞	1.09	1.24
0.02	64.25	∞	1.02	1.18
0.05	84.32	∞	1.00	1.12
0.09	93.80	∞	1.00	1.42
0.10	93.49	∞	1.00	1.59
0.11	92.71	∞	1.00	1.80
0.20	81.49	∞	1.00	4.66
0.30	71.42	∞	1.00	7.79
0.50	58.55	∞	1.03	12.36

For maximum phase margin, the value of K_D is 0.09. PM = 93.80 deg. GM = ∞, $M_r = 1$, and BW = 1.42 rad/sec.



(b) Sensitivity Plots:

The PD control reduces the peak value of the sensitivity function $\left| S_G^M(j\omega) \right|$

10-8 (a) Forward-path Transfer Function:

$$G(s) = \frac{100\left(K_P + \dfrac{K_I}{s}\right)}{s^2 + 10s + 100} \qquad \text{For } K_v = 10, \qquad K_v = \lim_{s\to 0} sG(s) = \lim_{s\to 0} s\frac{100\left(K_P s + K_I\right)}{s\left(s^2 + 10s + 100\right)} = K_I = 10$$

Thus $K_I = 10$.

(b) Let the complex roots of the characteristic equation be written as $s = -\sigma + j15$ an d $s = -\sigma - j15$.

The quadratic portion of the characteristic equation is $s^2 + 2\sigma s + \left(\sigma^2 + 225\right) = 0$

The characteristic equation of the system is $s^3 + 10s^2 + \left(100 + 100K_P\right)s + 1000 = 0$

The quadratic equation must satisfy the characteristic equation. Using long division and solve for zero remainder condition.

$$\begin{array}{r|l}
 & s + (10 - 2\sigma) \\
\hline
s^2 + 2\sigma s + \sigma^2 + 225 & s^3 + 10s^2 + \left(100 + 100K_P\right)s + 1000 \\
 & s^3 + 2\sigma s^2 + \left(\sigma^2 + 225\right)s \\
\hline
 & (10 - 2\sigma)s^2 + \left(100K_P - \sigma^2 - 125\right)s + 1000 \\
 & (10 - 2\sigma)s^2 + \left(20\sigma - 4\sigma^2\right)s + (10 - 2\sigma)\left(\sigma^2 + 225\right)
\end{array}$$

$$\overline{\left(100K_P + 3s^2 - 20s - 125\right)s + 2s^3 - 10s^2 + 450s - 1250}$$

For zero remainder, $2s^3 - 10s^2 + 450s - 1250 = 0$ (1)

and $100K_P + 3s^2 - 20s - 125 = 0$ (2)

The real solution of Eq. (1) is $s = 2.8555$. From Eq. (2),

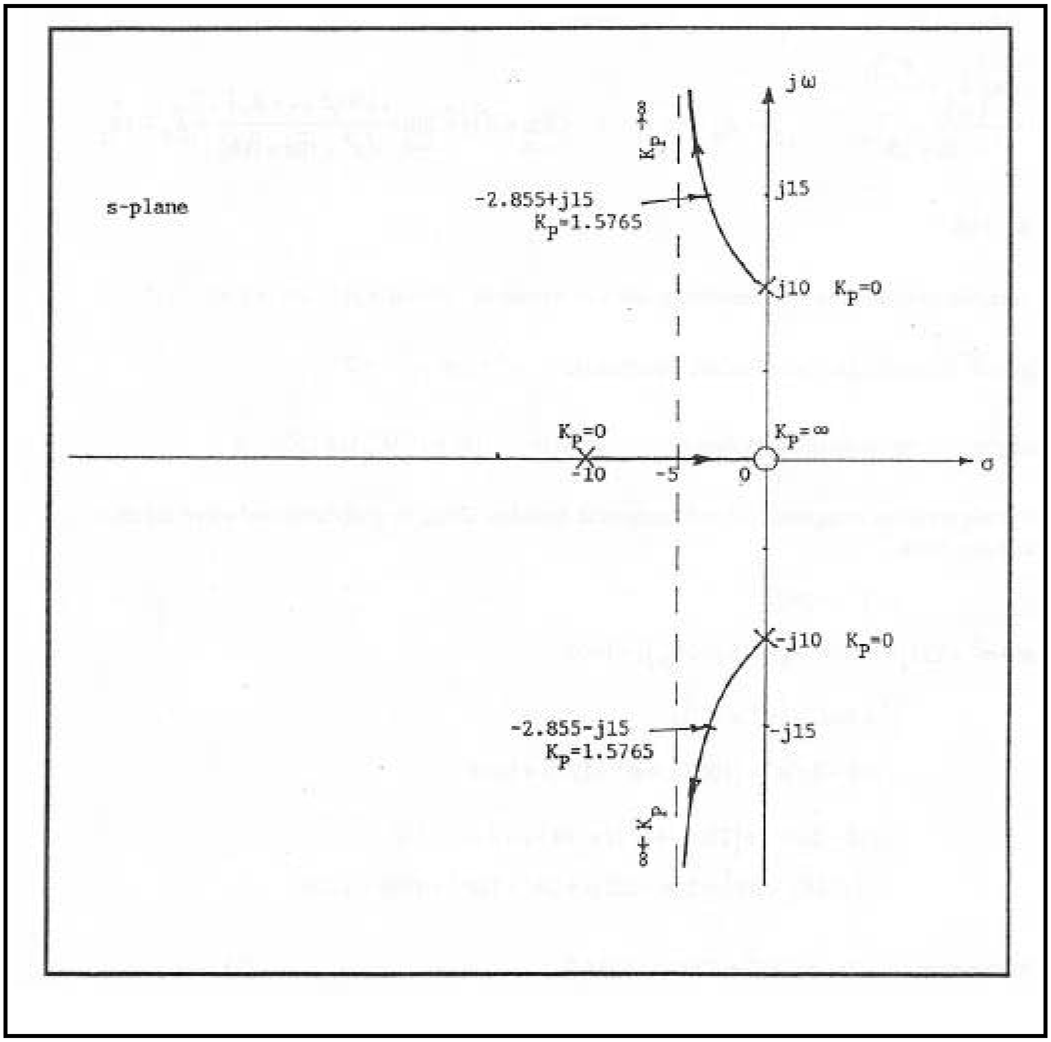
$$K_P = \frac{125 + 20s - 3s^2}{100} = 1.5765$$

The characteristic equation roots are: $s = -2.8555 + j15$, $-2.8555 - j15$, and $s = -10 + 2s = -4.289$

(c) Root Contours:

$$G_{eq}(s) = \frac{100K_P s}{s^3 + 10s^2 + 100s + 1000} = \frac{100K_P s}{(s+10)(s^2+100)}$$

Root Contours:



10-9 (a) Forward-path Transfer Function:

$$G(s) = \frac{100\left(K_P + \frac{K_I}{s}\right)}{s^2 + 10s + 100} \qquad \text{For } K_v = 10, \qquad K_v = \lim_{s \to 0} sG(s) = \lim_{s \to 0} s\frac{100\left(K_P s + K_I\right)}{s\left(s^2 + 10s + 100\right)} = K_I = 10$$

Thus the forward-path transfer function becomes

$$G(s) = \frac{10\left(1 + 0.1K_P s\right)}{s\left(1 + 0.1s + 0.01s^2\right)}$$

Attributes of the Frequency Response:

K_P	PM (deg)	GM (dB)	M_r	BW (rad/sec)
0.1	5.51	1.21	10.05	14.19

0.5	22.59	6.38	2.24	15.81
0.6	25.44	8.25	1.94	16.11
0.7	27.70	10.77	1.74	16.38
0.8	29.40	14.15	1.88	16.62
0.9	30.56	20.10	1.97	17.33
1.0	31.25	$\propto$	2.00	18.01
1.5	31.19	$\propto$	1.81	20.43
1.1	31.51	$\propto$	2.00	18.59
1.2	31.40	$\propto$	1.97	19.08

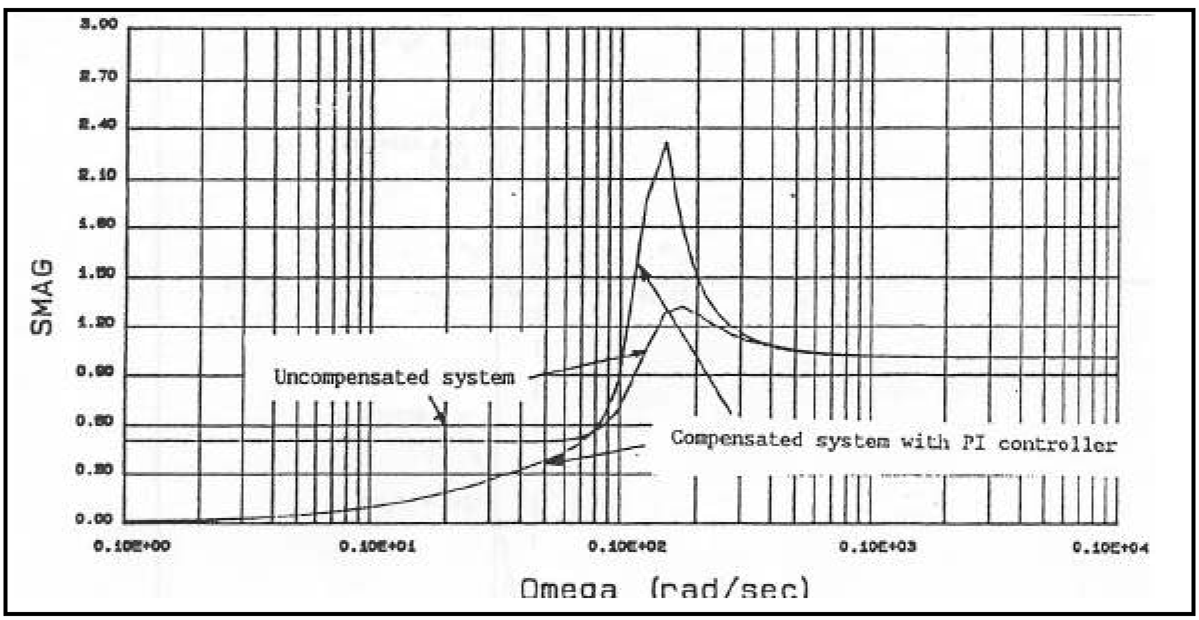
When $K_P = 1.1$ and $K_I = 10$, $K_v = 10$, the phase margin is 31.51 deg., and is maximum.

The corresponding roots of the characteristic equation roots are:

$$-5.4, \quad -2.3 + j13.41, \text{ and } \quad -2.3 - j13.41$$

Referring these roots to the root contours in Problem 10-8(c), the complex roots corresponds to a relative damping ratio that is near optimal.

(b) Sensitivity Function:



In the present case, the system with the PI controller has a higher maximum value for the sensitivity function.

10-10 (a) Forward-path Transfer Function:

$$G(s) = \frac{100\left(K_P s + K_I\right)}{s\left(s^2 + 10s + 100\right)}$$

For $K_v = 100$,

$$K_v = \lim_{s \to 0} sG(s) = \lim_{s \to 0} s \frac{100\left(K_P s + K_I\right)}{s\left(s^2 + 10s + 100\right)} = K_I = 100 \qquad \text{Thus } K_I = 100 .$$

(b) The characteristic equation is $s^3 + 10s^2 + \left(100 + 100K_P\right)s + 100K_I = 0$

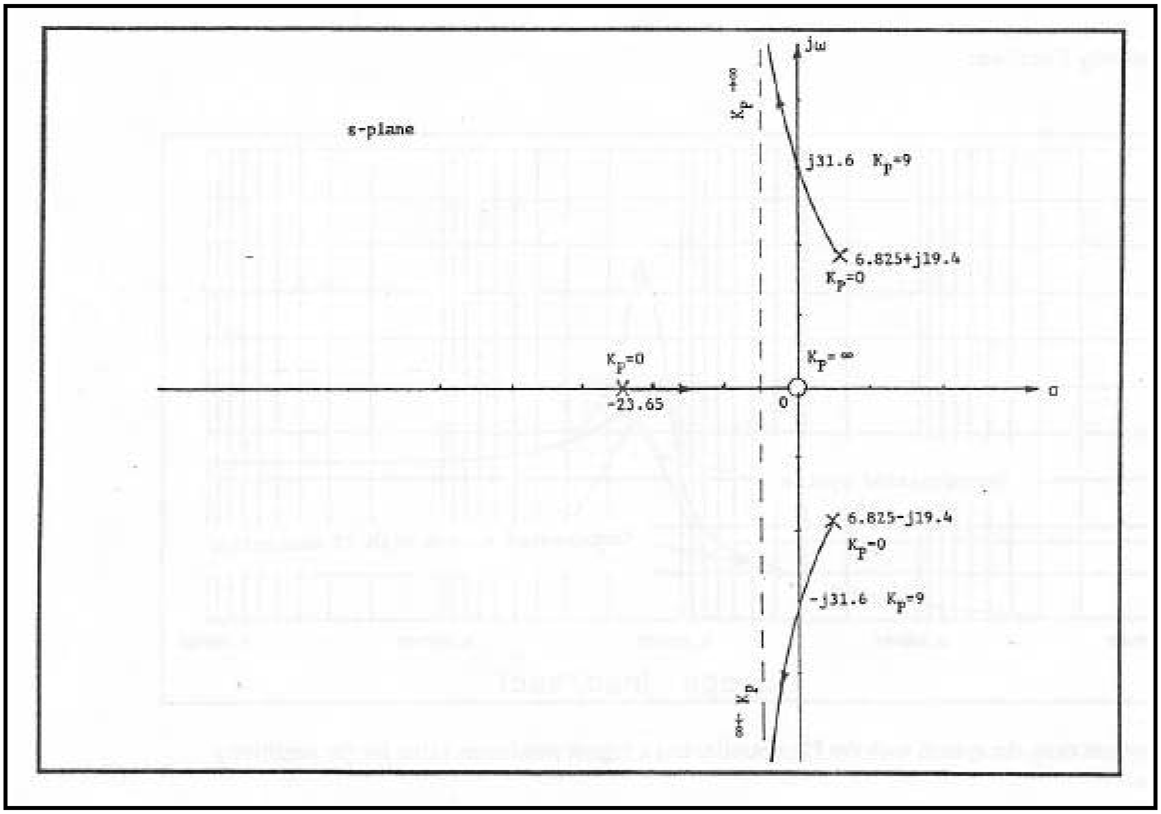
Routh Tabulation:

s^3	1	$100 + 100K_P$
s^2	10	10,000
s^1	$100K_P - 900$	0
s^0	10,000	

For stability, $100K_P - 900 > 0$ Thus $K_P > 9$

Root Contours:

$$G_{eq}(s) = \frac{100K_P s}{s^3 + 10s^2 + 100s + 10{,}000} = \frac{100K_P s}{(s + 23.65)(s - 6.825 + j19.4)(s - 6.825 - j19.4)}$$



(c) $K_I = 100$

$$G(s) = \frac{100\left(K_P s + 100\right)}{s\left(s^2 + 10s + 100\right)}$$

The following maximum overshoots of the system are computed for various values of K_P.

K_P	15	20	22	24	25	26	30	40	100	1000
y_{max}	1.794	1.779	1.7788	1.7785	1.7756	1.779	1.782	1.795	1.844	1.859

When $K_P = 25$, **minimum** $y_{max} = 1.7756$

10-11 (a) Forward-path Transfer Function:

$$G(s) = \frac{100\left(K_P s + K_I\right)}{s\left(s^2 + 10s + 100\right)} \qquad \text{For } K_v = \frac{100\, K_I}{100} = 10, \qquad K_I = 10$$

(b) Characteristic Equation: $s^3 + 10s^2 + 100\left(K_P + 1\right)s + 1000 = 0$

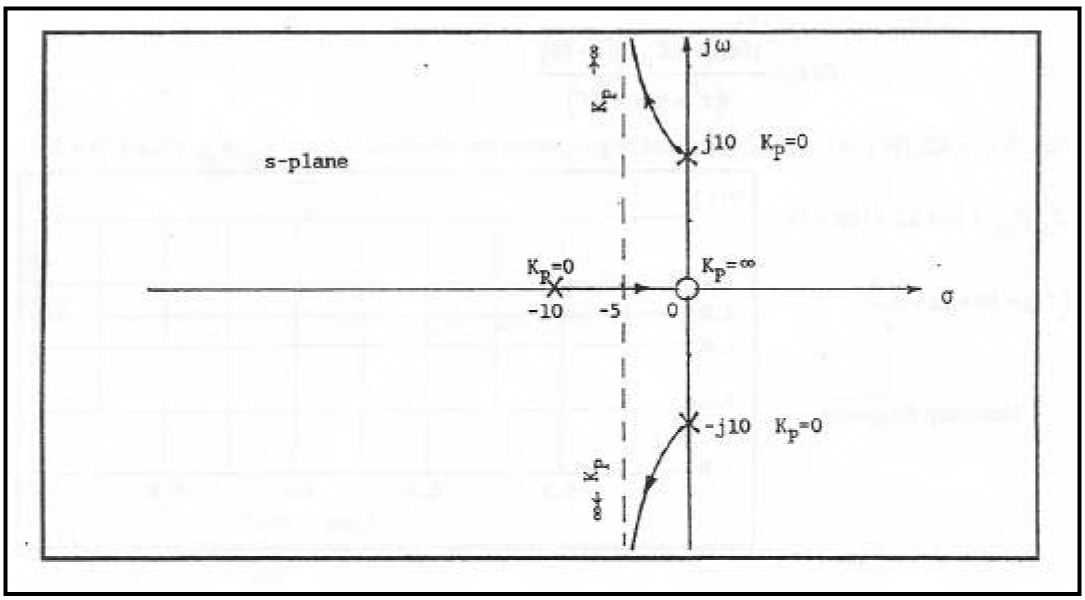
Routh Tabulation:

s^3	1	$100 + 100K_P$
s^2	10	1000
s^1	$100K_P$	0
s^0	1000	

For stability, $K_P > 0$

Root Contours:

$$G_{eq}(s) = \frac{100 K_P s}{s^3 + 10s^2 + 100s + 1000}$$



(c) The maximum overshoots of the system for different values of K_P ranging from 0.5 to 20 are computed and tabulated below.

K_P	0.5	1.0	1.6	1.7	1.8	1.9	2.0	3.0	5.0	10	20
y_{max}	1.393	1.275	1.2317	1.2416	1.2424	1.2441	1.246	1.28	1.372	1.514	1.642

When $K_P = 1.7$, maximum $y_{max} = 1.2416$

10-12

$$G_c(s) = K_P + K_D s + \frac{K_I}{s} = \frac{K_D s^2 + K_P s + K_I}{s} = \left(1 + K_{D1}s\right)\left(K_{P2} + \frac{K_{I2}}{s}\right)$$

where

$$K_P = K_{P2} + K_{D1}K_{I2} \qquad K_D = K_{D1}K_{P2} \qquad K_I = K_{I2}$$

Forward-path Transfer Function:

$$G(s) = G_c(s)G_p(s) = \frac{100\left(1+K_{D1}s\right)\left(K_{P2}s+K_{I2}\right)}{s\left(s^2+10s+100\right)} \qquad K_v = \lim_{s\to 0} sG(s) = K_{I2} = 100$$

Thus

$$K_I = K_{I2} = 100$$

Consider only the PI controller, (with $K_{D1} = 0$)

Forward-path Transfer Function:

$$G(s) = \frac{100\left(K_{P2}s+100\right)}{s\left(s^2+10s+100\right)}$$

Characteristic Equation:

$$s^3 + 10s^2 + \left(100 + 100K_{P2}\right)s + 10{,}000 = 0$$

For stability, $K_{P2} > 9$. Select $K_{P2} = 10$ for fast rise time.

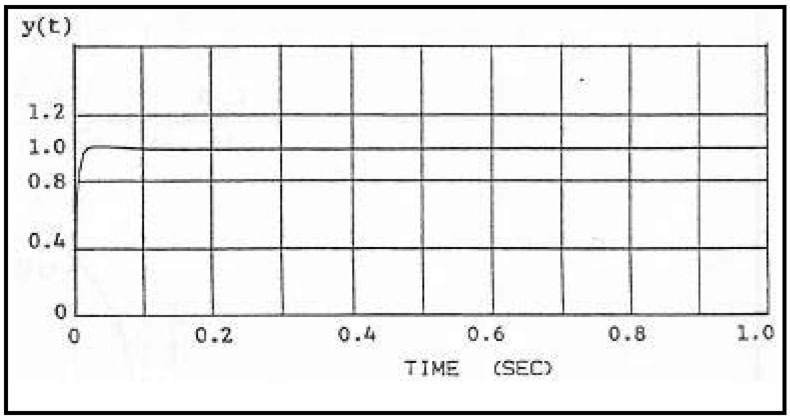
$$G(s) = \frac{1000\left(1+K_{D1}s\right)\left(s+10\right)}{s\left(s^2+10s+100\right)}$$

When $K_{D1} = 0.2$, the rise time and overshoot requirements are satisfied. $K_D = K_{D1}K_{P2} = 0.2 \times 10 = 2$

$$K_P = K_{P2} + K_{D1}K_{I2} = 10 + 0.2 \times 100 = 30$$

$$G_c(s) = 30 + 2s + \frac{100}{s}$$

Unit-step Response



10-13 Process Transfer Function:

$$G_p(s) = \frac{Y(s)}{U(s)} = \frac{e^{-0.2s}}{1+0.25s} \cong \frac{1}{\left(1+0.25s\right)\left(1+0.2s+0.02s^2\right)}$$

(a) PI Controller:

$$G(s) = G_c(s)G_p(s) \cong \frac{K_P + \dfrac{K_I}{s}}{\left(1+0.25s\right)\left(1+0.2s+0.02s^2\right)} = \frac{200\left(K_Ps+K_I\right)}{s\left(s+4\right)\left(s^2+10s+50\right)}$$

For $K_v = 2$, $\quad K_v = \lim_{s\to 0} sG(s) = \frac{200K_I}{4 \times 50} = K_I = 2 \quad$ Thus $K_I = 2$

Thus
$$G(s) = \frac{200\left(2+K_Ps\right)}{s\left(s+4\right)\left(s^2+10s+50\right)}$$

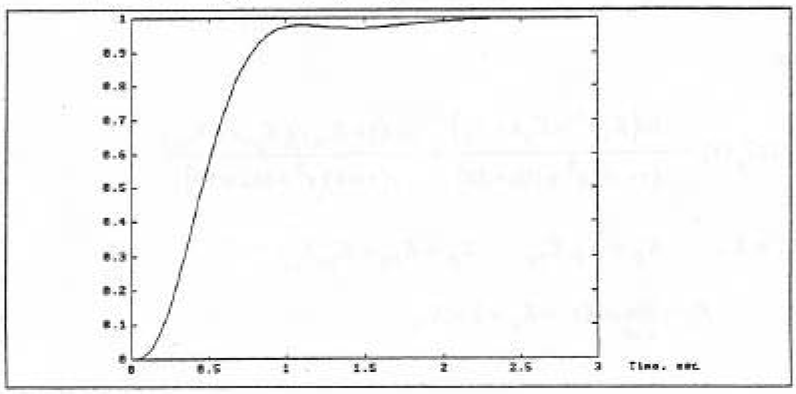
The following values of the attributes of the unit-step response are computed for the system with various values for K_P.

K_P	Max overshoot (%)	t_s (sec)	t_s (sec)

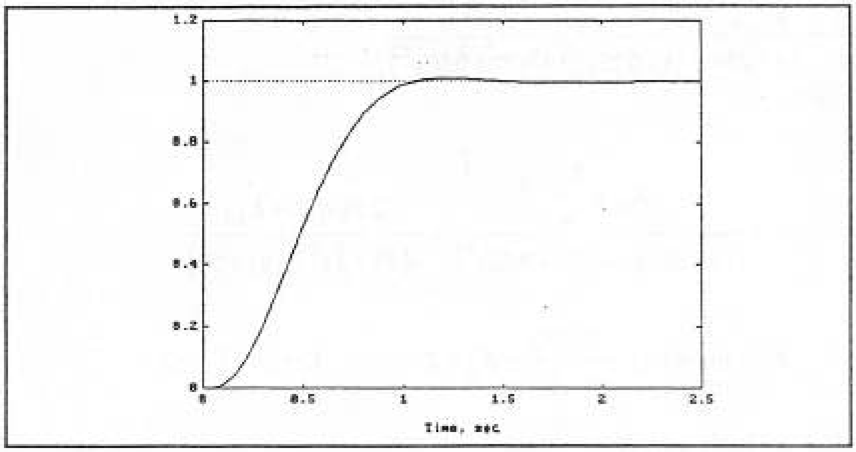
0.1	19.5	0.61	2.08
0.2	13.8	0.617	1.966
0.3	8.8	0.615	1.733
0.4	4.6	0.606	0.898
0.5	1.0	0.5905	0.878
0.6	0	0.568	0.851
0.7	0	0.541	1.464
0.8	0	0.5078	1.603
1.0	0	0.44	1.618

The settling time t_s is minimum (0.851 sec) when $K_P = 0.6$. Statistically, $K_P = 0.6$ is the best choice. The unit-step response is shown below. However, a better response is obtained when $K_P = 0.5$.

Unit-step Response: ($K_P = 0.6$, $K_I = 2$)



Unit-step Response: ($K_P = 0.5$, $K_I = 2$)



For stability check we perform the Routh tabulation. The characteristic equation with $K_I = 2$ is

$$s^4 + 14s^3 + 90s^2 + (200 + 200K_P)s + 400 = 0$$

Routh Tabulation:

s^4	1	90	400
s^3	14	$200 + 200K_P$	
s^2	$75.714 - 14.284K_P$	400	
s^1	$\dfrac{9542.8 + 12285.66K_P - 2857.14K_P^2}{75.714 - 14.284K_P}$		
s^0	400		

For the coefficients in the first row to be positive, from the s^2 row, $K_P < 5.3$. From the s^1 row,

$$9542.8 + 12285.66K_P - 2857.14K_P^2 > 0 \quad \text{or} \quad (K_P - 4.9718)(K_P + 0.6718) < 0$$

Thus $\quad K_P < 4.9718$ which is the condition for stability.

10-13 (b) PID Controller:

$$G(s) = G_c(s)G_p(s) = \frac{200(K_D s^2 + K_P s + K_I)}{s(s+4)(s^2 + 10s + 50)} = \frac{200(1 + K_{D1}s)(K_{P2}s + K_{I2})}{s(s+4)(s^2 + 10s + 50)}$$

where

$$K_I = K_{I2} \qquad K_D = K_{D1}K_{P2} \qquad K_P = K_{P2} + K_{D1}K_{I2}$$

$$K_v = \lim_{s \to 0} sG(s) = K_I = 2 = K_{I2}$$

$$G(s) = \frac{200(1+K_{D1}s)(K_{P2}s+2)}{s(s+4)(s^2+10s+50)} = \frac{200(K_D s^2 + K_P s + K_I)}{s(s+4)(s^2+10s+50)}$$

From the results in part (a), we set $K_P = 0.6$. The following attributes of the unit-step response show that adding derivative control does not provide any further improvement to the system response.

K_D	Max Overshoot (%)	t_r (sec)	t_s (sec)
0.1	1.1	0.9568	1.247
0.05	0.1	0.792	1.14
0.01	0	0.608	0.9075
0.005	0	0.588	0.8828
0.001	0	0.572	0.8753
0.0005	0	0.570	0.8778

10-14 Forward-path Transfer Function:

$$G(s) = G_c(s)G_p(s) = \frac{(K_P s + K_I)e^{-0.2s}}{s(1+0.25s)} \qquad K_v = \lim_{s \to 0} sG(s) = K_I = 2 \qquad \text{Thus } K_I = 2$$

The attributes of the frequency response for various values of K_P are computed and tabulated below.

K_P	PM (deg)	GM (deg)	M_r	BW (rad/sec)
0.1	49.72	10.91	1.196	3.55
0.2	54.5	12.58	1.092	3.54
0.3	59.0	13.15	1.027	3.56
0.5	67.07	11.88	1.000	3.81
0.6	70.50	10.92	1.000	4.20
0.7	73.41	9.98	1.000	5.09
0.8	75.65	9.10	1.000	6.62
0.9	76.93	8.27	1.000	7.99
1.0	77.04	7.50	1.000	9.05
1.1	75.81	6.78	1.033	9.90
2.0	31.08	2.03	4.029	13.64
2.4	8.51	0.52	12.55	14.52
2.5545	0	0	∞	

Maximum phase margin of 77.04 deg is obtained when $K_P = 1.0$. In Problem 10-13(a), $K_P = 0.6$ is chosen for 0 maximum overshoot and minimum settling time.

The critical value of K_P for stability is 2.5545. In Problem 10-13(a), the critical value of K_P is 4.9718.

10-15 (a)

$$G_p(s) = \frac{Z(s)}{F(s)} = \frac{1}{Ms^2 + K_s} = \frac{1}{150s^2 + 1} = \frac{0.00667}{s^2 + 0.00667}$$

The transfer function $G_p(s)$ has poles on the $j\omega$ axis. The natural undamped frequency is $\omega_n = 0.0816$ rad/sec.

(b) PID Controller:

$$G(s) = G_c(s)G_p(s) = \frac{0.00667\left(K_D s^2 + K_P s + K_I\right)}{s\left(s^2 + 0.00667\right)}$$

$K_v = \lim_{s\to 0} sG(s) = K_I = 100$ Thus $K_I = 100$

Characteristic Equation: $s^3 + 0.00667K_D s^2 + 0.00667\left(1 + K_P\right)s + 0.00667K_I = 0$

For $\zeta = 0.707$ and $\omega_n = 1$ rad / sec, the second-order term of the characteristic equation is $s^2 + 1.414s + 1 = 0$. Divide the characteristic equation by the second-order term.

$$
\begin{array}{l}
\qquad\qquad s + \left(0.00667K_D - 1.414\right) \\
s^2 + 1.414s + 1 \,\big|\, s^3 + 0.00667K_D s^2 + \left(0.00667 + 0.00667K_P\right)s + 0.00667K_I \\
\qquad\qquad s^3 + 1.414s^2 + 1 \\
\qquad\qquad\quad \left(0.00667K_D - 1.414\right)s^2 + \left(0.00667K_P - 0.99333\right)s + 0.00667K_I \\
\qquad\qquad\quad \left(0.00667K_D - 1.414\right)s^2 + \left(0.00943K_D - 2\right)s + 0.00667K_d - 1.414 \\
\qquad\qquad\qquad \left(0.00667K_P - 0.00943K_D + 1.00667\right)s + 0.00667K_I - 0.00667K_D + 1.41
\end{array}
$$

For zero remainder, $0.00667K_I - 0.00667K_D + 1.414 = 0$ (1)

and $0.00667K_P - 0.00943K_D + 1.00667 = 0$ (2)

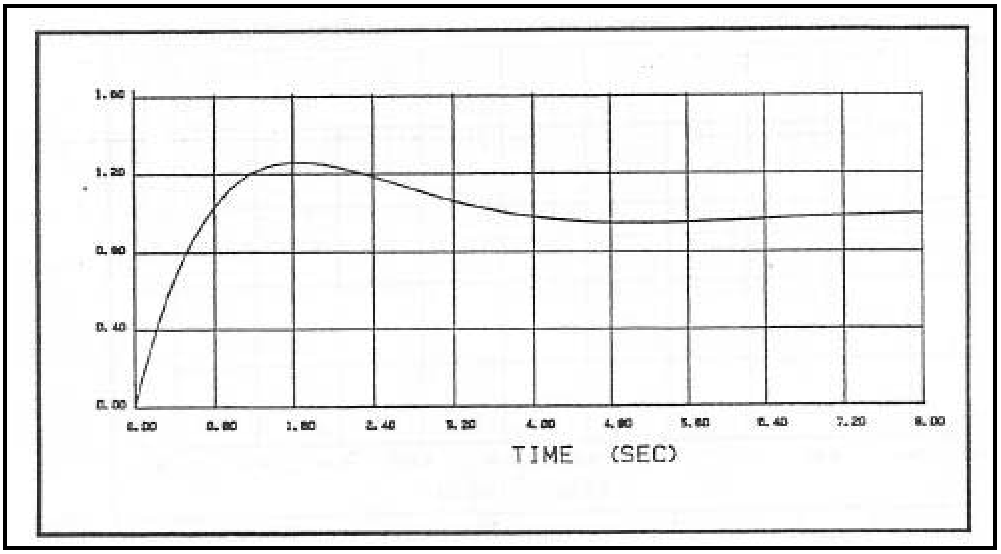
From Eq. (1), $K_D = \dfrac{2.081}{0.00667} = 312$

From Eq. (2), $K_P = \dfrac{0.00943K_D - 1.00667}{0.00667} - 290.18$

The forward-path transfer function becomes,

$$G(s) = \frac{2.081s^2 + 1.9355s + 0.667}{s\left(s^2 + 0.00667\right)}$$

Unit-step Response.



The unit-step response shows a maximum overshoot of 26%. Although the relative damping ratio of the complex roots is 0.707, the real pole of the third-order system transfer function is at −0.667 which adds to the overshoot.

(c)

$$G(s) = G_c(s)G_p(s) = \frac{0.00667\left(1+K_{D1}s\right)\left(K_{P2}s+K_{I2}\right)}{s\left(s^2+0.00667\right)}$$

For $K_v = 100$, $K_{I2} = K_I = 100$. Let us select $K_{P2} = 50$. Then

$$G(s) = \frac{0.00667\left(1+K_{D1}s\right)\left(50s+100\right)}{s\left(s^2+0.00667\right)}$$

For a small overshoot, K_{D1} must be relatively large. When $K_{D1} = 100$, the maximum overshoot is approximately 4.5%. Thus,

$$K_P = K_{P2} + K_{D1}K_{I2} = 50 + 100 \times 100 = 10050$$

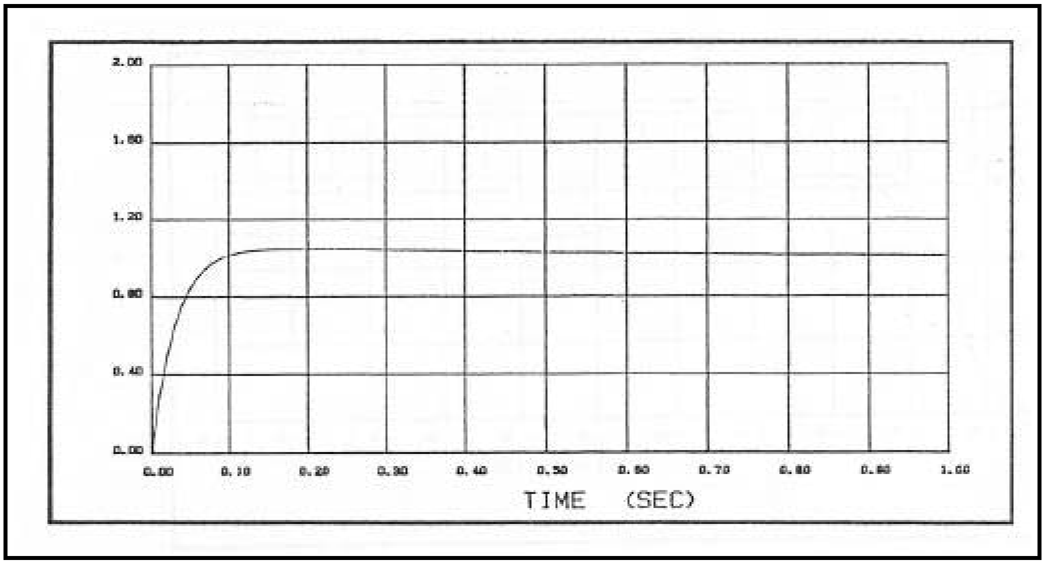
$$K_D = K_{D1}K_{P2} = 100 \times 50 = 5000$$

$$K_I = 100$$

System Characteristic Equation: $s^3 + 33.35s^2 + 67.04s + 0.667 = 0$

Roots: **– 0.01, – 2.138, – 31.2**

Unit-step Response.



10-16 (a)

$$G_p(s) = \frac{Z(s)}{F(s)} = \frac{1}{Ms^2 + K_s} = \frac{1}{150s^2 + 1} = \frac{0.00667}{s^2 + 0.00667}$$

The transfer function $G_p(s)$ has poles on the $j\omega$ axis. The natural undamped frequency is $\omega_n = 0.0816$ rad/sec.

(b) PID Controller:

$$G(s) = G_c(s)G_p(s) = \frac{0.00667\left(K_D s^2 + K_P s + K_I\right)}{s\left(s^2 + 0.00667\right)}$$

$$K_v = \lim_{s \to 0} sG(s) = K_I = 100 \qquad \text{Thus} \qquad K_I = 100$$

Characteristic Equation: $s^3 + 0.00667K_D s^2 + 0.00667\left(1 + K_P\right)s + 0.00667K_I = 0$

For $\zeta = 1$ and $\omega_n = 1$ rad/sec, the second-order term of the characteristic equation is $s^2 + 2s + 1$.
Dividing the characteristic equation by the seond-order term.

$$\begin{array}{r|l}
 & s + \left(0.00667K_D - 2\right) \\
\hline
s^2 + 2s + 1 & s^3 + 0.00667K_D s^2 + \left(0.00667 + 0.00667K_P\right)s + 0.00667K_I \\
 & s^3 + 2s^2 + s \\
\hline
 & \left(0.00667K_D - 2\right)s^2 + \left(0.00667K_P - 0.99333\right)s + 0.00667K_I \\
 & \left(0.00667K_D - 2\right)s^2 + \left(0.01334K_D - 4\right)s + 0.00667K_D - 2 \\
\hline
 & \left(0.00667K_P - 0.01334K_D + 3.00667\right)s + 0.00667K_I - 0.00667K_D + 2
\end{array}$$

For zero remainder,

$$0.00667K_P - 0.01334K_D + 3.00667 = 0 \qquad (1)$$

$$-0.00667K_D + 0.00667K_I + 2 = 0 \qquad (2)$$

From Eq. (2),

$$0.00667 K_D = 0.00667 K_I + 2 = 2.667 \qquad \text{Thus} \qquad K_D = 399.85$$

From Eq. (1),

$$0.00667 K_P = 0.01334 K_D - 3.00667 = 2.3273 \qquad \text{Thus} \qquad K_P = 348.93$$

Forward-path Transfer Function:

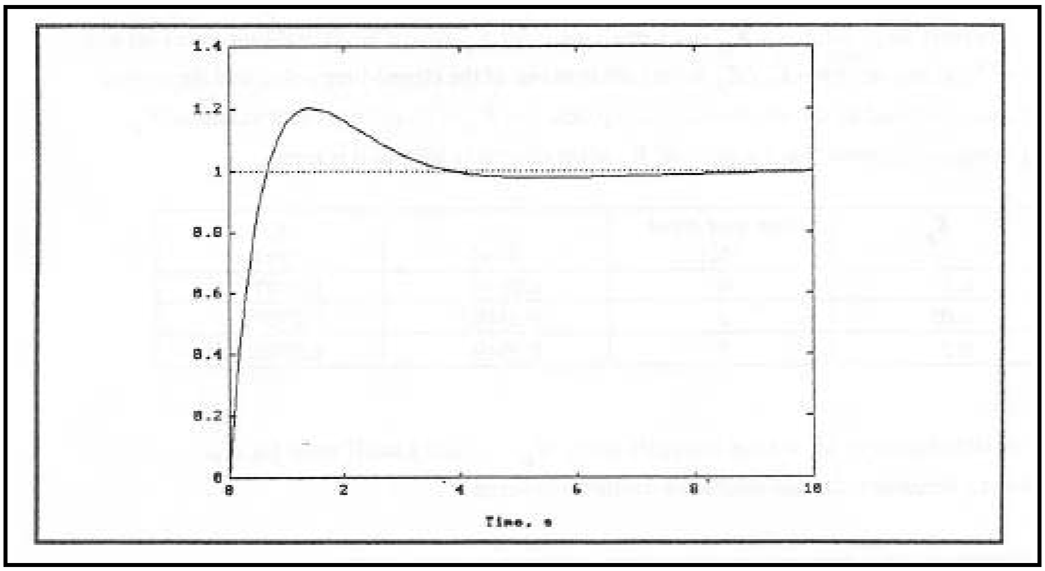
$$G(s) = \frac{0.00667\left(399.85s^2 + 348.93s + 100\right)}{s\left(s^2 + 0.00667\right)}$$

Characteristic Equation:

$$s^3 + 2.667s^2 + 2.334s + 0.667 = (s+1)^2(s+0.667) = 0$$

Roots: -1, -1, -0.667

Unit-step Response.



The maximum overshoot is 20%.

10-17 (a) Process Transfer Function:

$$G_p(s) = \frac{4}{s^2}$$

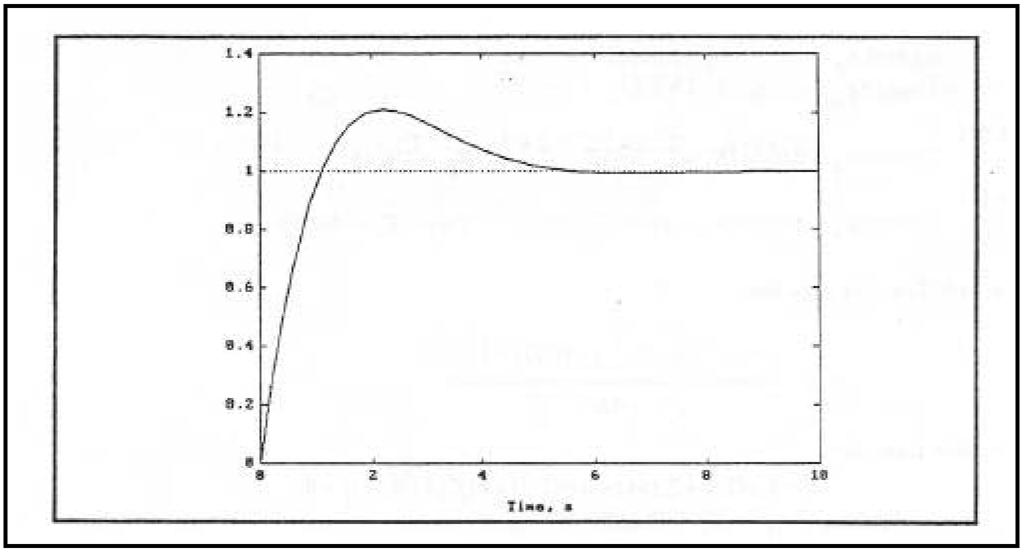
Forward-path Transfer Function

$$G(s) = G_c(s)G_p(s) = \frac{4\left(K_P + K_D s\right)}{s^2}$$

Characteristic Equation: $s^2 + 4K_D s + 4K_P = s^2 + 1.414s + 1 = 0$ for $\zeta = 0.707$, $\omega_n = 1$ rad/sec

$K_P = 0.25$ and $K_D = 0.3535$

Unit-step Response.



Maximum overshoot = 20.8%

(b) Select a relatively large value for K_D and a small value for K_P so that the closed-loop poles are real. The closed-loop zero at $s = -K_P / K_D$ is very close to one of the closed-loop poles, and the system dynamics are governed by the other closed-loop poles. Let $K_D = 10$ and use small values of K_P. The following results show that the value of K_P is not critical as long as it is small.

K_p	Max overshoot (%)	t_r (sec)	t_s (sec)
0.1	0	0.0549	0.0707
0.05	0	0.0549	0.0707
0.2	0	0.0549	0.0707

(c) For BW $\leq$ 40 rad/sec and $M_r = 1$, we can again select $K_D = 10$ and a small value for K_P. The following frequency-domain results substantiate the design.

K_p	PM (deg)	M_r	BW (rad/sec)
0.1	89.99	1	40
0.05	89.99	1	40
0.2	89.99	1	40

10-18 (a) Forward-path Transfer Function: **Characteristic Equation:**

$$G(s) = G_c(s)G_p(s) = \frac{10{,}000\left(K_P + K_D s\right)}{s^2\left(s+10\right)} \qquad s^3 + 10s^2 + 10{,}000K_D s + 10{,}000K_P = 0$$

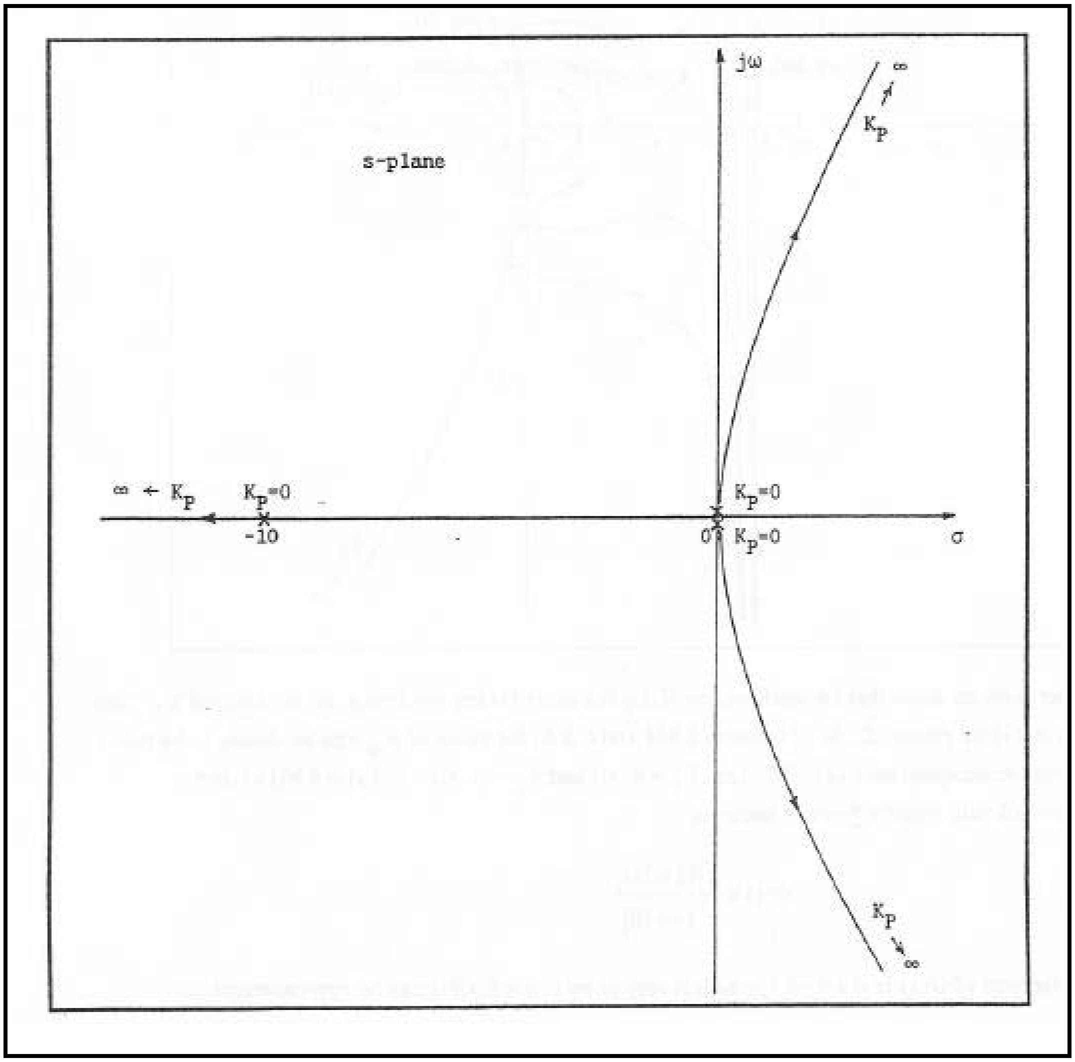
Routh Tabulation:

s^3	1	$10{,}000\ K_D$
s^2	10	$10{,}000\ K_P$
s^1	$10{,}000\ K_D - 1000\ K_P$	0
s^0	$10{,}000\ K_P$	

The system is stable for $K_P > 0$ and $K_D > 0.1K_P$

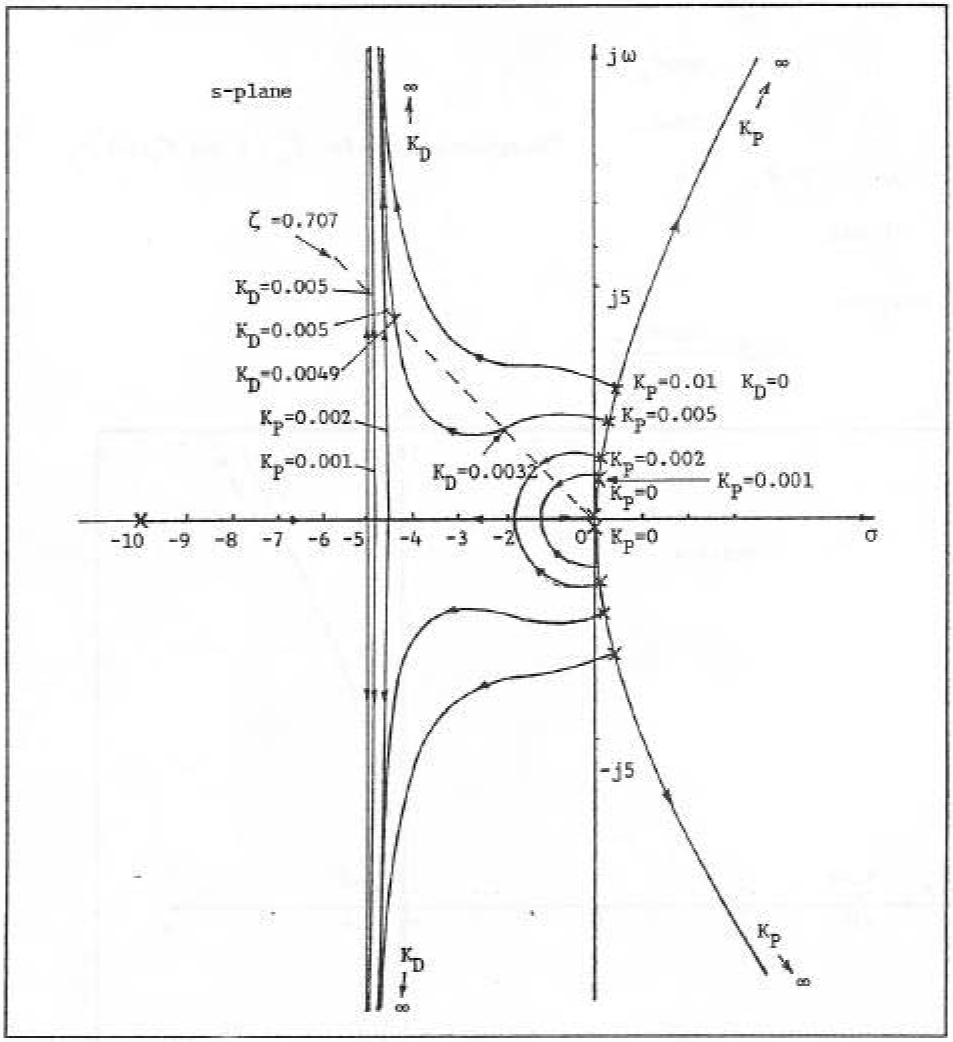
(b) Root Locus Diagram:

$$G(s) = \frac{10{,}000K_P}{s^2(s+10)}$$



Root Contours: $0 \le K_D < \infty$, $K_P = 0.001,\ 0.002,\ 0.005,\ 0.01$.

$$G_{eq}(s) = \frac{10{,}000\ K_D s}{s^3 + 10s^2 + 10{,}000\ K_P}$$



(c) The root contours show that for small values of K_P the design is insensitive to the variation of K_P. This means that if we choose K_P to be between 0.001 and 0.005, the value of K_D can be chosen to be 0.005 for a relative damping ratio of 0.707. Let $K_P = 0.001$ and $K_D = 0.005$. $G_c(s) = 0.001 + 0.005s$. The forward-path transfer function becomes

$$G(s) = \frac{10(1+5s)}{s^2(s+10)}$$

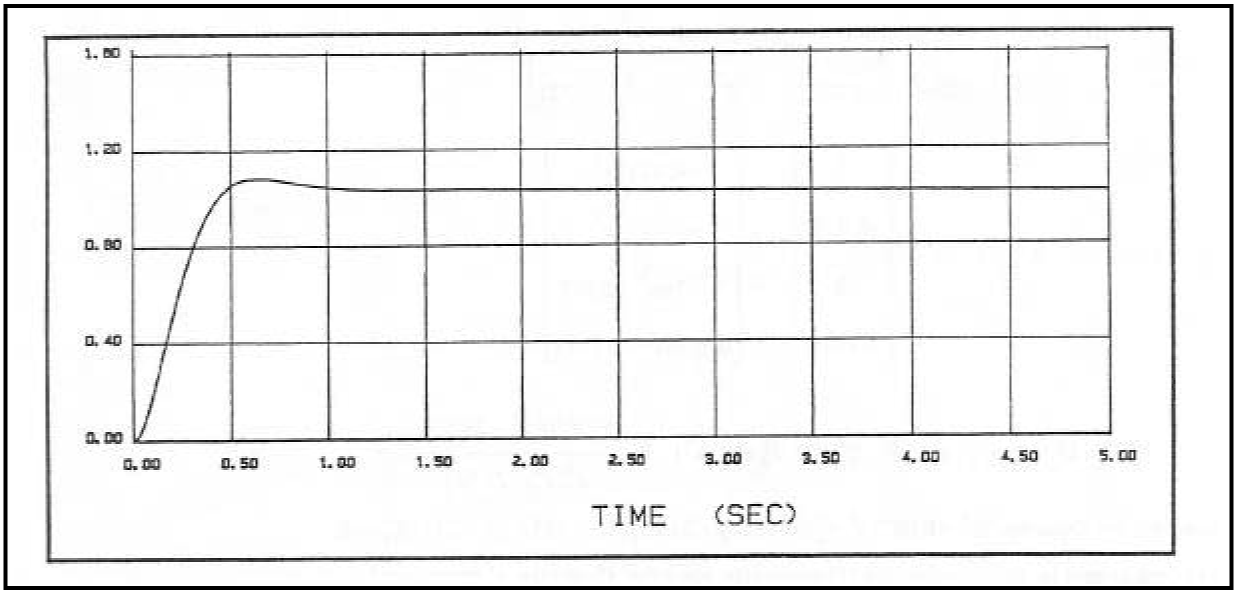
Since the zero of $G(s)$ is at $s = -0.2$, which is very close to $s = 0$, $G(s)$ can be approximated as:

$$G(s) \cong \frac{50}{s(s+10)}$$

For the second-order system, $\zeta = 0.707$. Using Eq. (7-104), the rise time is obtained as

$$t_r = \frac{1 - 0.4167\zeta + 2.917\zeta^2}{\omega_n} = 0.306 \text{ sec}$$

Unit-step Response:



(d) Frequency-domain Characteristics:

$$G(s) = \frac{10(1+5s)}{s^2(s+10)}$$

PM (deg)	GM (dB)	M_r	BW (rad/sec)
63	∞	1.041	7.156

10-19

$$\mathbf{A}^* = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 25.92 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -2.36 & 0 & 0 & 0 \end{bmatrix} \qquad s\mathbf{I} - \mathbf{A}^* = \begin{bmatrix} s & -1 & 0 & 0 \\ -25.92 & s & 0 & 0 \\ 0 & 0 & s & -1 \\ 2.36 & 0 & 0 & s \end{bmatrix}$$

$$\Delta = \left| s\mathbf{I} - \mathbf{A}^* \right| = s\begin{vmatrix} s & 0 & 0 \\ 0 & s & -1 \\ 0 & 0 & s \end{vmatrix} + \begin{vmatrix} -25.92 & 0 & 0 \\ 0 & s & -1 \\ 2.36 & 0 & s \end{vmatrix} = s^2\left(s^2 - 25.92\right)$$

$$\left(s\mathbf{I} - \mathbf{A}^*\right)^{-1} = \frac{1}{\Delta}\begin{bmatrix} s^3 & s^2 & 0 & 0 \\ 25.92s^2 & s^3 & 0 & 0 \\ -2.36s & -2.36 & s^3 - 25.92s & s^2 - 25.92 \\ -2.36s^2 & -2.36s & 0 & s^3 - 25.92s \end{bmatrix}$$

$$\left(s\mathbf{I}-\mathbf{A}^*\right)^{-1}\mathbf{B}=\left(s\mathbf{I}-\mathbf{A}^*\right)^{-1}\begin{bmatrix}0\\-0.0732\\0\\0.0976\end{bmatrix}=\frac{1}{\Delta}\begin{bmatrix}-0.0732s^2\\-0.0732s^3\\0.0976s^2-2.357\\0.0976s^3-2.357s\end{bmatrix}$$

$$Y(s)=\mathbf{D}\left(s\mathbf{I}-\mathbf{A}^*\right)^{-1}\mathbf{B}=\begin{bmatrix}0&0&1&0\end{bmatrix}\left(s\mathbf{I}-\mathbf{A}^*\right)^{-1}\mathbf{B}=\frac{0.0976\left(s^2-24.15\right)}{s^2\left(s^2-25.92\right)}$$

Characteristic Equation: $s^4+0.0976s^3+\left(0.0976K_P-25.92\right)s^2-2.357K_Ds-2.357K_P=0$

The system cannot be stabilized by the PD controller, since the s^3 and the s^1 terms involve K_D which require opposite signs for K_D.

10-20 Let us first attempt to compensate the system with a PI controller.

$$G_c(s)=K_P+\frac{K_I}{s}\qquad \text{Then}\qquad G(s)=G_c(s)\,G_p(s)=\frac{100\left(K_Ps+K_I\right)}{s\left(s^2+10s+100\right)}$$

Since the system with the PI controller is now a type 1 system, the steady-state error of the system due to a step input will be zero as long as the values of K_P and K_I are chosen so that the system is stable.

Let us choose the ramp-error constant $K_v=100$. Then, $K_I=100$. The following frequency-domain performance characteristics are obtained with $K_I=100$ and various value of K_P ranging from 10 to 100.

K_P	PM (deg)	GM (dB)	M_r	BW (rad/sec)
10	1.60	∞	29.70	50.13
20	6.76	∞	7.62	69.90
30	7.15	∞	7.41	85.40
40	6.90	∞	8.28	98.50
50	6.56	∞	8.45	106.56
100	5.18	∞	11.04	160.00

The maximum phase margin that can be achieved with the PI controller is only 7.15 deg when $K_P=30$.

Thus, the overshoot requirement cannot be satisfied with the PI controller alone.
Next, we try a PID controller.

$$G_c(s)=K_P+K_Ds+\frac{K_I}{s}=\frac{\left(1+K_{D1}s\right)\left(K_{P2}s+K_{I2}\right)}{s}=\frac{\left(1+K_{D1}s\right)\left(K_{P2}s+100\right)}{s}$$

Based on the PI-controller design, let us select $K_{P2}=30$. Then the forward-path transfer function becomes

$$G(s)=\frac{100\left(30s+100\right)\left(1+K_{D1}s\right)}{s\left(s^2+10s+100\right)}$$

The following attributes of the frequency-domain performance of the system with the PID controller are obtained for various values of K_{D1} ranging from 0.05 to 0.4.

K_{D1}	PM (deg)	GM (dB)	M_r	BW (rad/sec)
0.05	85.0	∞	1.04	164.3
0.10	89.4	∞	1.00	303.8
0.20	90.2	∞	1.00	598.6
0.30	90.2	∞	1.00	897.0

0.40	90.2	∝	1.00	1201.0

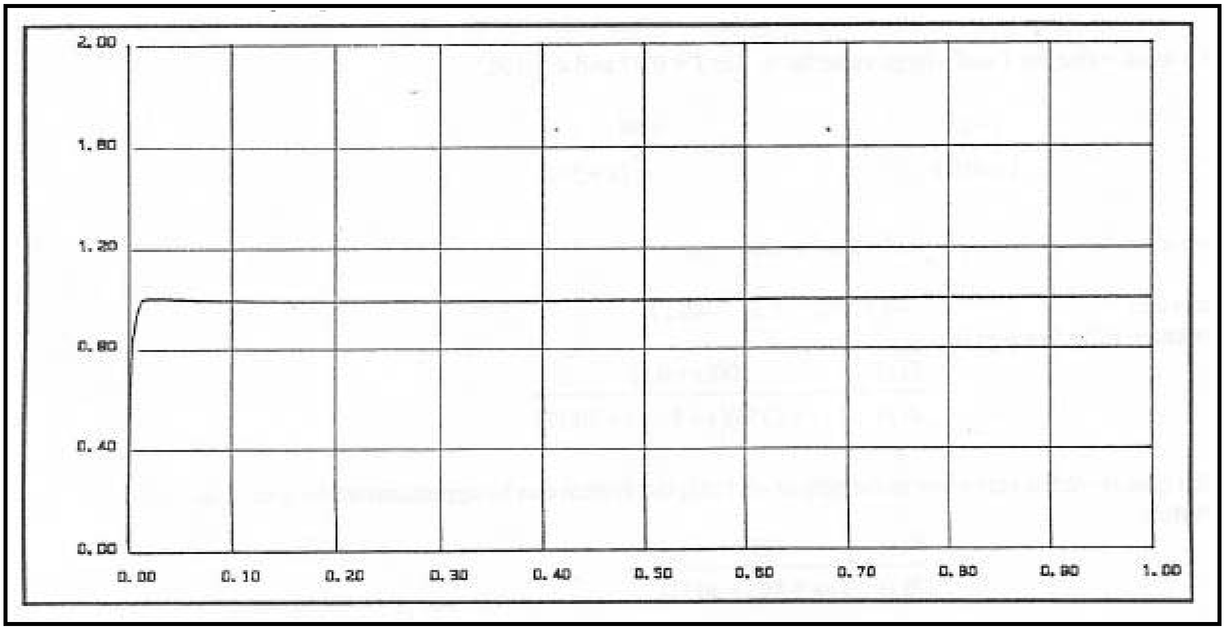
We see that for values of K_{D1} greater than 0.2, the phase margin no longer increases, but the bandwidth increases with the increase in K_{D1}. Thus we choose

$$K_{D1} = 0.2, \quad K_I = K_{I2} = 100, \quad K_D = K_{D1}K_{P2} = 0.2 \times 30 = 6,$$

$$K_P = K_{P2} + K_{D1}K_{I2} = 30 + 0.2 \times 100 = 50$$

The transfer function of the PID controller is $\quad G_c(s) = 50 + 6s + \frac{100}{s}$

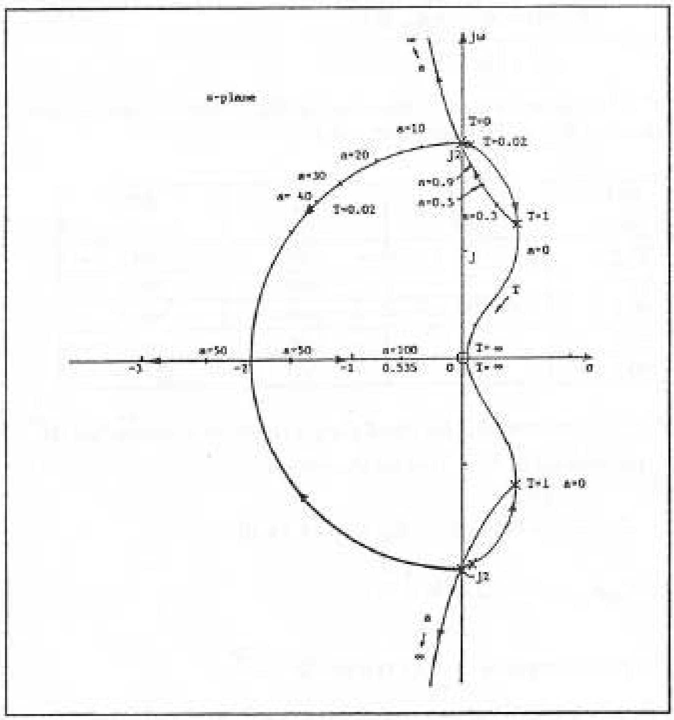
The unit-step response is show below. The maximum overshoot is zero, and the rise time is 0.0172 sec.



10-21 (a)

$$G_p(s) = \frac{4}{s^2} \qquad G(s) = G_c(s)G_p(s) = \frac{4(1+aTs)}{s^2(1+Ts)} \qquad G_{eq}(s) = \frac{4aTs}{Ts^3 + s^2 + 4}$$

Root Contours: (*T* is fixed and *a* varies)



Select a small value for T and a large value for a. Let $T = 0.02$ and $a = 100$.

$$G_c(s) = \frac{1 + 2s}{1 + 0.02s} \qquad G(s) = \frac{400(s+0.5)}{s^2(s+50)}$$

The characteristic equation is $s^3 + 50s^2 + 400s + 200 = 0$

The roots are: $-0.5355, \quad -9.3, \quad -40.17$

The system transfer function is

$$\frac{Y(s)}{R(s)} = \frac{400(s+0.5)}{(s+0.5355)(s+9.3)(s+40.17)}$$

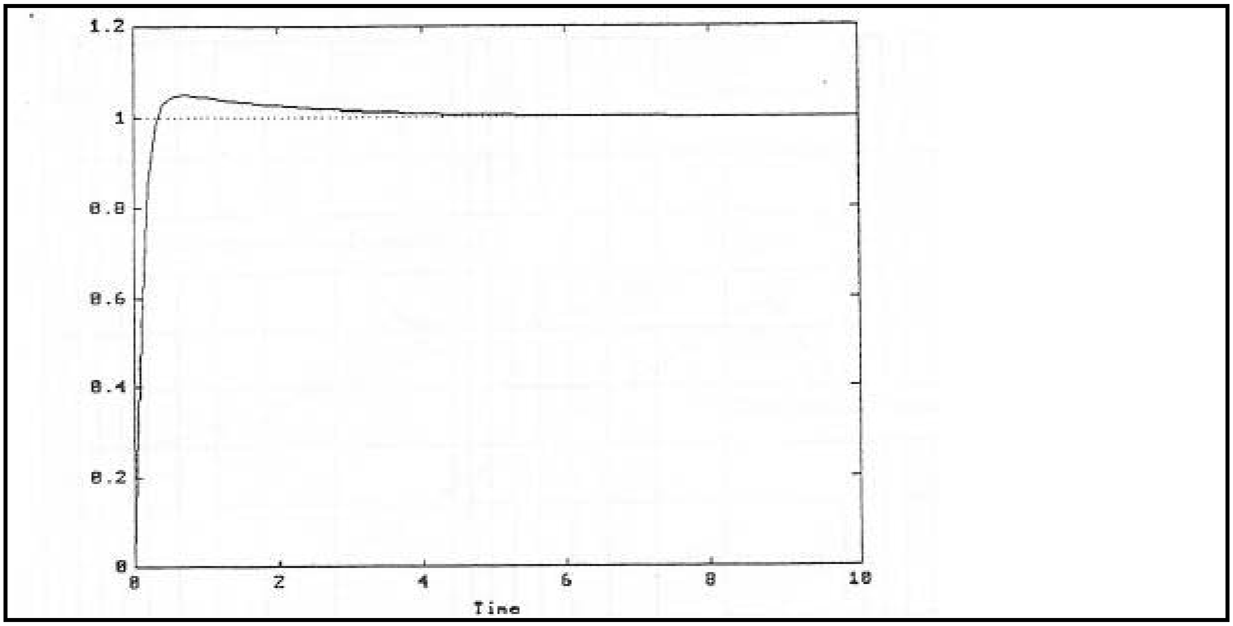
Since the zero at -0.5 is very close to the pole at -0.5355, the system can be approximated by a second-order system,

$$\frac{Y(s)}{R(s)} = \frac{373.48}{(s+9.3)(s+40.17)}$$

The unit-step response is shown below. The attributes of the response are:

Maximum overshoot = 5% $\qquad t_s = 0.6225$ sec $\qquad t_r = 0.2173$ sec

Unit-step Response.



The following attributes of the frequency-domain performance are obtained for the system with the phase-lead controller.

PM = 77.4 deg GM = infinite M_r = 1.05 BW = 9.976 rad/sec

10-21 (b) The Bode plot of the uncompensated forward-path transfer function is shown below. The diagram shows that the uncompensated system is marginally stable. The phase of $G(j\omega)$ is -180 deg at all frequencies. For the phase-lead controller we need to place ω_m at the new gain crossover frequency to realize the desired phase margin which has a theoretical maximum of 90 deg.
For a desired phase margin of 80 deg,

$$a = \frac{1+\sin 80^o}{1-\sin 80^o} = 130$$

The gain of the controller is $20 \log_{10} a = 42$ dB. The new gain crossover frequency is at

$$|G(j\omega)| = -\frac{42}{2} = -21 \text{ dB}$$

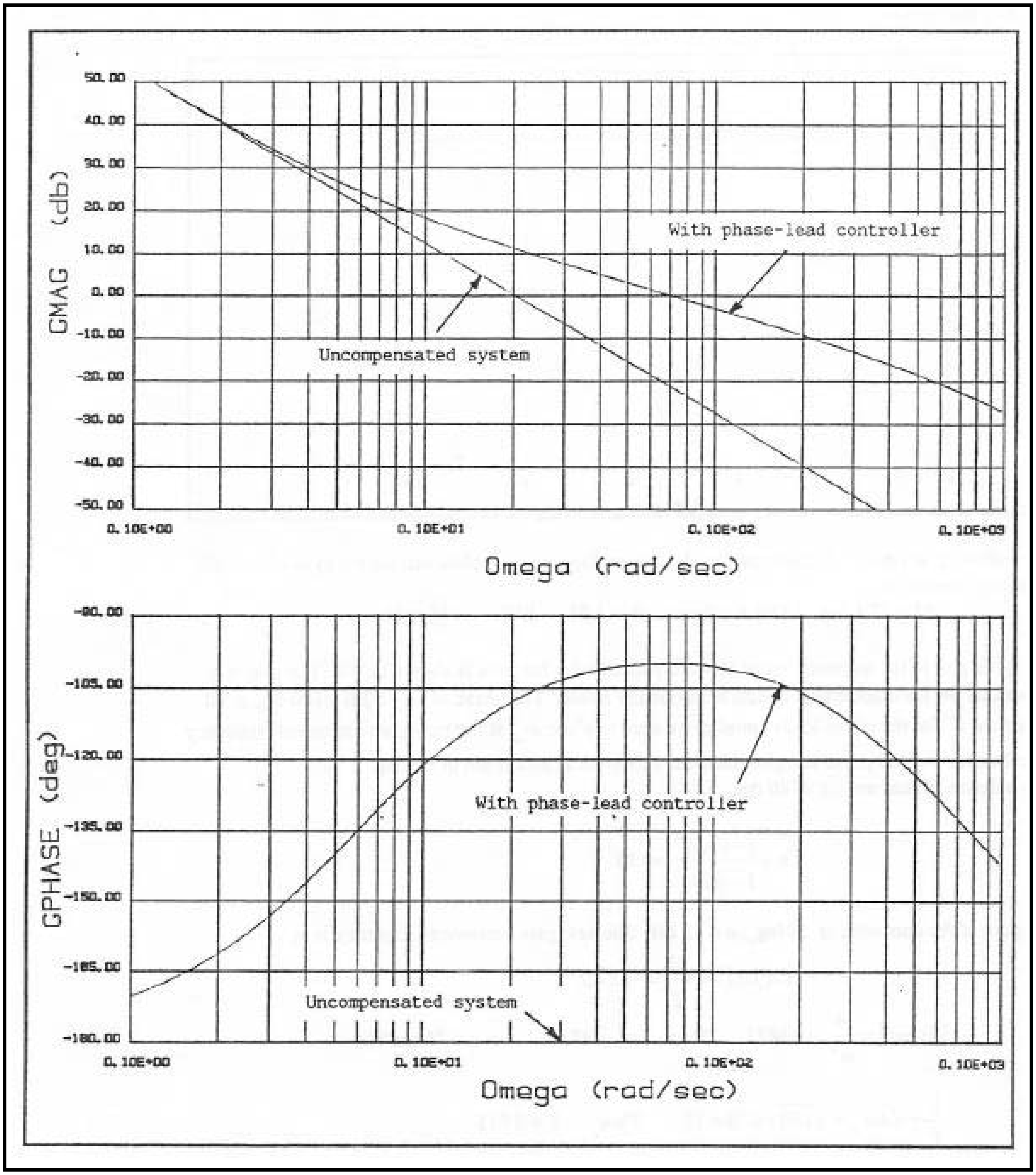
Or $\quad |G(j\omega)| = \dfrac{4}{\omega^2} = 0.0877 \quad$ Thus $\quad \omega^2 = 45.61 \quad \omega = 6.75$ rad/sec

$$\frac{1}{T} = \sqrt{a}\,\omega_m = \sqrt{130} \times 6.75 = 77 \qquad \text{Thus} \qquad T = 0.013$$

$$\frac{1}{aT} = 0.592 \qquad \text{Thus} \qquad aT = 1.69$$

$$G_c(s) = \frac{1+aTs}{1+Ts} = \frac{1+1.702s}{1+0.0131s} \qquad G(s) = \frac{4(1+1.702s)}{s^2(1+0.0131s)}$$

Bode Plot.



10-22 (a) **Forward-path Transfer Function:**

$$G(s) = G_c(s)G_p(s) = \frac{1000(1+aTs)}{s(s+10)(1+Ts)} = \frac{1000a\left(s+\dfrac{1}{aT}\right)}{s(s+10)\left(s+\dfrac{1}{T}\right)}$$

Set $1/aT = 10$ so that the pole of $G(s)$ at $s = -10$ is cancelled. The characteristic equation of the system becomes

$$s^2 + \frac{1}{T}s + 1000a = 0$$

$\omega_n = \sqrt{1000a}$ $\qquad 2\zeta\omega_n = \frac{1}{T} = 2\sqrt{1000a}$ $\qquad$ Thus $a = 40$ and $T = 0.0025$

Controller Transfer Function: **Forward-path Transfer Function:**

$$G_c(s) = \frac{1 + 0.01s}{1 + 0.0025s} \qquad\qquad G(s) = \frac{40{,}000}{s(s+400)}$$

The attributes of the unit-step response of the compensated system are:

Maximum overshoot = 0 $\qquad t_r = 0.0168$ sec $\qquad t_s = 0.02367$ sec

(b) Frequency-domain Design

The Bode plot of the uncompensated forward-path transfer function is made below.

$$G(s) = \frac{1000}{s(s+10)}$$

The attributes of the system are PM = 17.96 deg, GM = infinite. M_r = 3.117, and BW = 48.53 rad/sec.

To realize a phase margin of 75 deg, we need more than 57 deg of additional phase. Let us add an additional 10 deg for safety. Thus, the value of ϕ_m for the phase-lead controller is chosen to be 67 deg. The value of a is calculated from

$$a = \frac{1 + \sin 67^o}{1 - \sin 67^o} = 24.16$$

The gain of the controller is $20\log_{10} a = 20\log_{10} 24.16 = 27.66$ dB. The new gain crossover frequency is at

$$\left|G(j\omega'_m)\right| = -\frac{27.66}{2} = -13.83 \text{ dB}$$

From the Bode plot ω'_m is found to be 70 rad/sec. Thus,

$$\frac{1}{T} = \sqrt{a}T = \sqrt{24.16} \times 70 = 344 \qquad \text{or} \qquad T = 0.0029 \qquad aT = 0.0702$$

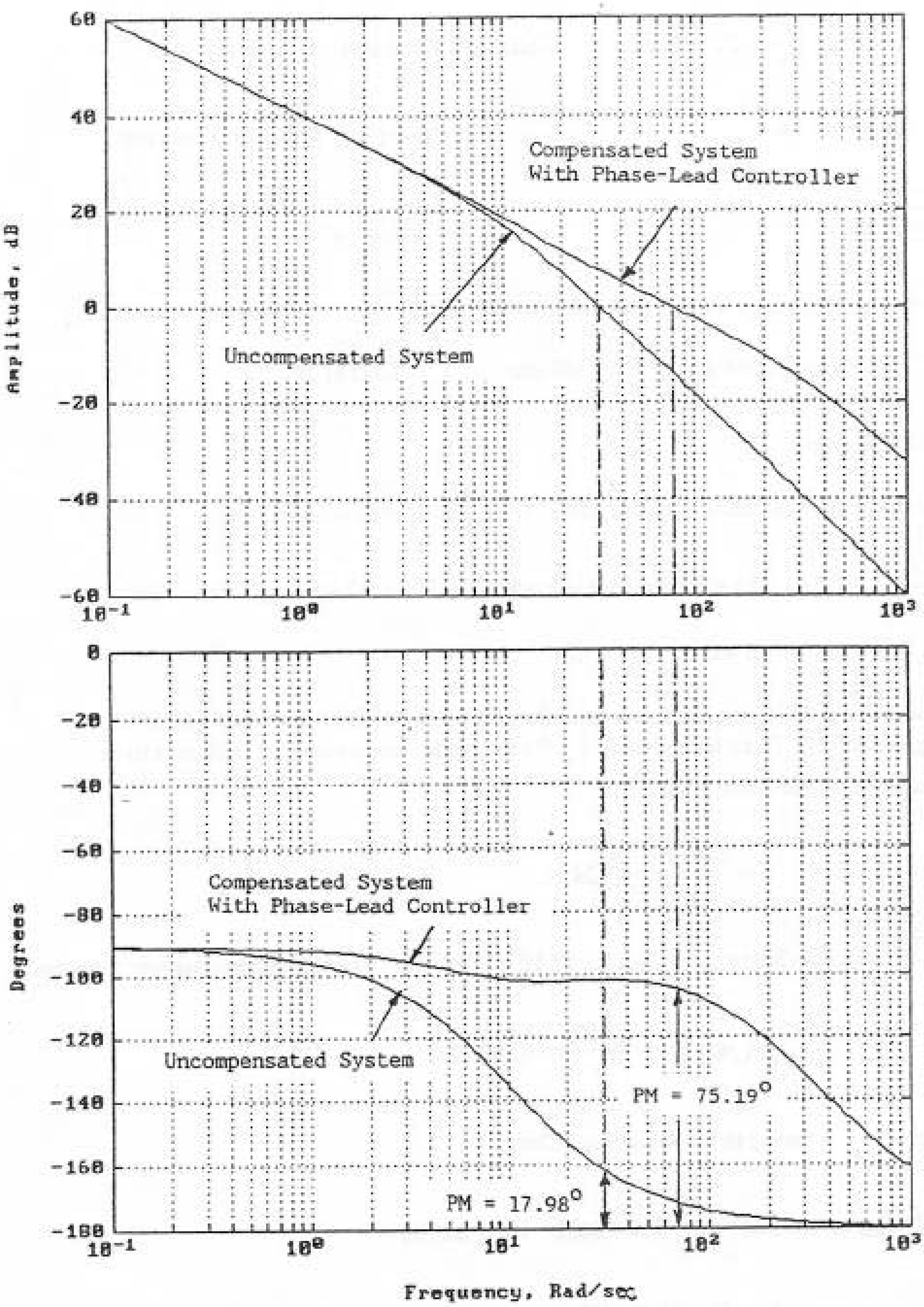
Thus $$G_c(s) = \frac{1 + aTs}{1 + Ts} = \frac{1 + 0.0702s}{1 + 0.0029s}$$

The compensated system has the following frequency-domain attributes:

PM = 75.19 deg $\qquad$ GM = infinite $\qquad M_r$ = 1.024 BW = 91.85 rad/sec

The attributes of the unit-step response are:

Rise time t_r = 0.02278 sec Settling time t_s = 0.02828 sec $\qquad$ Maximum overshoot = 3.3%



10-23 (a) Forward-path Transfer Function: (N = 10)

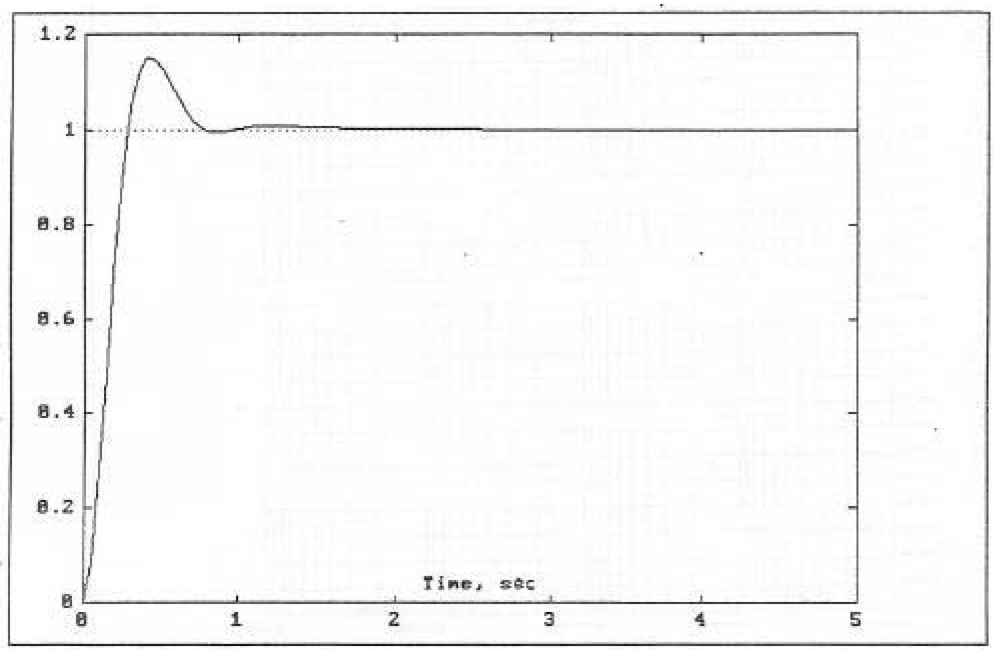
$$G(s) = G_c(s)G_p(s) = \frac{200(1+aTs)}{s(s+1)(s+10)(1+Ts)}$$

Starting with a = 1000, we vary T first to stabilize the system. The following time-domain attributes are obtained by varying the value of T.

T	**Max Overshoot (%)**	t_r	t_s
0.0001	59.4	0.370	5.205
0.0002	41.5	0.293	2.911
0.0003	29.9	0.315	1.83
0.0004	22.7	0.282	1.178
0.0005	18.5	0.254	1.013
0.0006	16.3	0.230	0.844
0.0007	15.4	0.210	0.699
0.0008	15.4	0.192	0.620
0.0009	15.5	0.182	0.533
0.0010	16.7	0.163	0.525

The maximum overshoot is at a minimum when T = 0.0007 or T = 0.0008. The maximum overshoot is 15.4%.

Unit-step Response. (T = 0.0008 sec a = 1000)



10-23 (b) Frequency-domain Design.

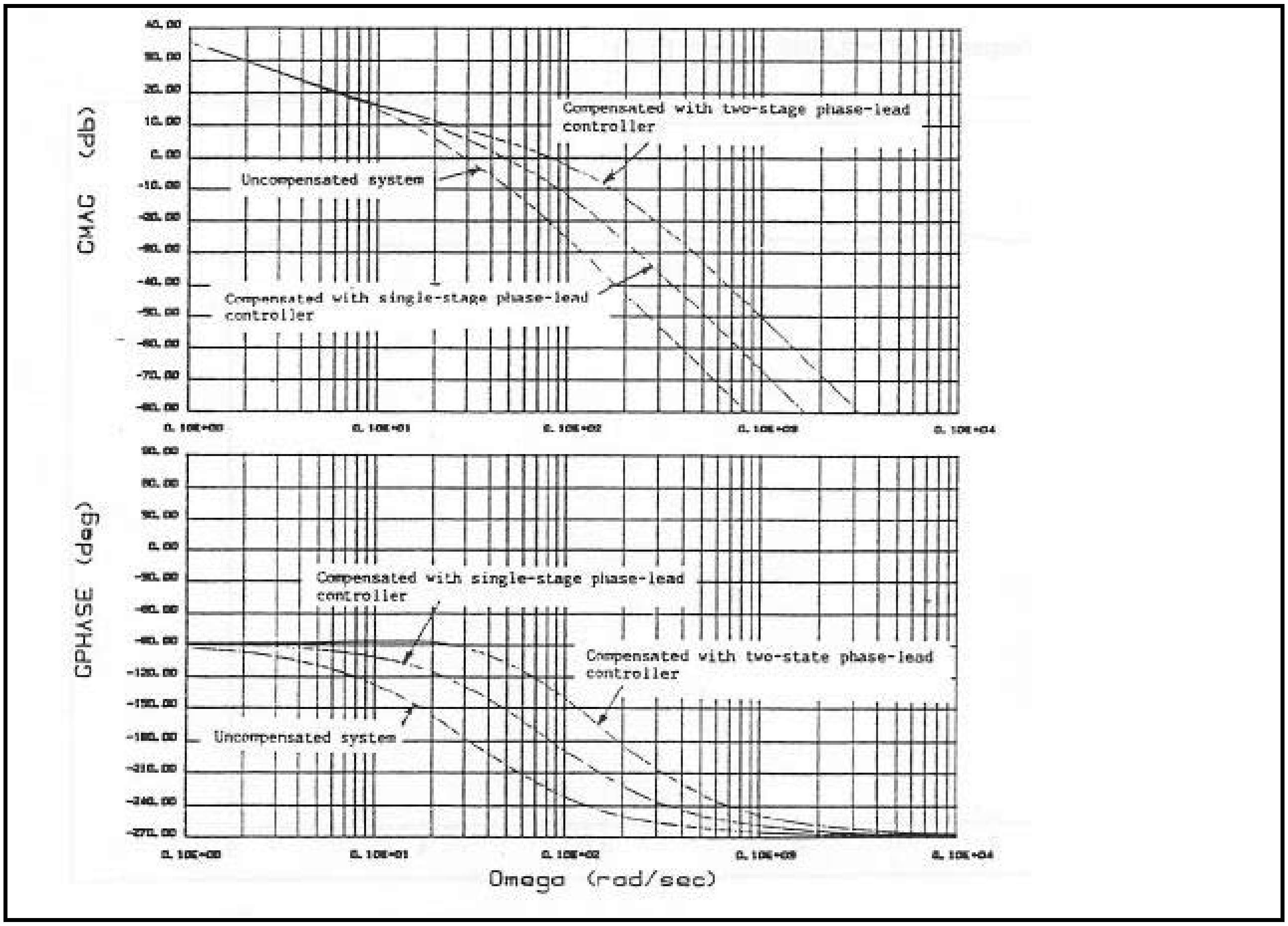
Similar to the design in part (a), we set a = 1000, and vary the value of T between 0.0001 and 0.001. The attributes of the frequency-domain characteristics are given below.

T	**PM (deg)**	**GM (dB)**	M_r	**BW (rad/sec)**
0.0001	17.95	60.00	3.194	4.849
0.0002	31.99	63.53	1.854	5.285
0.0003	42.77	58.62	1.448	5.941
0.0004	49.78	54.53	1.272	6.821
0.0005	53.39	51.16	1.183	7.817
0.0006	54.69	48.32	1.138	8.869
0.0007	54.62	45.87	1.121	9.913

0.0008	53.83	43.72	1.125	10.92
0.0009	52.68	41.81	1.140	11.88
0.0010	51.38	40.09	1.162	12.79

The phase margin is at a maximum of 54.69 deg when T = 0.0006. The performance worsens if the value of a is less than 1000.

10-24 (a) Bode Plot.



The attributes of the frequency response are:

PM = 4.07 deg GM = 1.34 dB M_r = 23.24 BW = 4.4 rad/sec

10-24 (b) Single-stage Phase-lead Controller.

$$G(s) = \frac{6(1+aTs)}{s(1+0.2s)(1+0.5s)(1+Ts)}$$

We first set a = 1000, and vary T. The following attributes of the frequency-domain characteristics are obtained.

T	PM (deg)	M_r
0.0050	17.77	3.21
0.0010	43.70	1.34
0.0007	47.53	1.24
0.0006	48.27	1.22

0.0005	48.06	1.23
0.0004	46.01	1.29
0.0002	32.08	1.81
0.0001	19.57	2.97

The phase margin is maximum at 48.27 deg when T = 0.0006.
Next, we set T = 0.0006 and reduce a from 1000. We can show that the phase margin is not very sensitive to the variation of a when a is near 1000. The optimal value of a is around 980, and the corresponding phase margin is 48.34 deg.
With a = 980 and T = 0.0006, the attributes of the unit-step response are:

Maximum overshoot = 18.8% t_r = 0.262 sec t_s = 0.851 sec

(c) Two-stage Phase-lead Controller. (*a* = 980, *T* = 0.0006)

$$G(s) = \frac{6(1+0.588s)\left(1+bT_2 s\right)}{s(1+0.2s)(1+0.5s)(1+0.0006s)\left(1+T_2 s\right)}$$

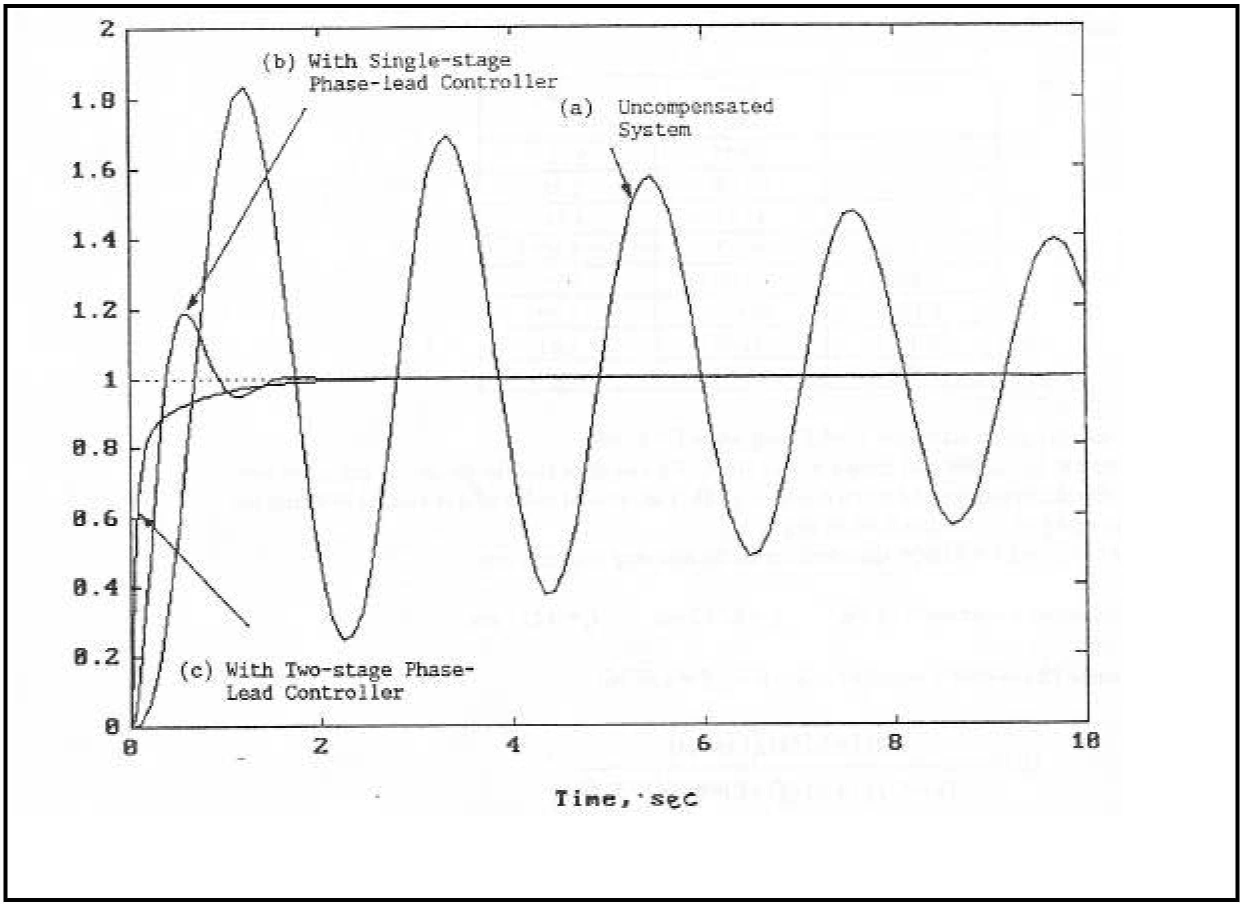
Again, let b = 1000, and vary T_2. The following results are obtained in the frequency domain.

T_2	PM (deg)	M_r
0.0010	93.81	1.00
0.0009	94.89	1.00
0.0008	96.02	1.00
0.0007	97.21	1.00
0.0006	98.43	1.00
0.0005	99.61	1.00
0.0004	100.40	1.00
0.0003	99.34	1.00
0.0002	91.98	1.00
0.0001	73.86	1.00

Reducing the value of b from 1000 reduces the phase margin. Thus, the maximum phase margin of 100.4 deg is obtained with b = 1000 and T_2 = 0.0004. The transfer function of the two-stage phase-lead controller is

$$G_c(s) = \frac{(1+0.588s)(1+0.4s)}{(1+0.0006s)(1+0.0004s)}$$

(c) Unit-step Responses.



10-25 (a) The loop transfer function of the system is

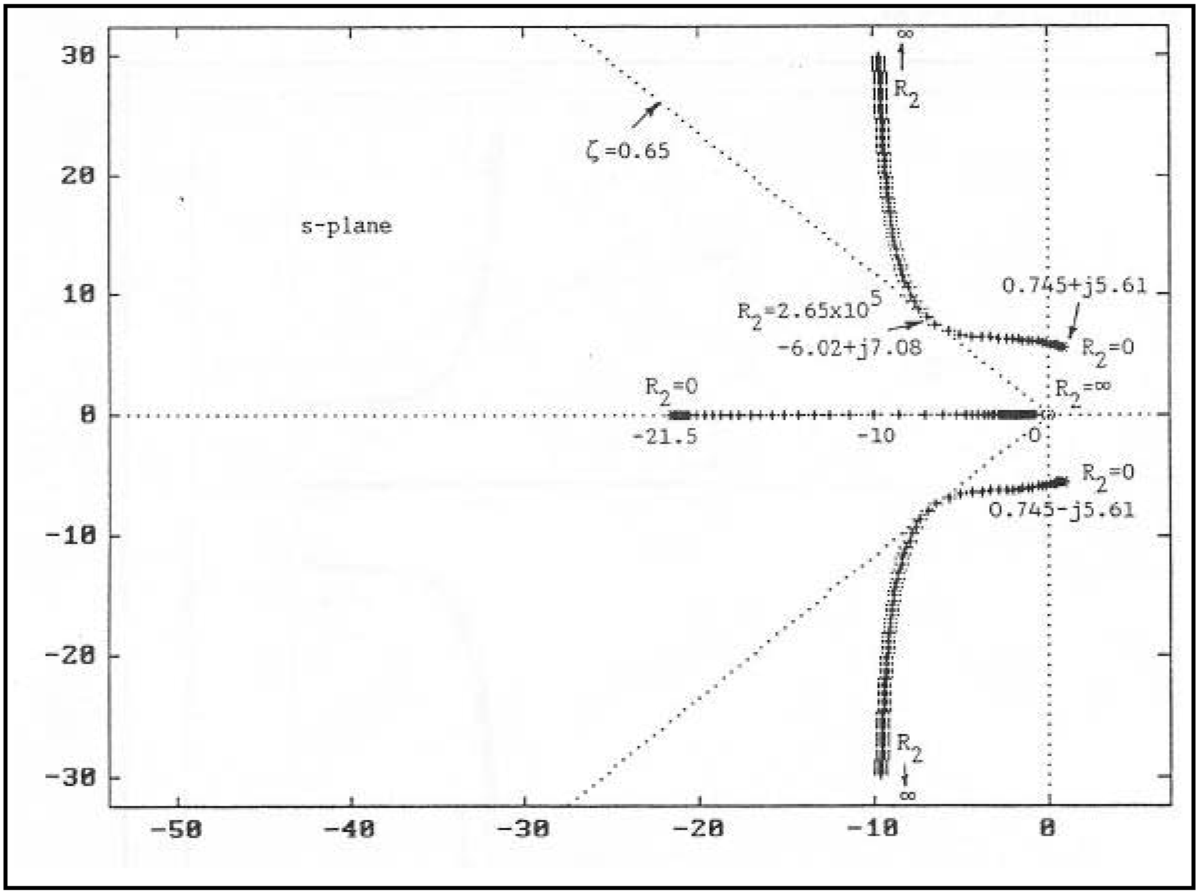
$$G(s)\,H(s) = \frac{10K_pK_aK_e}{Ns\left(1+0.05s\right)}\frac{1+R_2Cs}{R_1Cs} = \frac{68.76}{s\left(1+0.05s\right)}\frac{1+R_2\times 10^{-6}s}{2s}$$

The characteristic equation is $s^3 + 20s^2 + 6.876\times 10^{-4}R_2 s + 687.6 = 0$

For root locus plot with R_2 as the variable parameter, we have

$$G_{eq}(s) = \frac{6.876\times 10^{-4}R_2 s}{s^3+20s^2+687.6} = \frac{6.876\times 10^{-4}R_2 s}{\left(s+21.5\right)\left(s-0.745+j5.61\right)\left(s-0.745-j5.61\right)}$$

Root Locus Plot.



When $R_2 = 2.65 \times 10^5$, the roots are at $-6.02 \pm j7.08$, and the relative damping ratio is 0.65 which is maximum. The unit-step response is plotted at the end together with those of parts (b) and (c).

(b) Phase-lead Controller.

$$G(s)H(s) = \frac{68.76(1+aTs)}{s(1+0.05s)(1+Ts)}$$

Characteristic Equation: $Ts^3 + (1+20T)s^2 + (20+1375.2aT)s + 1375.2 = 0$

With $T = 0.01$, the characteristic equation becomes

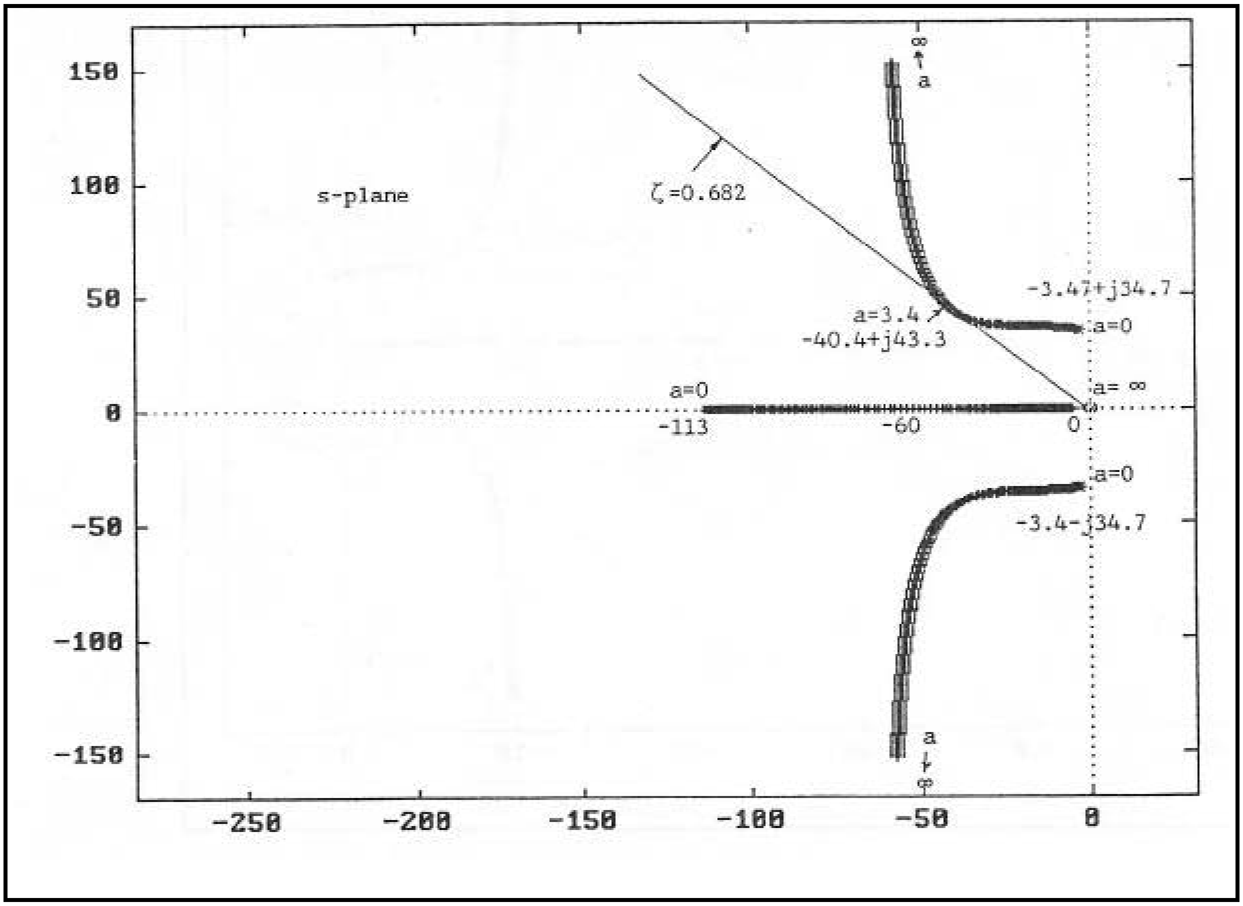
$$s^3 + 120s^2 + (2000 + 1375.2a)s + 137520 = 0$$

The last equation is conditioned for a root contour plot with a as the variable parameter.
Thus

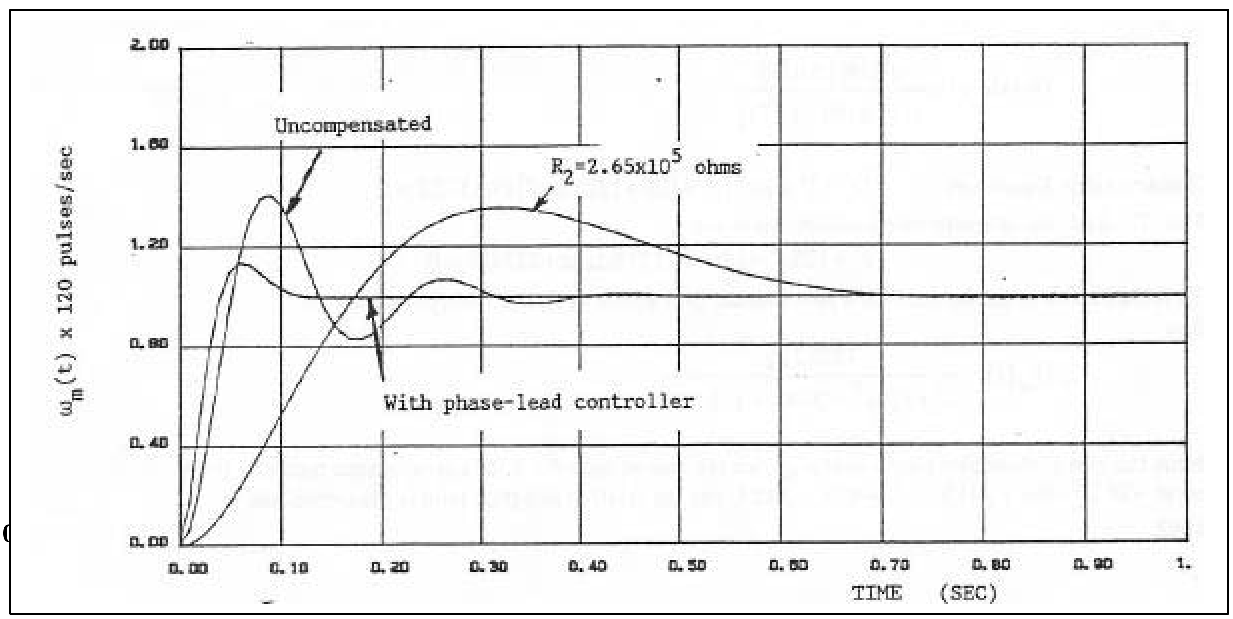
$$G_{eq}(s) = \frac{1375.2as}{s^3 + 120s^2 + 2000s + 137{,}520}$$

From the root contour plot on the next page we see that when $a = 3.4$ the characteristic equation roots are at -39.2, $-40.4 + j43.3$, and $-40.4 - j43.3$, and the relative damping ratio is maximum and is 0.682.

Root Contour Plot (*a* varies).



Unit-step Responses.



$$G_c(s) = \frac{1 + aTs}{1 + Ts} = \frac{1 + 0.04036\ s}{1 + 0.00923\ s}$$

10-26 (a) Time-domain Design of Phase-lag Controller.

Process Transfer Function:

$$G_p(s) = \frac{200}{s(s+1)(s+10)}$$

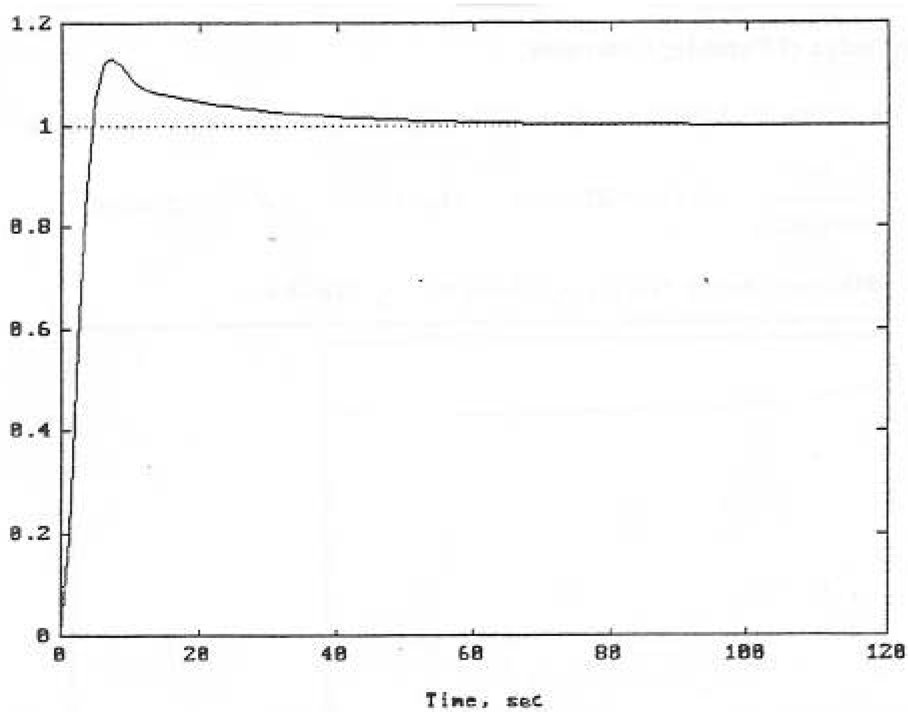
For the uncompensated system, the two complex characteristic equation roots are at $s = -0.475 + j0.471$ and $-0.475 - j0.471$ which correspond to a relative damping ratio of 0.707, when the forward path gain is 4.5 (as against 200). Thus, the value of a of the phase-lag controller is chosen to be

$$a = \frac{4.5}{200} = 0.0225 \qquad \text{Select } T = 1000 \qquad \text{which is a large number.}$$

Then

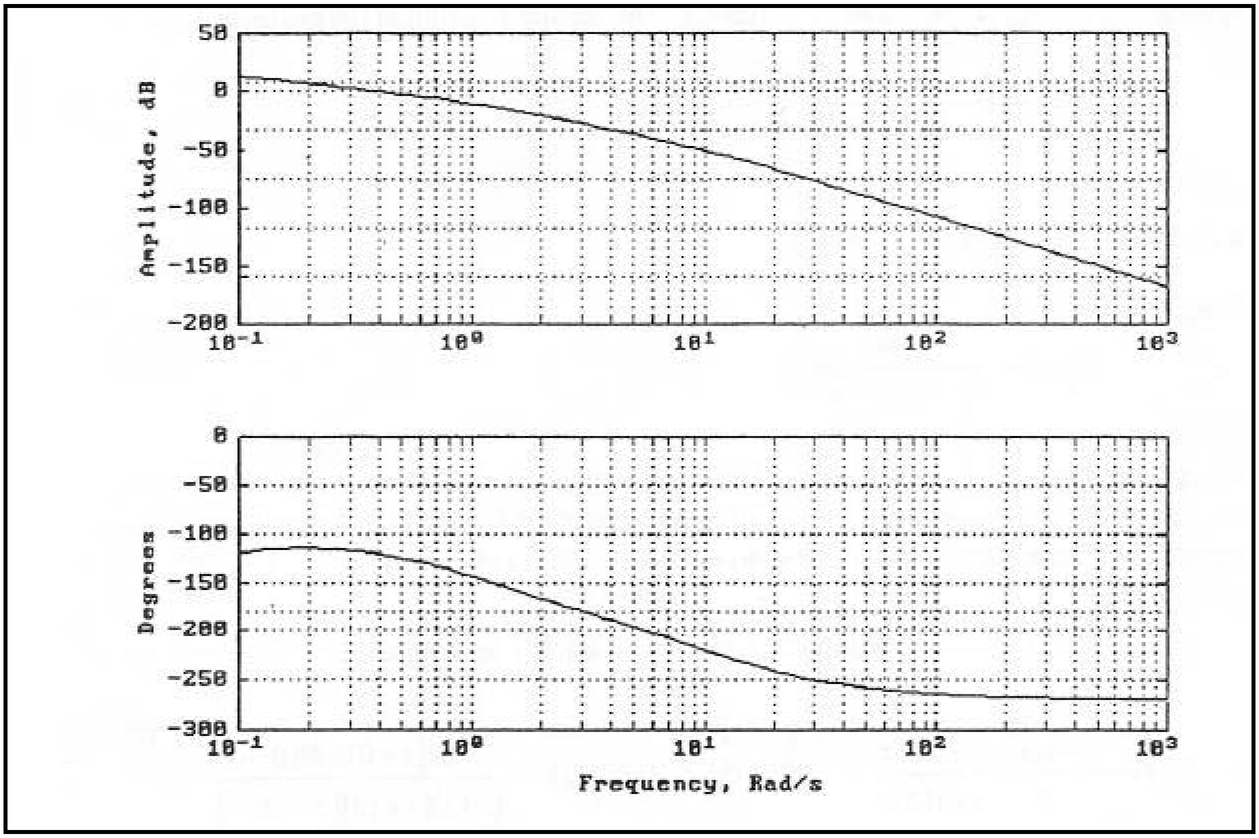
$$G_c(s) = \frac{1+aTs}{1+Ts} = \frac{1+22.5s}{1+1000s} \qquad G(s) = G_c(s)G_p(s) = \frac{4.5(s+0.0889))}{s(s+1)(s+10)(s+0.001)}$$

Unit-step Response.



Maximum overshoot = 13.6 t_r = 3.238 sec t_s = 18.86 sec

Bode Plot (with phase-lag controller, a = 0.0225, T = 1000)



PM = 59 deg. GM = 27.34 dB M_r = 1.1 BW = 0.6414 rad/sec

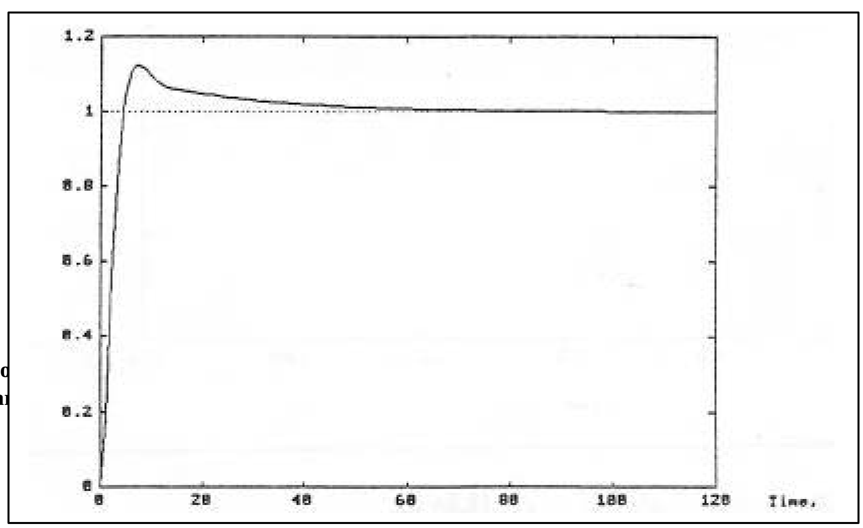
10-26 (b) Frequency-domain Design of Phase-lag Controller.

For PM = 60 deg, we choose a = 0.02178 and T = 1130.55. The transfer function of the phase-lag controller is

$$G_c(s) = \frac{1 + 24.62s}{1 + 1130.55s}$$

GM = 27.66 dB M_r = 1.093 BW = 0.619 rad/sec

Unit-step Response. Max overshoot = 12.6%, t_r = 3.297 sec t_s = 18.18 sec



10-27 (a) Time-do
Forwar

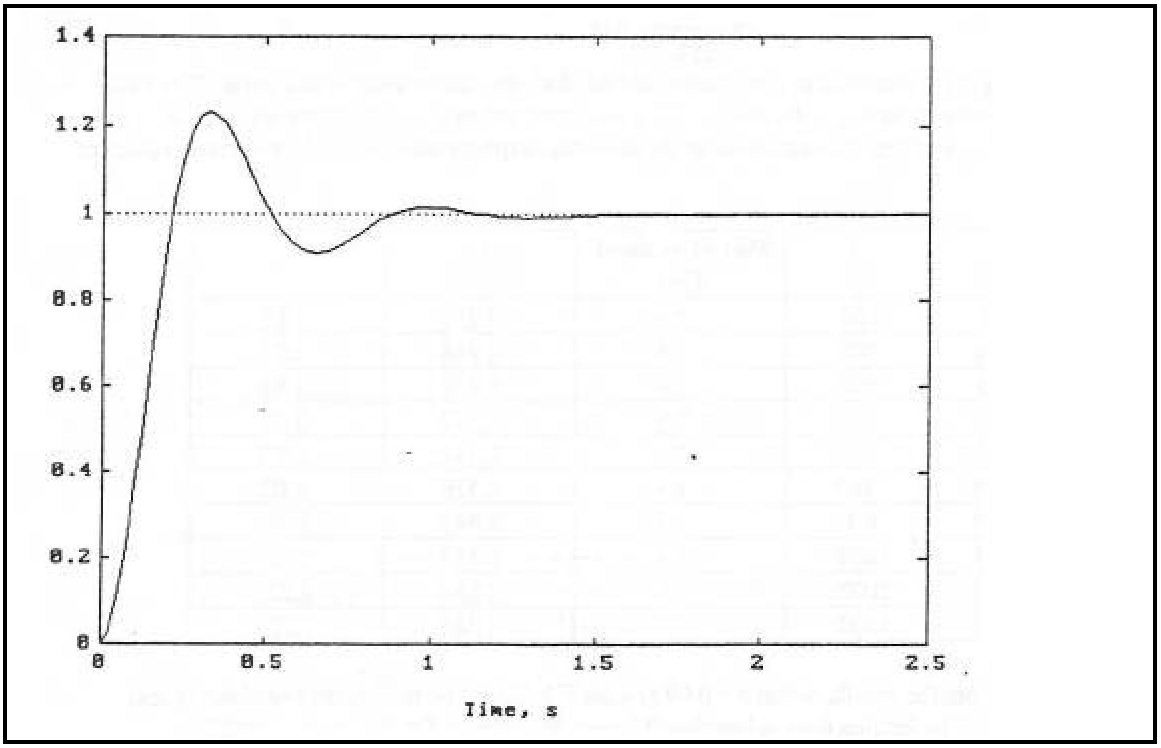
K = 250

With $K = 250$, the system without compensation is marginally stable. For $a > 1$, select a small value for T and a large value for a. Let $a = 1000$. The following results are obtained for various values of T ranging from 0.0001 to 0.001. When $T = 0.0004$, the maximum overshoot is near minimum at 23%.

T	Max Overshoot (%)	t_r (sec)	t_s (sec)
0.0010	33.5	0.0905	0.808
0.0005	23.8	0.1295	0.6869
0.0004	23.0	0.1471	0.7711
0.0003	24.4	0.1689	0.8765
0.0002	30.6	0.1981	1.096
0.0001	47.8	0.2326	2.399

As it turns out $a = 1000$ is near optimal. A higher or lower value for a will give larger overshoot.

Unit-step Response.



(b) Frequency-domain Design of Phase-lead Controller

$$G(s) = \frac{250(1+aTs)}{s^2(s+5)^2(1+Ts)}$$

Setting $a = 1000$, and varying T, the following attributes are obtained.

T	PM (deg)	M_r	BW (rad/sec)
0.00050	41.15	1.418	16.05
0.00040	42.85	1.369	14.15
0.00035	43.30	1.355	13.16
0.00030	43.10	1.361	12.12
0.00020	38.60	1.513	10.04

When $a = 1000$, the best value of T for a maximum phase margin is 0.00035, and PM = 43.3 deg. As it turns out varying the value of a from 1000 does not improve the phase margin. Thus the transfer function of the controller is

$$G_c(s) = \frac{1+aTs}{1+Ts} = \frac{1+0.35\,s}{1+0.00035\,s} \qquad \text{and} \qquad G(s) = \frac{250(1+0.35s)}{s(s+5)^2(1+0.00035s)}$$

10-27 (c) Time-domain Design of Phase-lag Controller

Without compensation, the relative damping is critical when K = 18.5. Then, the value of a is chosen to be

$$a = \frac{18.5}{250} = 0.074$$

We can use this value of a as a reference, and conduct the design around this point. The value of T is preferrably to be large. However, if T is too large, rise and settling times will suffer. The following performance attributes of the unit-step response are obtained for various values of a and T.

a	T	Max Overshoot (%)	t_r	t_s
0.105	500	2.6	1.272	1.82
0.100	500	2.9	1.348	1.82
0.095	500	2.6	1.422	1.82
0.090	500	2.5	1.522	2.02
0.090	600	2.1	1.532	2.02
0.090	700	1.9	1.538	2.02
0.090	800	1.7	1.543	2.02
0.090	1000	1.4	1.550	2.22
0.090	2000	0.9	1.560	2.22
0.090	3000	0.7	1.566	2.22

As seen from the results, when a = 0.09 and for $T \geq$ 2000, the maximum overshoot is less than 1% and the settling time is less than 2.5 sec. We choose T = 2000 and a = 0.09. The corresponding frequency-domain characteristics are:

PM = 69.84 deg   GM = 20.9 dB M_r = 1.004   BW = 1.363 rad/sec

(d) Frequency-domain Design of Phase-lag Controller

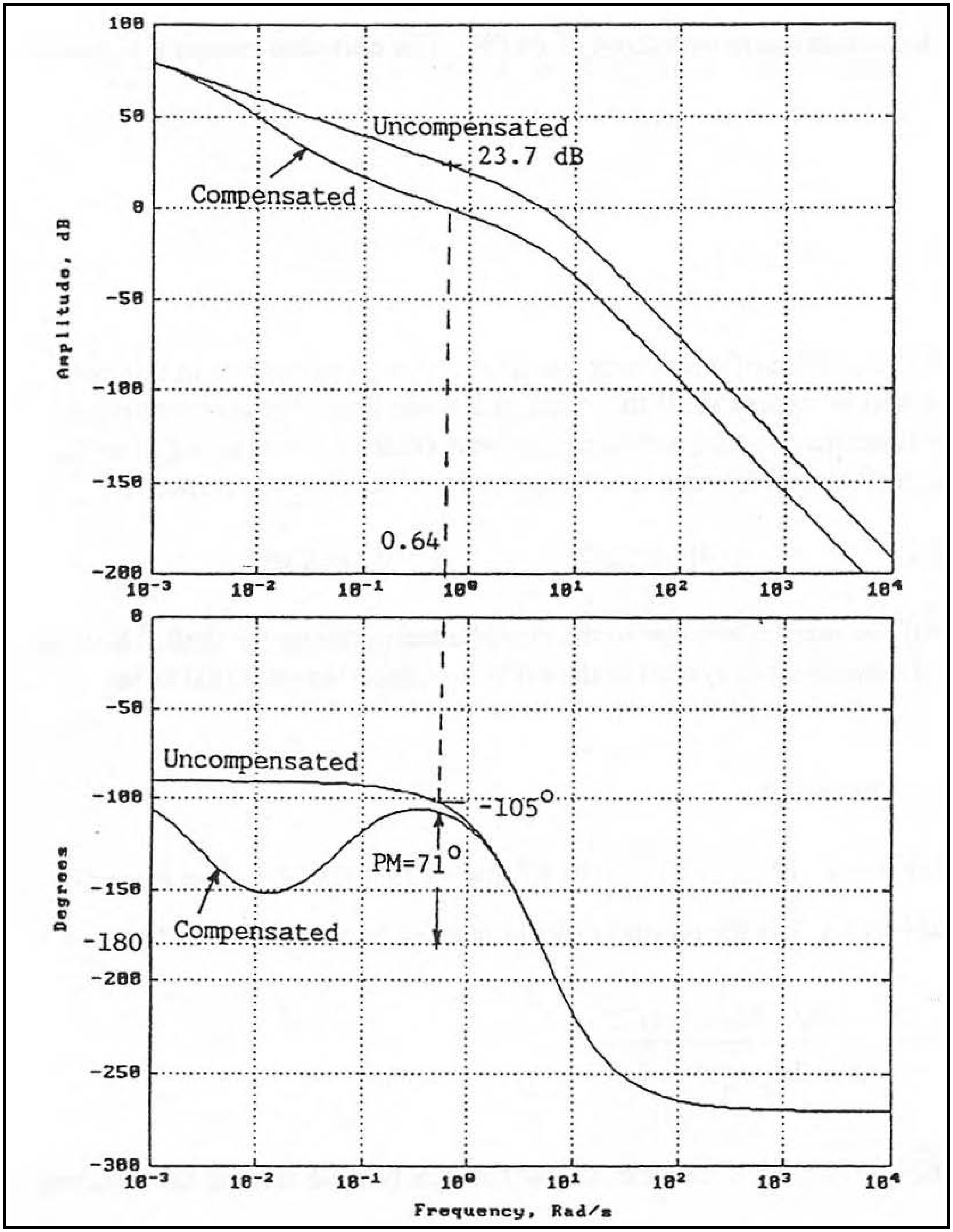
$$G(s) = \frac{250(1+aTs)}{s(s+5)^2(1+Ts)} \qquad a < 1$$

The Bode plot of the uncompensated system is shown below. Let us add a safety factor by requiring that the desired phase margin is 75 degrees. We see that a phase margin of 75 degrees can be realized if the gain crossover is moved to 0.64 rad/sec. The magnitude of $G(j\omega)$ at this frequency is 23.7 dB. Thus the phase-lag controller must provide an attenuation of $-$23.7 dB at the new gain crossover frequency. Setting

$$20\log_{10} a = -23.7 \text{ dB} \qquad \text{we have} \qquad a = 0.065$$

We can set the value of $1/aT$ to be at least one decade below 0.64 rad/sec, or 0.064 rad/sec. Thus, we get T = 236. Let us choose T = 300. The transfer function of the phase-lag controller becomes

$$G_c(s) = \frac{1+aTs}{1+Ts} = \frac{1+19.5\,s}{1+300\,s}$$



The attributes of the frequency response of the compensated system are:

PM = 71 deg GM = 23.6 dB M_r = 1.065 BW = 0.937 rad/sec

The attributes of the unit-step response are:

Maximum overshoot = 6% t_r = 2.437 sec t_s = 11.11 sec

Comparing with the phase-lag controller designed in part (a) which has a = 0.09 and T = 2000, the time response attributes are:

Maximum overshoot = 0.9% t_r = 1.56 sec t_s = 2.22 sec

The main difference is in the large value of T used in part (c) which resulted in less overshoot, rise and settling times.

10-28 Forward-path Transfer Function (No compensation)

$$G(s) = G_p(s) = \frac{6.087\times 10^7}{s\left(s^3 + 423.42s^2 + 2.6667\times 10^6 s + 4.2342\times 10^8\right)}$$

The uncompensated system has a maximum overshoot of 14.6%. The unit-step response is shown below.

(a) Phase-lead Controller

$$G_c(s) = \frac{1 + aTs}{1 + Ts} \qquad (a > 1)$$

By selecting a small value for T, the value of a becomes the critical design parameter in this case. If a is too small, the overshoot will be excessive. If the value of a is too large, the oscillation in the step response will be objectionable. By trial and error, the best value of a is selected to be 6, and T = 0.001. The following performance attributes are obtained for the unit-step response.

Maximum overshoot = 0% t_r = 0.01262 sec t_s = 0.1818 sec

However, the step response still has oscillations due to the compliance in the motor shaft. The unit-step response of the phase-lead compensated system is shown below, together with that of the uncompensated system.

(b) Phase-lead and Second-order Controller

The poles of the process $G_p(s)$ are at -161.3, $-131 + j1614.7$ and $-131 - j1614.7$. The second-order term is $s^2 + 262s + 2{,}624{,}417.1$. Let the second-order controller transfer function be

$$G_{c1}(s) = \frac{s^2 + 262s + 2{,}624{,}417.1}{s^2 + 2\zeta_p \omega_n s + \omega_n^2}$$

The value of ω_n is set to $\sqrt{2{,}624{,}417.1} = 1620$ rad/sec, so that the steady-state error is not affected.

Let the two poles of $G_{c1}(s)$ be at $s = -1620$ and -1620. Then, $\zeta_p = 405$.

$$G_{c1}(s) = \frac{s^2 + 262s + 2{,}624{,}417.1}{s^2 + 3240s + 2{,}624{,}417.1}$$

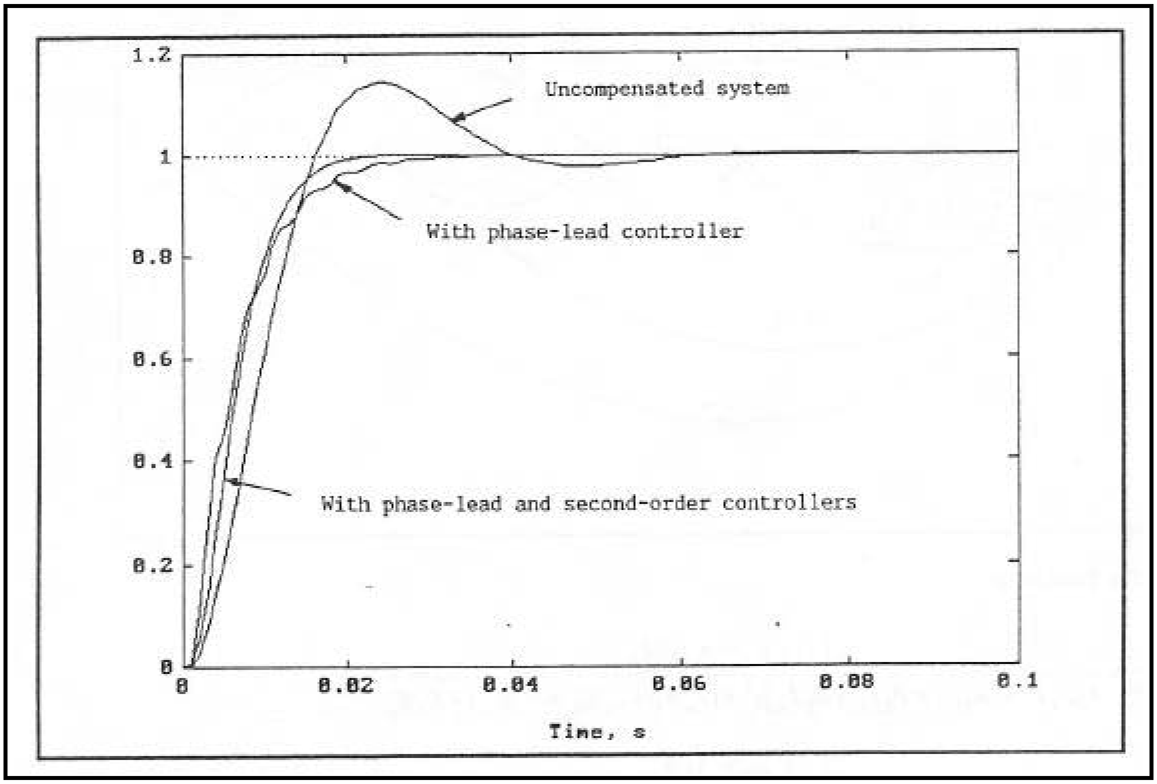
$$G(s) = G_c(s)G_{c1}(s)G_p(s) = \frac{6.087\times 10^{10}\left(1 + 0.006s\right)}{s\left(s + 161.3\right)\left(s^2 + 3240s + 2{,}624{,}417.1\right)\left(1 + 0.001s\right)}$$

The unit-step response is shown below, and the attributes are:

Maximum overshoot = 0.2 t_r = 0.01012 sec t_s = 0.01414 sec

The step response does not have any ripples.

Unit-step Responses



10-29 (a) System Equations.

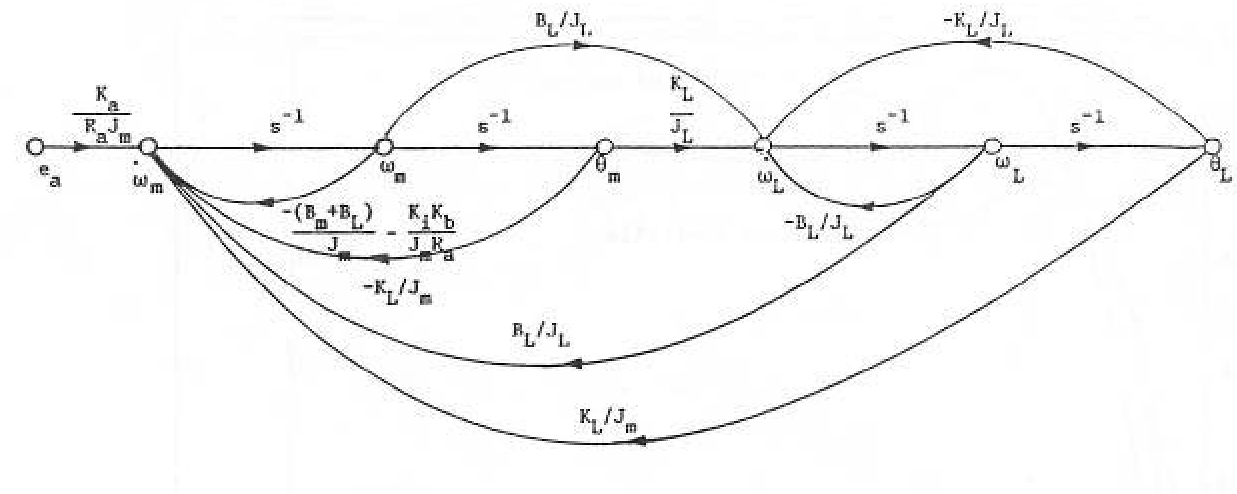
$$e_a = R_a i_a + K_b w_m \qquad T_m = K_i i_a \qquad T_m = J_m \frac{dw_m}{dt} + B_m w_m + K_L\left(q_m - q_L\right) + B_L\left(w_m - w_L\right)$$

$$K_L\left(q_m - q_L\right) + B_L\left(w_m - w_L\right) = J_L \frac{dw_L}{dt}$$

State Equations in Vector-matrix Form:

$$\begin{bmatrix} \dfrac{dq_L}{dt} \\ \dfrac{dw_L}{dt} \\ \dfrac{dq_m}{dt} \\ \dfrac{dw_m}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\dfrac{K_L}{J_L} & -\dfrac{B_L}{J_L} & \dfrac{K_L}{J_L} & \dfrac{B_L}{J_L} \\ 0 & 0 & 0 & 1 \\ \dfrac{K_L}{J_m} & \dfrac{B_L}{J_m} & -\dfrac{K_L}{J_m} & -\dfrac{B_m + B_l}{J_m} - \dfrac{K_i K_b}{J_m R_a} \end{bmatrix} \begin{bmatrix} q_L \\ w_L \\ q_m \\ w_m \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \dfrac{K_a}{R_a J_m} \end{bmatrix} e_a$$

State Diagram:



Transfer Functions:

$$\frac{\Omega_m(s)}{E_a(s)} = \frac{K_i\left(s^2 + B_L s + K_L\right)/R_a}{J_m J_L s^2 + \left(K_e J_L + B_L J_L + B_L J_m\right)s^2 + \left(J_m K_L + J_L K_L + K_e B_L\right)s + K_L K_e}$$

$$\frac{\Omega_L(s)}{E_a(s)} = \frac{K_i\left(B_L s + K_L\right)/R_a}{J_m J_L s^3 + \left(K_e J_L + B_L J_L + B_L J_m\right)s^2 + \left(J_m K_L + J_L K_L + K_e B_L\right)s + K_L K_e}$$

$$\frac{\Omega_m(s)}{E_a(s)} = \frac{133.33\left(s^2 + 10s + 3000\right)}{s^3 + 318.15s^2 + 60694.13s + 58240} = \frac{133.33\left(s^2 + 10s + 3000\right)}{\left(s + 0.9644\right)\left(s + 158.59 + j187.71\right)\left(s + 158.59 - j187.71\right)}$$

$$\frac{\Omega_L(s)}{E_a(s)} = \frac{1333.33\left(s + 300\right)}{\left(s + 0.9644\right)\left(s + 158.59 + j187.71\right)\left(s + 158.59 - j187.71\right)}$$

(b) Design of PI Controller.

$$G(s) = \frac{\Omega_L(s)}{E(s)} = \frac{1333.33K_P\left(s + \frac{K_I}{K_P}\right)\left(s + 300\right)}{s\left(s + 0.9644\right)\left(s^2 + 317.186s + 60388.23\right)}$$

$$K_v = \lim_{s \to 0} sG(s) = \frac{1333.33 \times 300\ K_I}{0.9644 \times 60388.23} = 6.87\,K_I = 100 \qquad \text{Thus} \quad K_I = 14.56$$

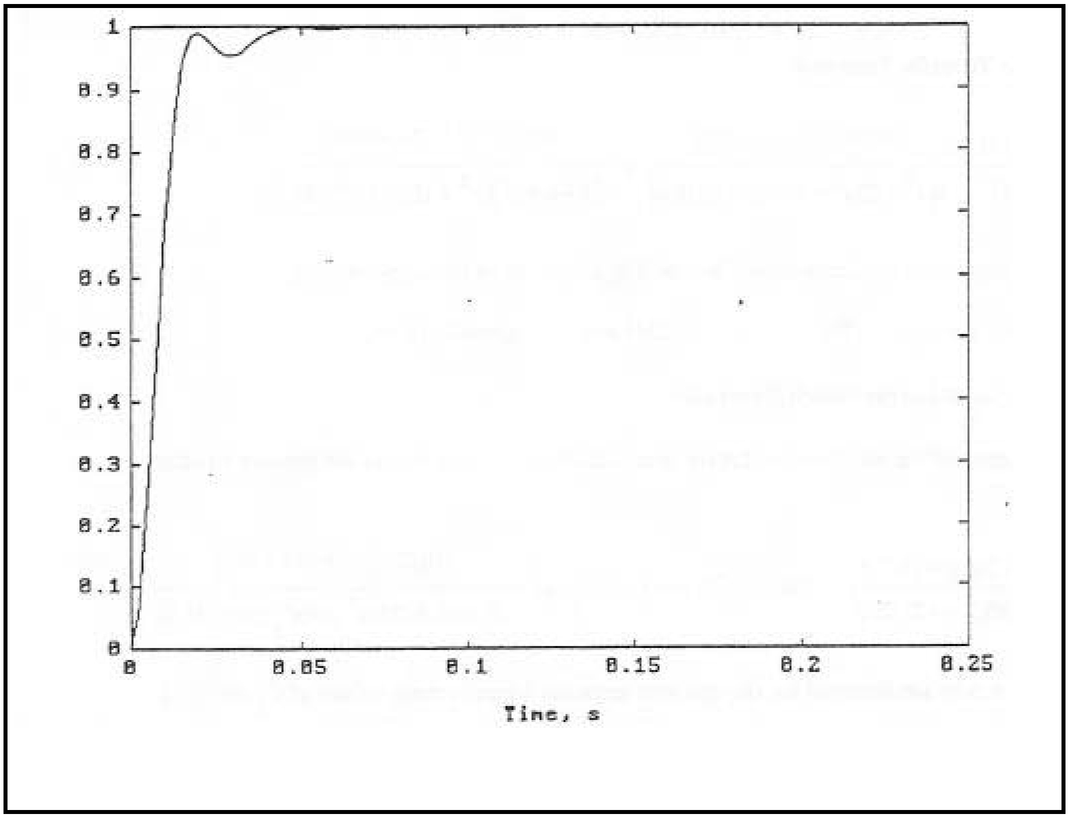
With $K_I = 14.56$, we study the effects of varying K_P. The following results are obtained.

K_P	t_r (sec)	t_s (sec)	Max Overshoot (%)
20	0.00932	0.02778	4.2
18	0.01041	0.01263	0.7

17	0.01113	0.01515	0
16	0.01184	0.01515	0
15	0.01303	0.01768	0
10	0.02756	0.04040	0.6

With $K_I = 14.56$ and K_P ranging from 15 to 17, the design specifications are satisfied.

Unit-step Response:



(c) Frequency-domain Design of PI Controller (K_I = 14.56)

$$G(s) = \frac{1333.33\left(K_P s + 14.56\right)\left(s + 300\right)}{s\left(s^3 + 318.15s^2 + 60694.13s + 58240\right)}$$

The following results are obtained by setting $K_I = 14.56$ and varying the value of K_P.

K_P	PM (deg)	GM (dB)	M_r	BW (rad/sec)	Max Overshoot (%)	t_r (sec)	t_s (sec)
20	65.93	$\propto$	1.000	266.1	4.2	0.00932	0.02778
18	69.76	$\propto$	1.000	243	0.7	0.01041	0.01263
17	71.54	$\propto$	1.000	229	0	0.01113	0.01515
16	73.26	$\propto$	1.000	211.6	0	0.01184	0.01515
15	74.89	$\propto$	1.000	190.3	0	0.01313	0.01768
10	81.11	$\propto$	1.005	84.92	0.6	0.0294	0.0404
8	82.66	$\propto$	1.012	63.33	1.3	0.04848	0.03492
7	83.14	$\propto$	1.017	54.19	1.9	0.03952	0.05253
6	83.29	$\propto$	1.025	45.81	2.7	0.04697	0.0606
5	82.88	$\propto$	1.038	38.12	4.1	0.05457	0.0606

From these results we see that the phase margin is at a maximum of 83.29 degrees when $K_P = 6$. However, the maximum overshoot of the unit-step response is 2.7%, and M_r is slightly greater than one. In part (b), the optimal value of K_P from the standpoint of minimum value of the maximum overshoot is between 15 and 17. Thus, the phase margin criterion is not a good indicator in the present case.

10-30 (a) Forward-path Transfer Function

$$G_p(s) = \frac{K\Theta_m(s)}{T_m(s)} = \frac{100K\left(s^2 + 10s + 100\right)}{s\left(s^3 + 20s^2 + 2100s + 10{,}000\right)} = \frac{10{,}000\left(s^2 + 10s + 100\right)}{s\left(s + 4.937\right)\left(s^2 + 15.06s + 2025.6\right)}$$

The unit-step response is plotted as shown below. The attributes of the response are:

Maximum overshoot = 57% $\quad t_r$ = 0.01345 sec $\quad t_s$ = 0.4949 sec

(b) Design of the Second-order Notch Controller

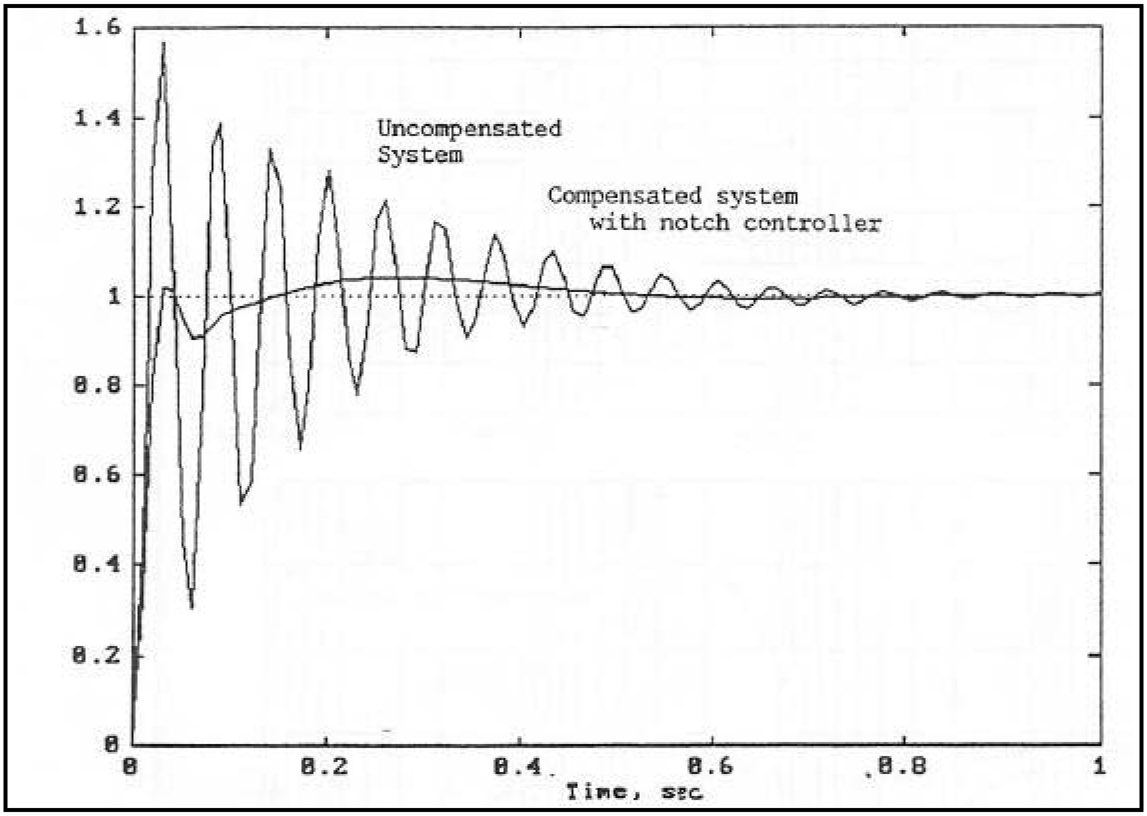
The complex zeros of the notch controller are to cancel the complex poles of the process transfer function. Thus

$$G_c(s) = \frac{s^2 + 15.06s + 2025.6}{s^2 + 90z_p s + 2025.6} \qquad \text{and} \qquad G(s) = G_c(s)G_p(s) = \frac{10{,}000\left(s^2 + 10s + 100\right)}{s\left(s + 4.937\right)\left(s^2 + 90z_p s + 2025.6\right)}$$

The following results are obtained for the unit-step response when various values of z_p are used. The maximum overshoot is at a minimum of 4.1% when $z_p = 1.222$. The unit-step response is plotted below, along with that of the uncompensated system.

z_p	$2zw_n$	**Max Overshoot (%)**
2.444	200	7.3
2.333	210	6.9
2.222	200	6.5
1.667	150	4.9
1.333	120	4.3
1.222	110	4.1
1.111	100	5.8
1.000	90	9.8

Unit-step Response



10-30 (c) Frequency-domain Design of the Notch Controller

The forward-path transfer function of the uncompensated system is

$$G(s) = \frac{10000\left(s^2 + 10s + 100\right)}{s\left(s + 4.937\right)\left(s^2 + 15.06s + 2025.6\right)}$$

The Bode plot of $G(j\omega)$ is constructed in the following. We see that the peak value of $\left|G(j\omega)\right|$ is approximately 22 dB. Thus, the notch controller should provide an attentuation of -22 dB or 0.0794 at the resonant frequency of 45 rad/sec. Using Eq. (10-155), we have

$$\left|G_c(j45)\right| = \frac{\zeta_z}{\zeta_p} = \frac{0.167}{\zeta_p} = 0.0794 \qquad \text{Thus} \qquad \zeta_p = 2.1024$$

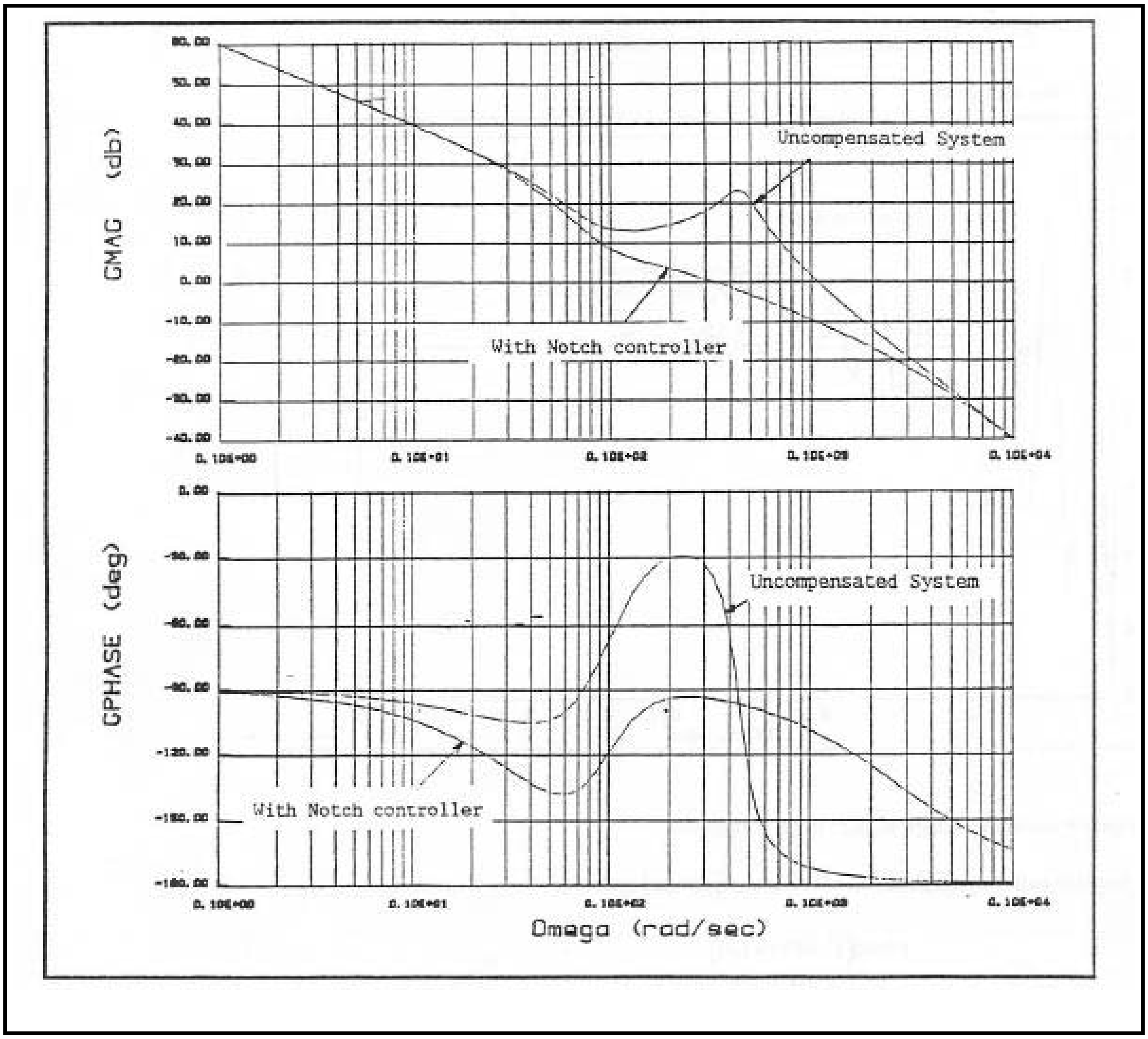
Notch Controller Transfer Function

$$G_c(s) = \frac{s^2 + 15.06s + 2025.6}{s^2 + 189.216s + 2025.6}$$

Forward-path Transfer Function

$$G(s) = \frac{10{,}000\left(s^2 + 10s + 100\right)}{s\left(s + 4.937\right)\left(s^2 + 189.22s + 2025.6\right)}$$

Bode Plots



Attributes of the frequency response: PM = 80.37 deg GM = infinite $M_r = 1.097$ BW = 66.4 rad/sec

Attributes of the frequency response of the system designed in part (b):

PM = 59.64 deg GM = infinite $M_r = 1.048$ BW = 126.5 rad/sec

10-31 (a) Process Transfer Function

$$G_p(s) = \frac{500(s+10)}{s(s^2+10s+1000)}$$

The Bode plot is constructed below. The frequency-domain attributes of the uncompensated system are:

PM = 30 deg GM = infinite $M_r = 1.86$ and BW = 3.95 rad/sec

The unit-step response is oscillatory.

(b) Design of the Notch Controller

For the uncompensated process, the complex poles have the following constants:

$$\omega_n = \sqrt{1000} = 31.6 \text{ rad/sec} \qquad 2\zeta\omega_n = 10 \qquad \text{Thus} \qquad \zeta = 0.158$$

The transfer function of the notch controller is

$$G_c(s) = \frac{s^2 + 2\zeta_z \omega_n s + \omega_n^2}{s^2 + 2\zeta_p s + \omega_n^2}$$

For the zeros of $G_c(s)$ to cancel the complex poles of $G_p(s)$, $\zeta_z = \zeta = 0.158$.

From the Bode plot, we see that to bring down the peak resonance of $\left|G(j\omega_n)\right|$ in order to smooth out the magnitude curve, the notch controller should provide approximately -26 dB of attenuation. Thus, using Eq. (10-155),

$$\frac{\zeta_z}{\zeta_p} = 10^{\frac{-26}{20}} = 0.05 \qquad \text{Thus} \qquad \zeta_p = \frac{0.158}{0.05} = 3.1525$$

The transfer function of the notch controller is

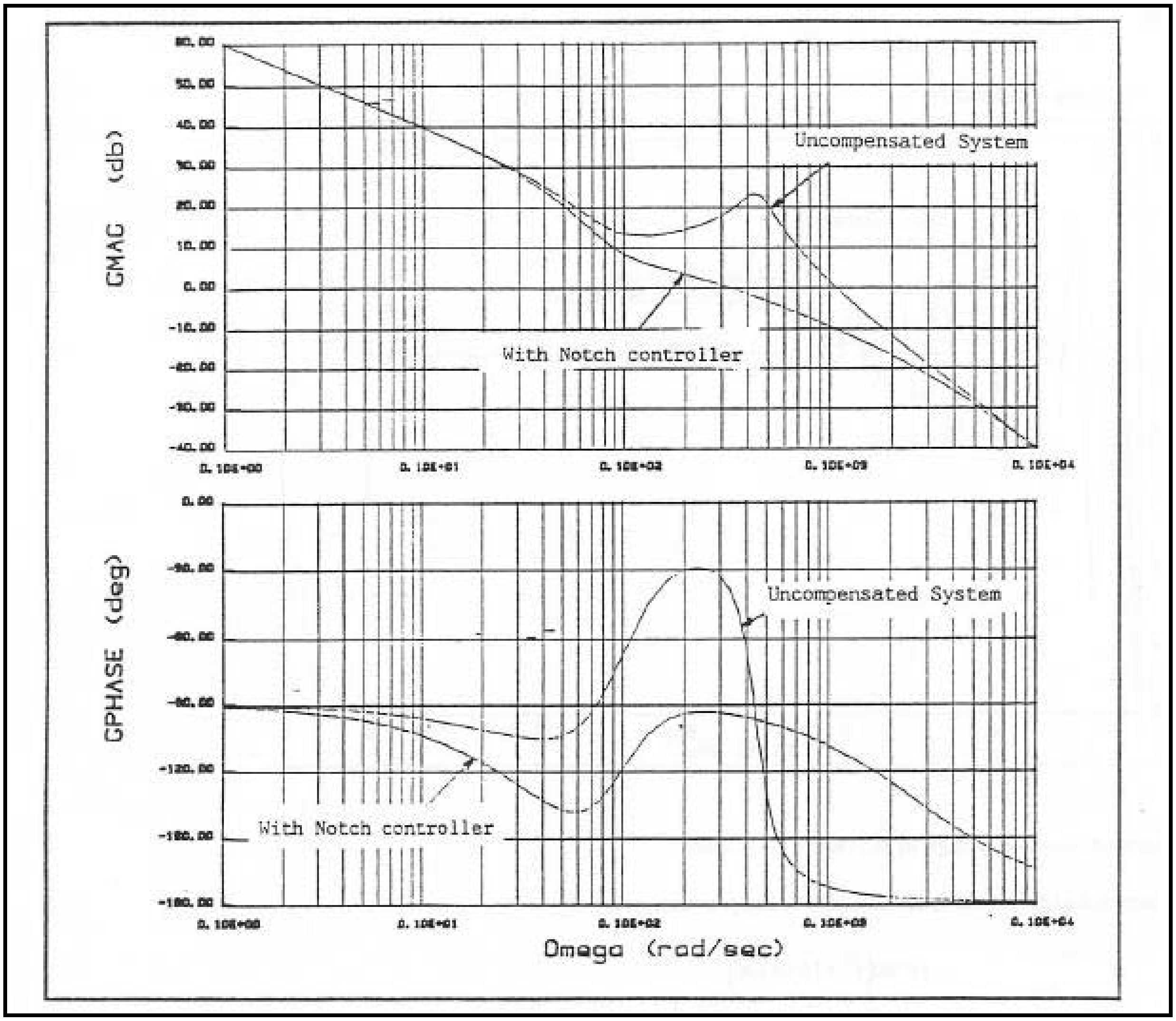
$$G_c(s) = \frac{s^2 + 10s + 1000}{s^2 + 199.08s + 1000} \qquad\qquad G(s) = G_c(s)G_p(s) = \frac{500(s+10)}{s\left(s^2 + 199.08s + 1000\right)}$$

The attributes of the compensated system are:

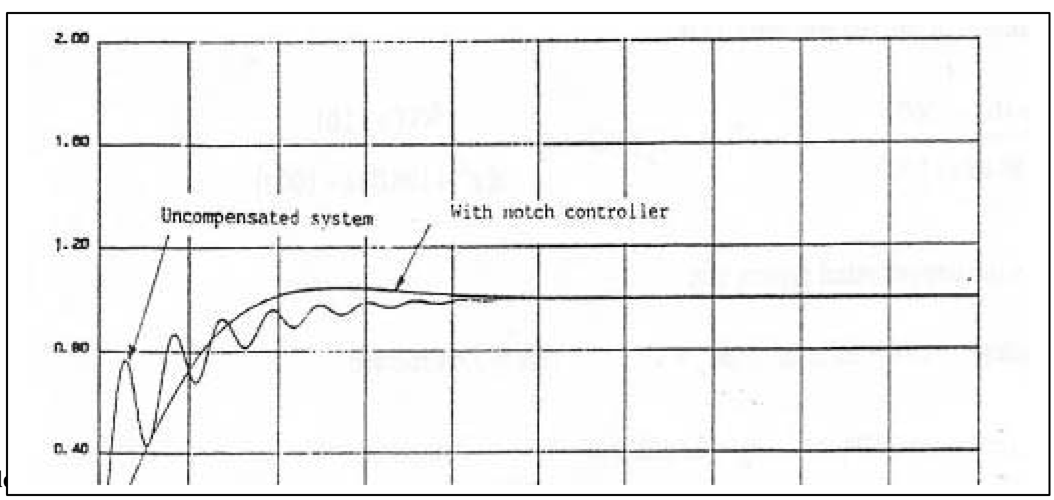
PM = 72.38 deg GM = infinite $M_r = 1$ BW = 5.44 rad/sec

Maximum overshoot = 3.4% $t_r = 0.3868$ sec $t_s = 0.4848$ sec

Bode Plots



Step Responses



10-31 (c) Time-d

With $\zeta_z = 0.158$ and $\omega_n = 31.6$, the forward-path transfer function of the compensated system is

$$G(s) = G_c(s)G_p(s) = \frac{500(s+10)}{s\left(s^2 + 63.2\zeta_p s + 1000\right)}$$

The following attributes of the unit-step response are obtained by varying the value of ζ_p.

ζ_p	$2\zeta\omega_n$	**Max Overshoot (%)**	t_r **(sec)**	t_s **(sec)**
1.582	100	0	0.4292	0.5859
1.741	110	0	0.4172	0.5657
1.899	120	0	0.4074	0.5455
2.057	130	0	0.3998	0.5253
2.215	140	0.2	0.3941	0.5152
2.500	158.25	0.9	0.3879	0.4840
3.318	209.7	4.1	0.3884	0.4848

When $\zeta_p = 2.5$ the maximum overshoot is 0.9%, the rise time is 0.3879 sec and the setting time is 0.4840 sec. These performance attributes are within the required specifications.

10-32 Let the transfer function of the controller be

$$G_c(s) = \frac{20{,}000\left(s^2 + 10s + 50\right)}{(s+1000)^2}$$

Then, the forward-path transfer function becomes

$$G(s) = G_c(s)\,G_p(s) = \frac{20{,}000K\left(s^2 + 10s + 50\right)}{s\left(s^2 + 10s + 100\right)(s+1000)^2}$$

For $G_{cf}(s) = 1$, $\quad K_v = \lim_{s \to 0} sG(s) = \frac{10^6 K}{10^8} = 50 \quad$ Thus the nominal $K = 5000$

For $\pm$ 20% variation in K, $K_{\min} = 4000$ and $K_{\max} = 6000$. To cancel the complex closed-loop poles, we let

$$G_{cf}(s) = \frac{50(s+1)}{s^2 + 10s + 50}$$ where the $(s + 1)$ term is added to reduce the rise time.

Closed-loop Transfer Function:

$$\frac{Y(s)}{R(s)} = \frac{10^6 K(s+1)}{s\left(s^2 + 10s + 100\right)\left(s + 1000^2\right) + 20{,}000K\left(s^2 + 10s + 50\right)}$$

Characteristic Equation:

***K* = 4000:** $s^5 + 2010s^4 + 1{,}020{,}100s^3 + 9.02 \times 10^7 s^2 + 9 \times 10^8 + 4 \times 10^9 = 0$

Roots: $-97.7,\ -648.9,\ -1252.7,\ -5.35 + j4.6635,\ -5.35 - j4.6635$

Max overshoot $\cong$ 6.7% **Rise time < 0.04 sec**

***K* = 5000:** $s^5 + 2010s^4 + 1{,}020{,}100s^3 + 1.1 \times 10^8 s^2 + 1.1 \times 10^9 s + 5 \times 10^9 = 0$

Roots: $-132.46,\ 587.44,\ -1279.6,\ -5.272 + j4.7353,\ -5.272 - j4.7353$

Max overshoot $\cong$ 4% **Rise time < 0.04 sec**

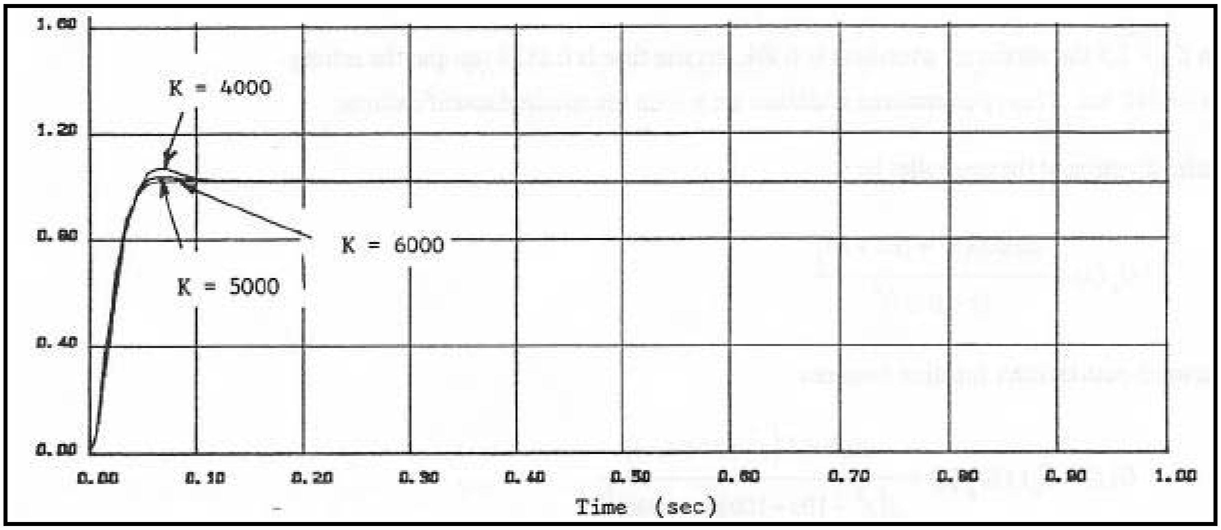
K = 6000 $s^5 + 2010 s^4 + 1{,}020{,}100 s^3 + 1.3 \times 10^8 s^2 + 1.3 \times 10^9 s + 6 \times 10^9 = 0$

Roots: $-176.77, \quad -519.37, \quad -1303.4, \quad -5.223 + j4.7818, \quad -5.223 - j4.7818$

Max overshoot $\cong$ 2.5% **Rise time < 0.04 sec**

Thus all the required specifications stay within the required tolerances when the value of K varies by plus and minus 20%.

Unit-step Responses



10-33 Let the transfer function of the controller be

$$G_c(s) = \frac{200\left(s^2 + 10s + 50\right)}{\left(s + 100\right)^2}$$

The forward-path transfer function becomes

$$G(s) = G_c(s)G_p(s) = \frac{200{,}000K\left(s^2 + 10s + 50\right)}{s\left(s + a\right)\left(s + 100\right)^2}$$

For a = 10,

$$K_v = \lim_{s \to 0} sG(s) = \frac{10^7 K}{10^5} = 100K = 100 \qquad \text{Thus} \qquad K = 1$$

Characteristic Equations: (K = 1)

***a* = 10:** $s^4 + 210 s^3 + 2.12 \times 10^5 s^2 + 2.1 \times 10^6 s + 10^7 = 0$

Roots: $-4.978 + j4.78, \quad -4.978 - j4.78, \quad -100 + j447.16 \quad -100 - j447.16$

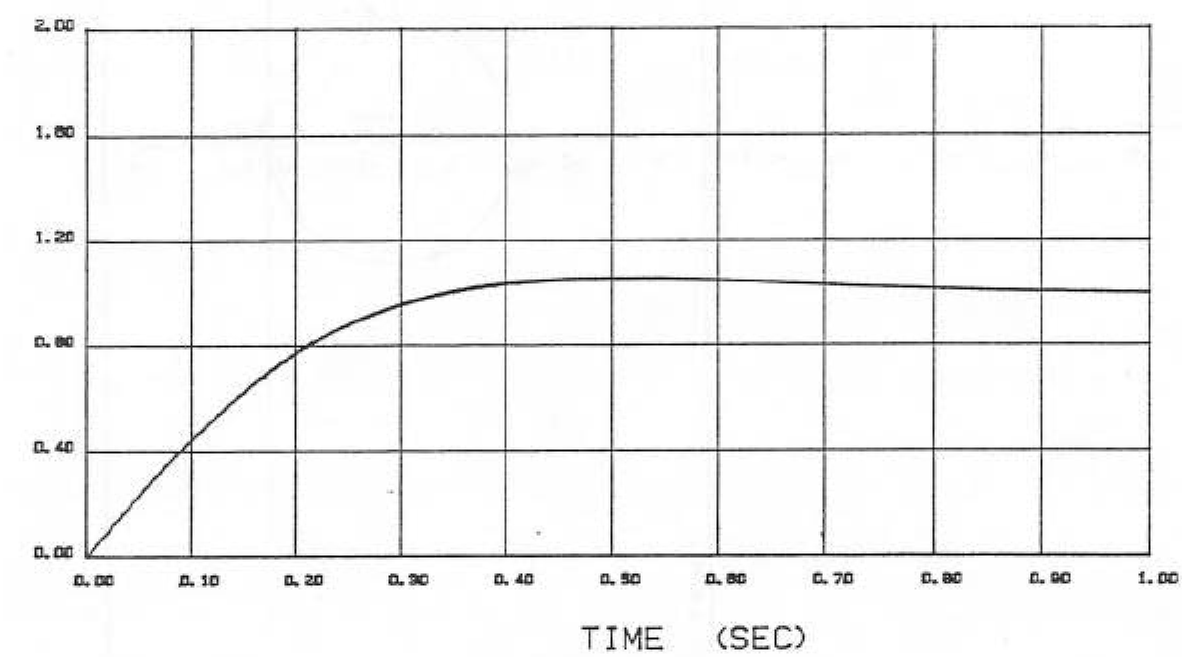
***a* = 8:** $s^4 + 208 s^3 + 2.116 \times 10^5 s^2 + 2.08 \times 10^6 s + 10^7 = 0$

Roots: $-4.939 + j4.828, \quad -4.939 - j4.828, \quad -99.06 + j446.97, \quad -99.06 - j446.97$

***a* = 12:** $s^4 + 212 s^3 + 2.124 \times 10^5 + 2.12 \times 10^6 s + 10^7 = 0$

Roots: $-5.017 + j4.73, \quad -5.017 - j4.73, \quad -100.98 + j447.36, \quad -100.98 - j447.36$

Unit-step Responses: All three responses for *a* = 8, *a* = 10, and 12 are similar.



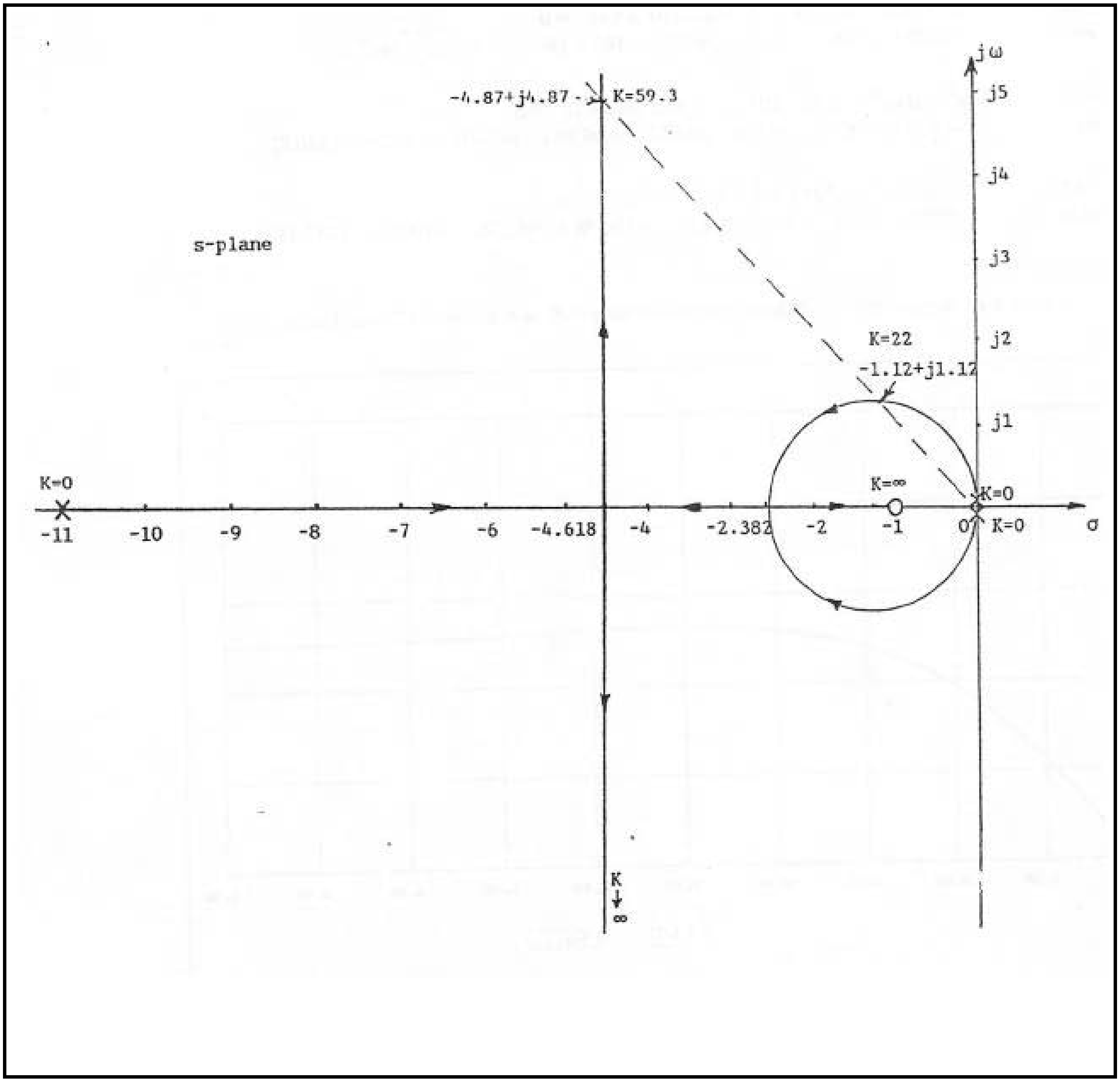
10-34 Forward-path Transfer Function:

$$G(s) = \frac{Y(s)}{E(s)} = \frac{K}{s(s+1)(s+10) + KK_t s} \qquad K_v = \lim_{s \to 0} sG(s) = \frac{K}{10 + KK_t} = 1$$

Characteristic Equation: $s^3 + 11s^2 + (10 + KK_t)s + K = s^3 + 11s^2 + Ks + K = 0$

For root loci, $G_{eq}(s) = \dfrac{K(s+1)}{s^2(s+11)}$

Root Locus Plot (*K* varies)



The root loci show that a relative damping ratio of 0.707 can be realized by two values of K. K = 22 and 59.3. As stipulated by the problem, we select K = 59.3.

10-35 Forward-path Transfer Function:

$$G(s) = \frac{10K}{s(s+1)(s+10) + 10K_t s} \qquad K_v = \lim_{s \to 0} sG(s) = \frac{10K}{10 + 10K_t} = \frac{K}{1+K_t} = 1 \qquad \text{Thus} \quad K_t = K - 1$$

Characteristic Equation: $s(s+1)(s+10)+10K_t+10K = s^3+11s^2+10Ks+10K = 0$

When $K = 5.93$ and $K_t = K - 1 = 4.93$, the characteristic equation becomes

$$s^3 + 11s^2 + 10.046s + 4.6 = 0$$

The roots are: -10.046, $-0.47723 + j0.47976$, $-0.47723 - j0.47976$

10-36 Forward-path Transfer Function:

$$G(s) = \frac{K(1+aTs)}{s((1+Ts)(s^2+10s+KK_t)}$$

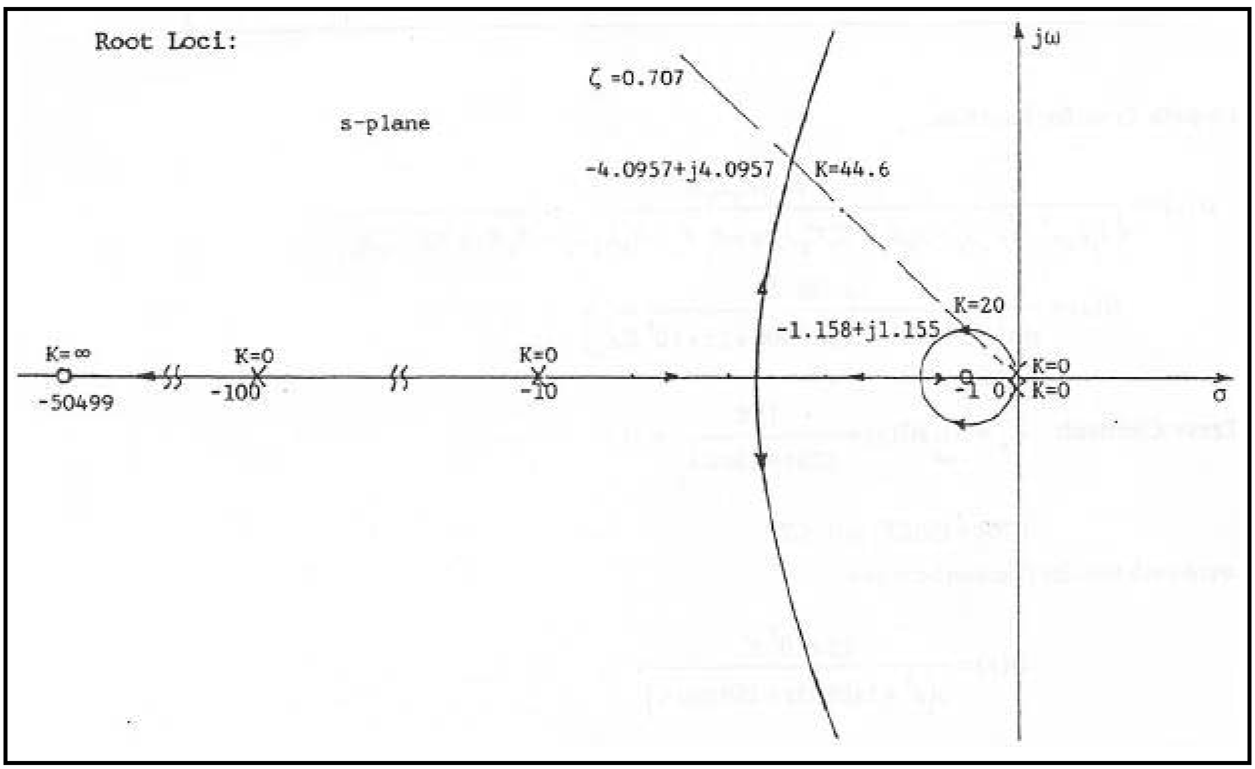
$$K_v = \lim_{s\to 0} sG(s) = \frac{1}{K_t} = 100 \qquad \text{Thus} \quad K_t = 0.01$$

Let $T = 0.01$ and $a = 100$. The characteristic equation of the system is written:

$$s^4 + 110s^3 + 1000s^2 + K\left(0.001s^2 + 101s + 100\right) = 0$$

To construct the root contours as K varies, we form the following equivalent forward-path transfer function:

$$G_{eq}(s) = \frac{0.001K\left(s^2+101{,}000s+100{,}000\right)}{s^2(s+10)(s+100)} = \frac{0.001K(s+1)(s+50499)}{s^2(s+10)(s+100)}$$



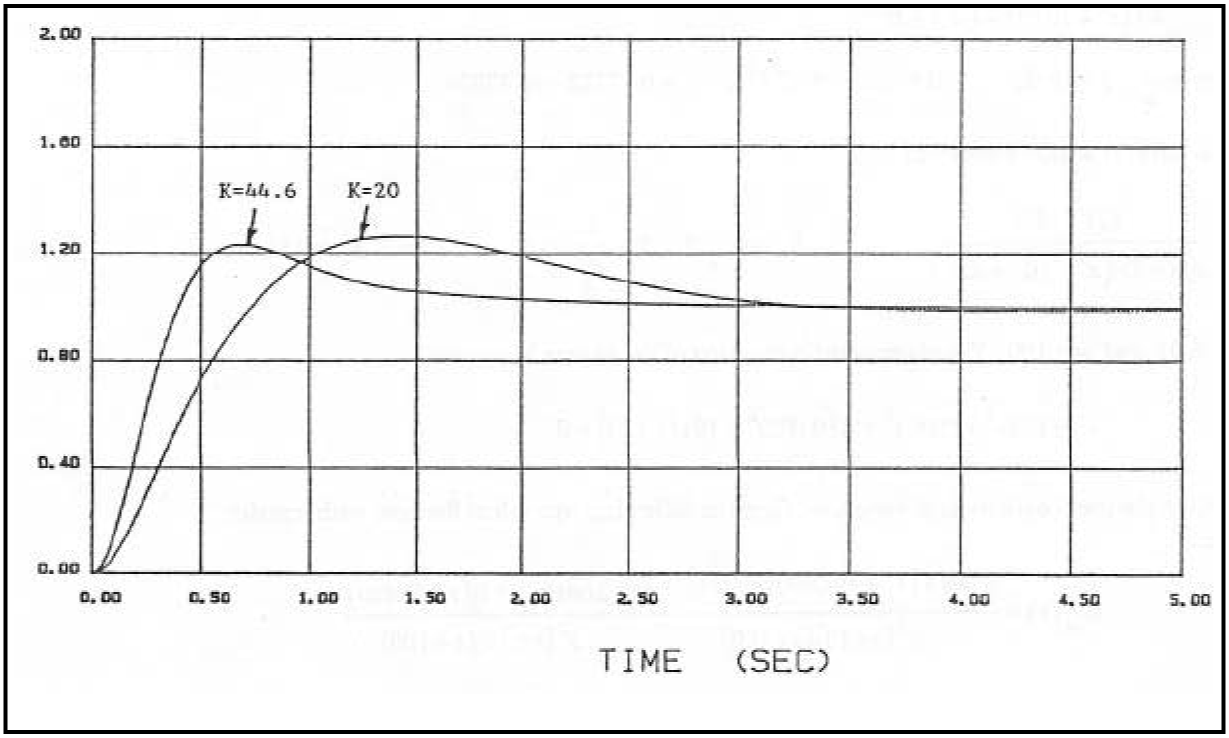
From the root contour diagram we see that two sets of solutions exist for a damping ratio of 0.707. These are:

K = 20: **Complex roots:** $-1.158 + j1.155$, $-1.158 - j1.155$

K = 44.6: **Complex roots:** $-4.0957 + j4.0957$, $-4.0957 - j4.0957$

The unit-step responses of the system for K = 20 and 44.6 are shown below.

Unit-step Responses:



10-37 **Forward-path Transfer Function:**

$$G(s) = \frac{K_s K K_1 K_i N}{s\left[J_t L_a s^2 + \left(R_a J_t + L_a B_t + K_1 K_2 J_t\right)s + R_a B_t + K_1 K_2 B_t + K_b K_i + K K_1 K_i K_t\right]}$$

$$G(s) = \frac{1.5\times 10^7 K}{s\left(s^2 + 3408.33s + 1{,}204{,}000 + 1.5\times 10^8 KK_t\right)}$$

Ramp Error Constant: $K_v = \lim_{s\to 0} sG(s) = \dfrac{15K}{1.204 + 150KK_t} = 100$

Thus $1.204 + 150KK_t = 0.15K$

The forward-path transfer function becomes

$$G(s) = \frac{1.5\times 10^7 K}{s\left(s^2 + 3408.33s + 150{,}000K\right)}$$

Characteristic Equation: $s^3 + 3408.33s + 150{,}000Ks + 1.5\times 10^7 K = 0$

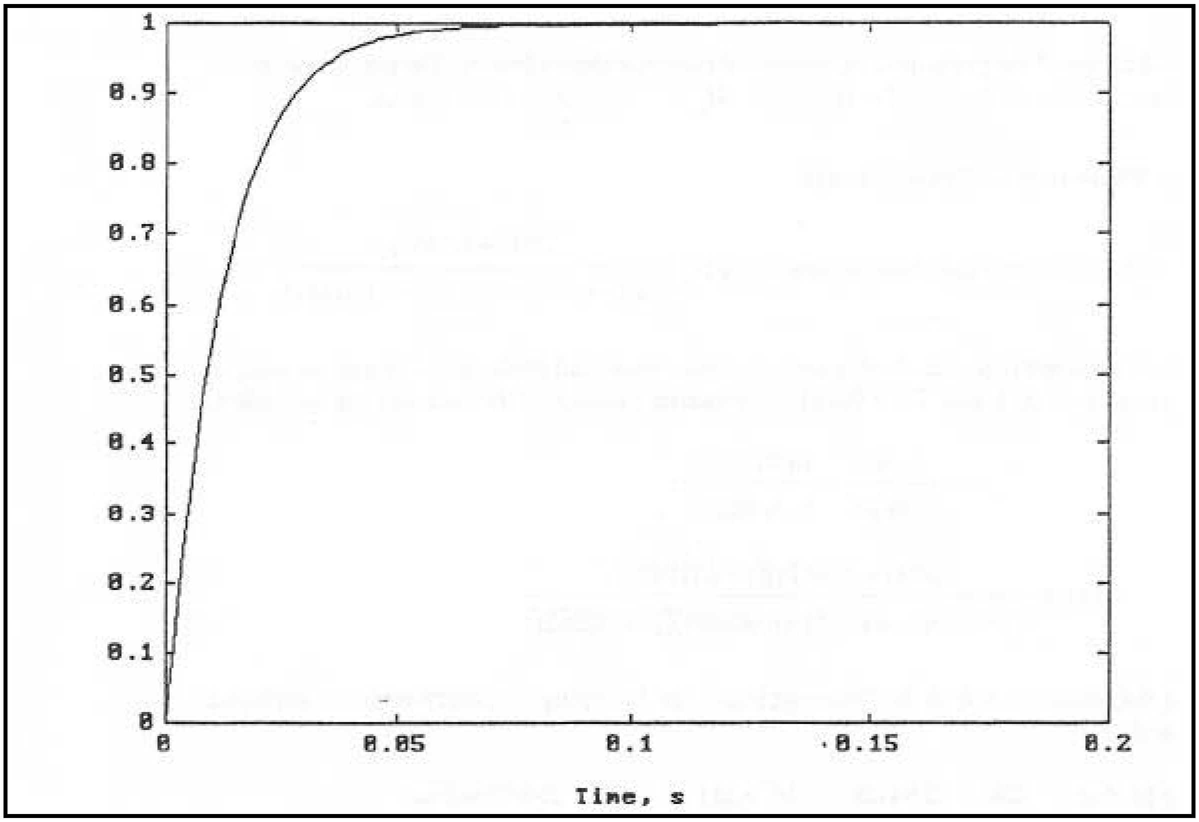
When $K = 38.667$ the roots of the characteristic equation are at

-0.1065, $-1.651 + j1.65$, $-1.651 - j1.65$ ($\zeta \cong 0.707$ for the complex roots)

The forward-path transfer function becomes

$$G(s) = \frac{5.8\times 10^8}{s\left(s^2 + 3408.33s + 5.8\times 10^6\right)}$$

Unit-step Response



Unit-step response attributes: Maximum overshoot = 0 Rise time = 0.0208 sec Settling time = 0.0283 sec

10-38 (a) Disturburnce-to-Output Transfer Function

$$\left.\frac{Y(s)}{T_L(s)}\right|_{r=0} = \frac{2(1+0.1s)}{s(1+0.01s)(1+0.1s)+20K} \qquad G_c(s)=1$$

For $T_L(s) = 1/s$

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) = \frac{1}{10K} \le 0.01 \qquad \text{Thus} \qquad K \ge 10$$

10-38 (b) Performance of Uncompensated System. $K = 10$, $G_c(s) = 1$

$$G(s) = \frac{200}{s(1+0.01s)(1+0.1s)}$$

The Bode diagram of $G(j\omega)$ is shown below. The system is unstable. The attributes of the frequency response are: PM = −9.65 deg GM = −5.19 dB.

(c) Single-stage Phase-lead Controller Design

To realize a phase margin of 30 degrees, a = 14 and T = 0.00348.

$$G_c(s) = \frac{1+aTs}{1+Ts} = \frac{1+0.0487s}{1+0.00348s}$$

The Bode diagram of the phase-lead compensated system is shown below. The performance attributes are: PM = 30 deg GM = 10.66 dB $M_r = 1.95$ BW = 131.6 rad/sec.

(d) Two-stage Phase-lead Controller Design

Starting with the forward-path transfer function $G(s) = \dfrac{200(1+0.0487s)}{s(1+0.1s)(1+0.01s)(1+0.00348s)}$

The problem becomes that of designing a single-stage phase-lead controller. For a phase margin or 55 degrees, a = 7.385 and T = 0.00263. The transfer function of the second-stage controller is

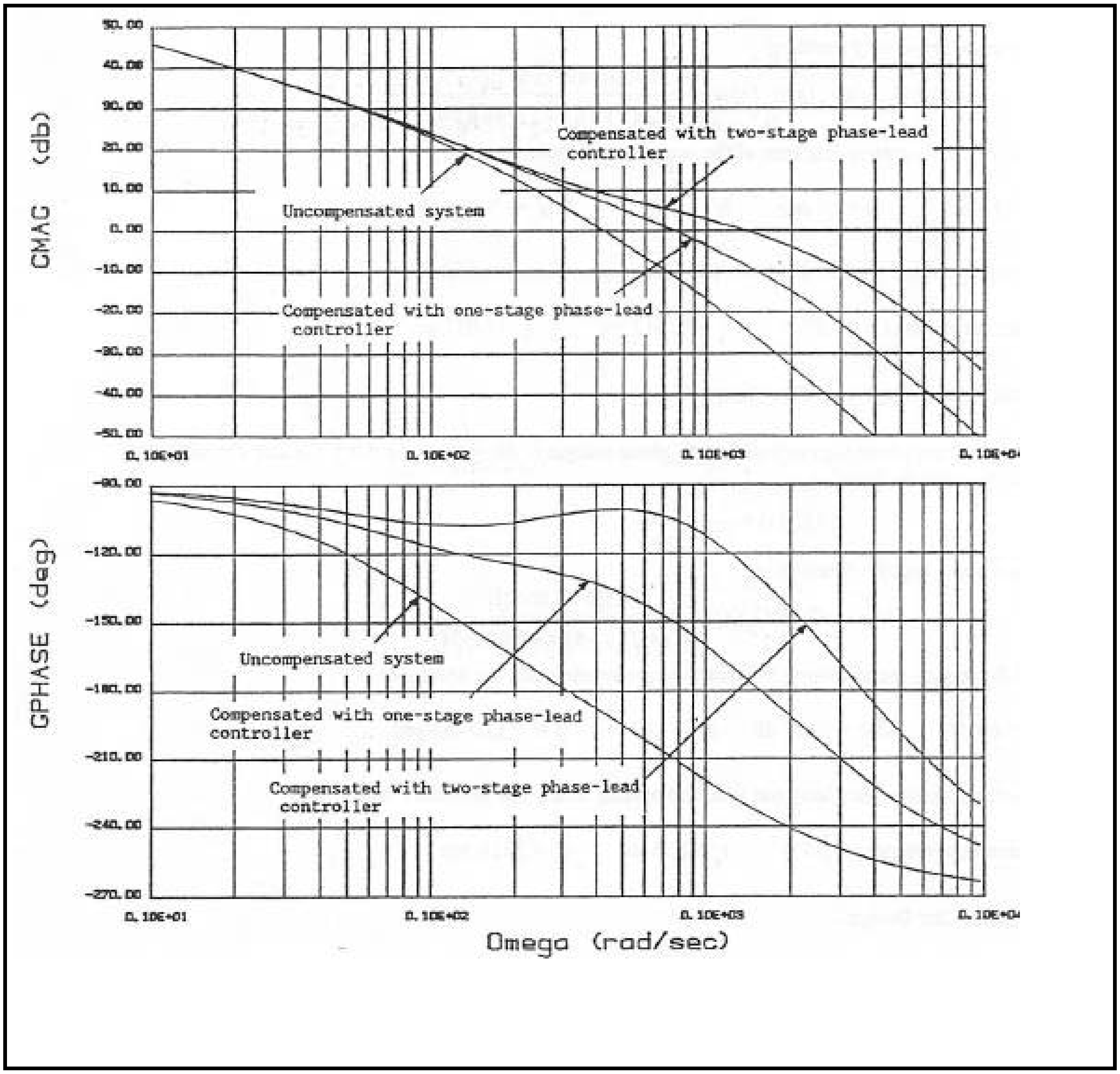
$$G_{c1}(s) = \frac{1+aTs}{1+Ts} = \frac{1+0.01845s}{1+0.00263s}$$

Thus $G(s) = \dfrac{200(1+0.0487s)(1+0.01845s)}{s(1+0.1s)(1+0.01s)(1+0.00348s)(1+0.00263s)}$

The Bode diagram is shown on the following page. The following frequency-response attributes are obtained:

PM = 55 deg GM = 12.94 dB $M_r = 1.11$ BW = 256.57 rad/sec

Bode Plot [parts (b), (c), and (d)]



10-39 (a) Two-stage Phase-lead Controller Design.

The uncompensated system is unstable. PM = −43.25 deg and GM = −18.66 dB.
With a single-stage phase-lead controller, the maximum phase margin that can be realized affectively is 12 degrees. Setting the desired PM at 11 deg, we have the parameters of the single-stage phase-lead controller as $a = 128.2$ and $T_1 = 0.00472$. The transfer function of the single-stage controller is

$$G_{c1}(s) = \frac{1 + aT_1 s}{1 + T_1 s} = \frac{1 + 0.6057\ s}{1 + 0.00472\ s}$$

Starting with the single-stage-controller compensated system, the second stage of the phase-lead controller is designed to realize a phase margin of 60 degrees. The parameters of the second-stage controller are: $b = 16.1$ and $T_2 = 0.0066$. Thus,

$$G_{c2}(s) = \frac{1 + bT_2 s}{1 + T_2 s} = \frac{1 + 0.106\ s}{1 + 0.0066\ s}$$

$$G_c(s) = G_{c1}(s) G_{c2}(s) = \frac{1 + 0.6057\ s}{1 + 0.00472\ s}\frac{1 + 0.106\ s}{1 + 0.0066\ s}$$

Forward-path Transfer Function:

$$G(s) = G_{c1}(s)G_{c2}(s)G_p(s) = \frac{1{,}236{,}598.6\,(s+1.651)(s+9.39)}{s(s+2)(s+5)(s+211.86)(s+151.5)}$$

Attributes of the frequency response of the compensated system are:

GM = 19.1 dB PM = 60 deg $M_r = 1.08$ BW = 65.11 rad/sec

The unit-step response is plotted below. The time-response attributes are:

Maximum overshoot = 10.2% $t_s = 0.1212$ sec $t_r = 0.037$ sec

(b) Single-stage Phase-lag Controller Design.

With a single-stage phase-lag controller, for a phase margin of 60 degrees, a = 0.0108 and T = 1483.8. The controller transfer function is

$$G_c(s) = \frac{1 + 16.08\, s}{1 + 1483.8\, s}$$

The forward-path transfer function is

$$G(s) = G_c(s)G_p(s) = \frac{6.5(s + 0.0662)}{s(s+2)(s+5)(s+0.000674)}$$

From the Bode plot, the following frequency-response attributes are obtained:

PM = 60 deg GM = 20.27 dB $M_r = 1.09$ BW = 1.07 rad/sec

The unit-step response has a long rise time and settling time. The attributes are:

Maximum overshoot = 12.5% $t_s = 12.6$ sec $t_r = 2.126$ sec

(c) Lead-lag Controller Design.

For the lead-lag controller, we first design the phase-lag portion for a 40-degree phase margin. The result is a = 0.0238 and $T_1 = 350$. The transfer function of the controller is

$$G_{c1}(s) = \frac{1 + 8.333\, s}{1 + 350\, s}$$

The phase-lead portion is designed to yield a total phase margin of 60 degrees. The result is b = 4.8 and $T_2 = 0.2245$. The transfer function of the phase-lead controller is

$$G_{c2}(s) = \frac{1 + 1.076\, s}{1 + 0.2245\, s}$$

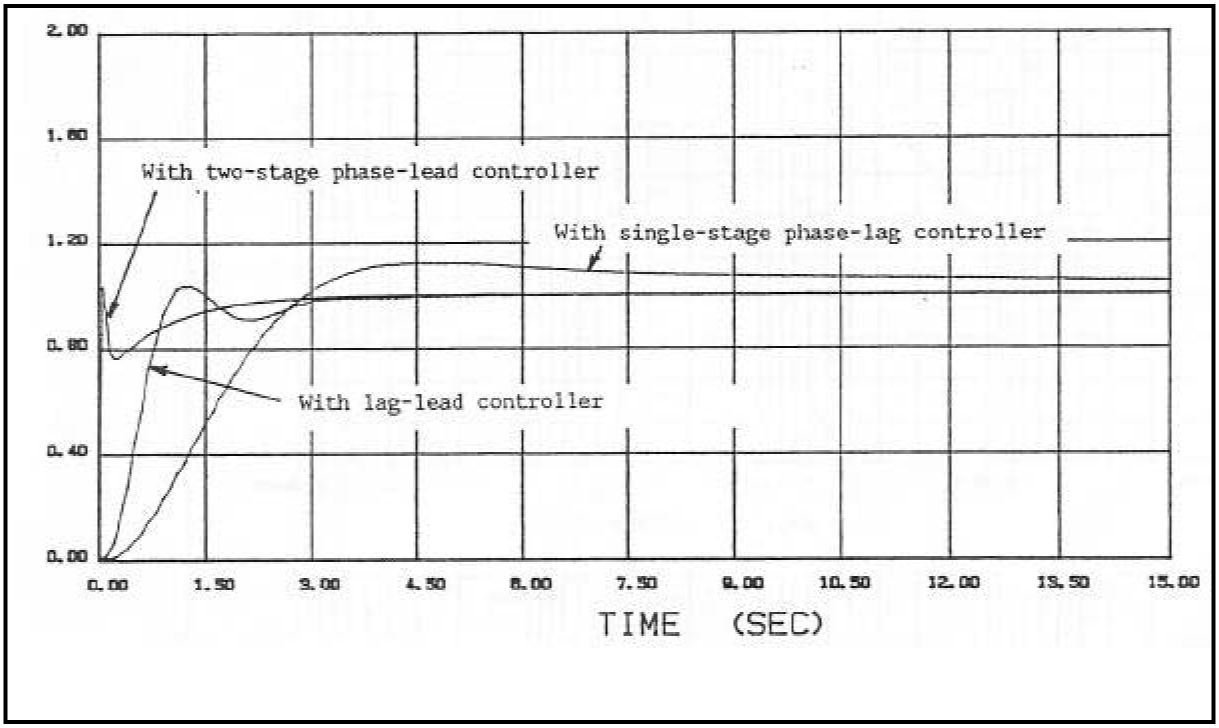
The forward-path transfer function of the lead-lag compensated system is

$$G(s) = \frac{68.63(s+0.12)(s+0.929)}{s(s+2)(s+5)(s+0.00286)(s+4.454)}$$

Frequency-response attributes: PM = 60 deg GM = 13.07 dB $M_r = 1.05$ BW = 3.83 rad/sec

Unit-step response attributes: Maximum overshoot = 5.9% $t_s = 1.512$ sec $t_r = 0.7882$ sec

Unit-step Responses.



10-40 (a) The uncompensated system has the following frequency-domain attributes:

PM = 3.87 deg GM = 1 dB $M_r = 7.73$ BW = 4.35 rad/sec

The Bode plot of $G_p(j\omega)$ shows that the phase curve drops off sharply, so that the phase-lead controller would not be very effective. Consider a single-stage phase-lag controller. The phase margin of 60 degrees is realized if the gain crossover is moved from 2.8 rad/sec to 0.8 rad/sec. The attenuation of the phase-lag controller at high frequencies is approximately -15 dB. Choosing an attenuation of -17.5 dB, we calculate the value of a from

$$20 \log_{10} a = -17.5 \text{ dB} \qquad \text{Thus } a = 0.1334$$

The upper corner frequency of the phase-lag controller is chosen to be at $1/aT = 0.064$ rad/sec. Thus, $1/T = 0.00854$ or $T = 117.13$. The transfer function of the phase-lag controller is

$$G_c(s) = \frac{1 + 15.63 s}{1 + 117.13}$$

The forward-path transfer function is

$$G(s) = G_c(s)G_p(s) = \frac{5(1+15.63s)(1-0.05s)}{s(1+0.1s)(1+0.5s)(1+117.13s)(1+0.05s)}$$

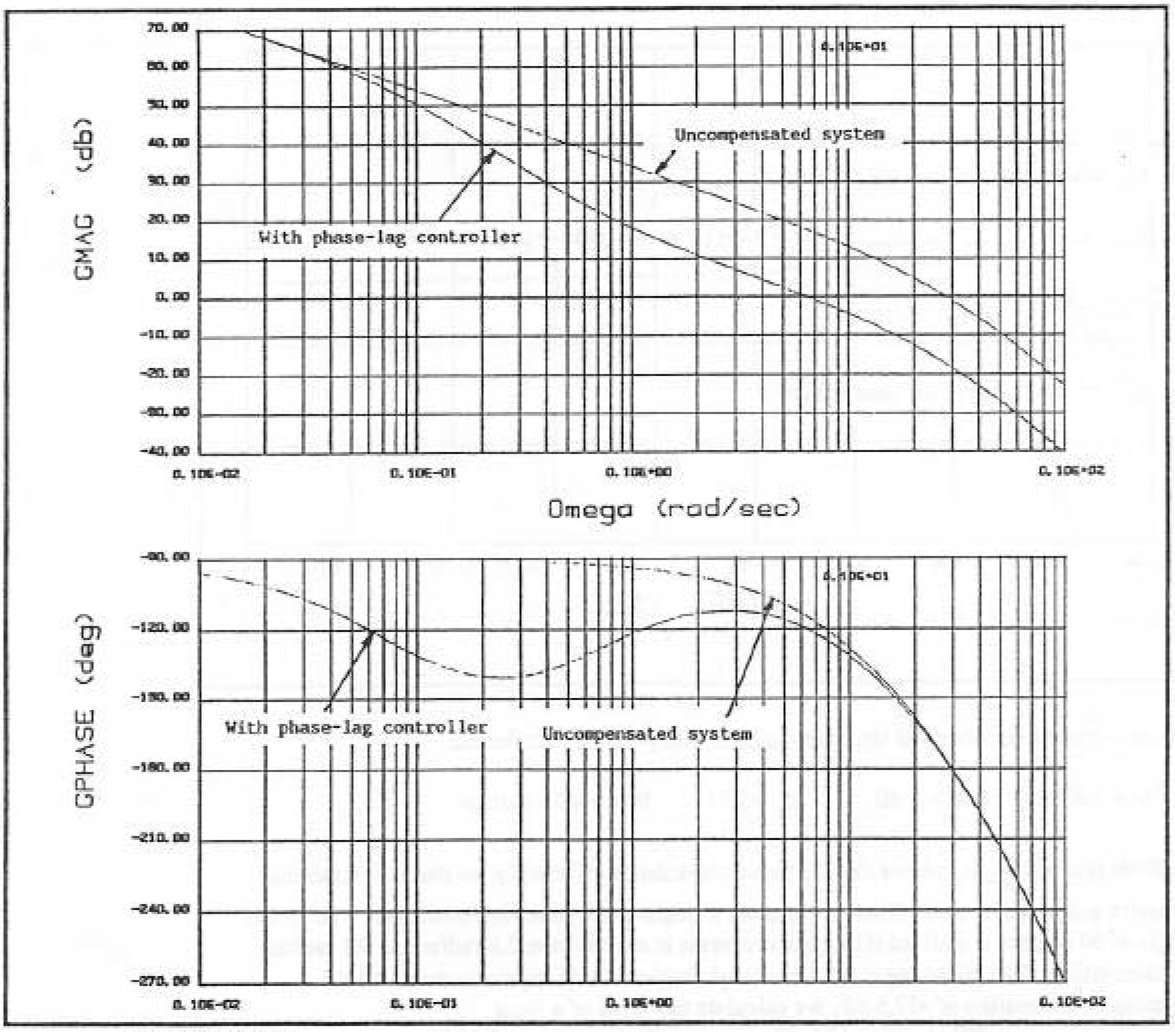
From the Bode plot of $G(j\omega)$, the following frequency-domain attributes are obtained:

PM = 60 deg GM = 18.2 dB $M_r = 1.08$ BW = 1.13 rad/sec

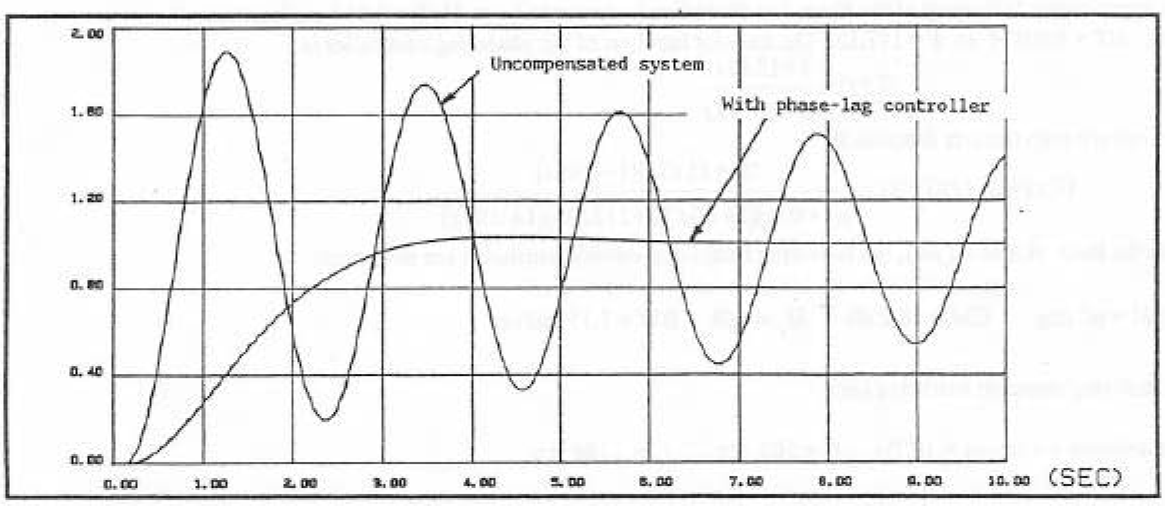
The unit-step response attributes are:

maximum overshoot = 10.7% $t_s = 10.1$ sec $t_r = 2.186$ sec

Bode Plots



Step Responses



10-40 (b) Using the exact expression of the time delay, the same design holds. The time and frequency domain attributes are not much affected.

10-41 (a) Uncompensated System.

Forward-path Transfer Function: $G(s) = \dfrac{10}{(1+s)(1+10s)(1+2s)(1+5s)}$

The Bode plot of $G(jW)$ is shown below.

The performance attributes are: PM = −10.64 deg GM = −2.26 dB
The uncompensated system is unstable.

(b) PI Controller Design.

Forward-path Transfer Function: $G(s) = \dfrac{10(K_p s + K_I)}{s(1+s)(1+10s)(1+2s)(1+5s)}$

Ramp-error Constant: $K_v = \lim_{s \to 0} sG(s) = 10K_I = 0.1$ Thus $K_I = 0.01$

$$G(s) = \frac{0.1(1+100K_p s)}{s(1+s)(1+10s)(1+2s)(1+5s)}$$

The following frequency-domain attributes are obtained for various values of K_P.

K_P	PM (deg)	GM (dB)	M_r	BW (rad/sec)
0.01	24.5	5.92	2.54	0.13
0.02	28.24	7.43	2.15	0.13
0.05	38.84	11.76	1.52	0.14
0.10	50.63	12.80	1.17	0.17
0.12	52.87	12.23	1.13	0.18
0.15	53.28	11.22	1.14	0.21
0.16	52.83	10.88	1.16	0.22
0.17	51.75	10.38	1.18	0.23
0.20	49.08	9.58	1.29	0.25

The phase margin is maximum at 53.28 degrees when $K_P = 0.15$.
The forward-path transfer function of the compensated system is

$$G(s) = \frac{0.1(1+15s)}{s(1+s)(1+10s)(1+5s)(1+2s)}$$

The attributes of the frequency response are:

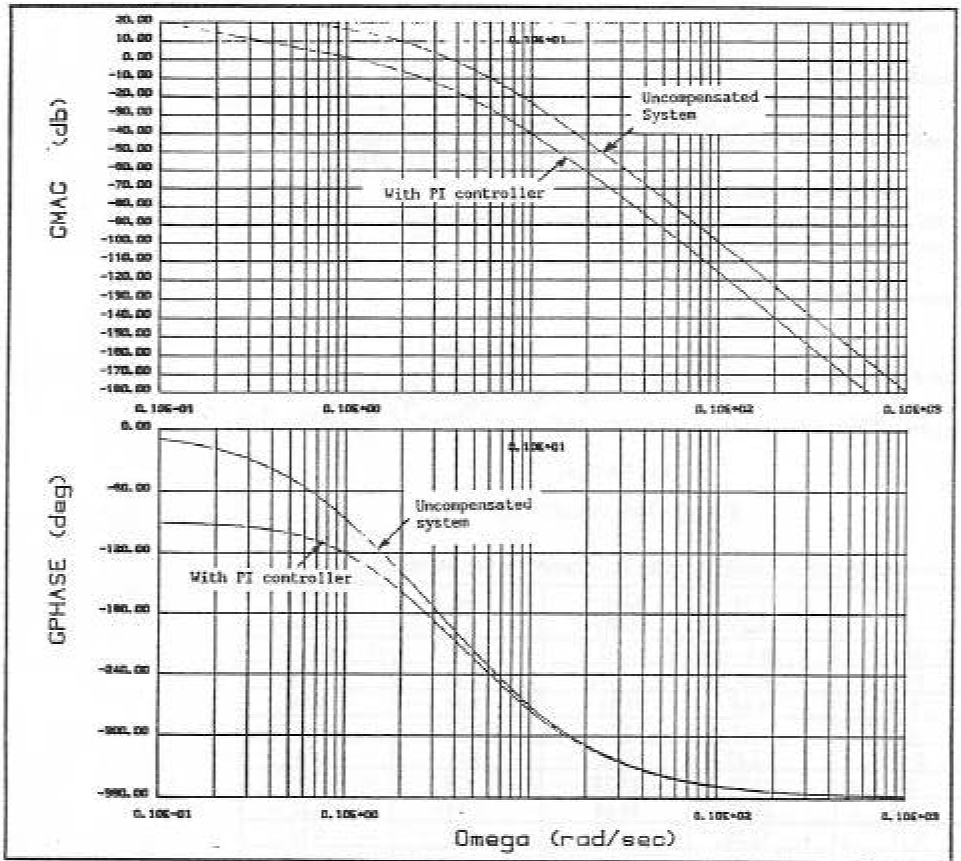
PM = 53.28 deg GM = 11.22 dB $M_r = 1.14$ BW = 0.21 rad/sec

The attributes of the unit-step response are:

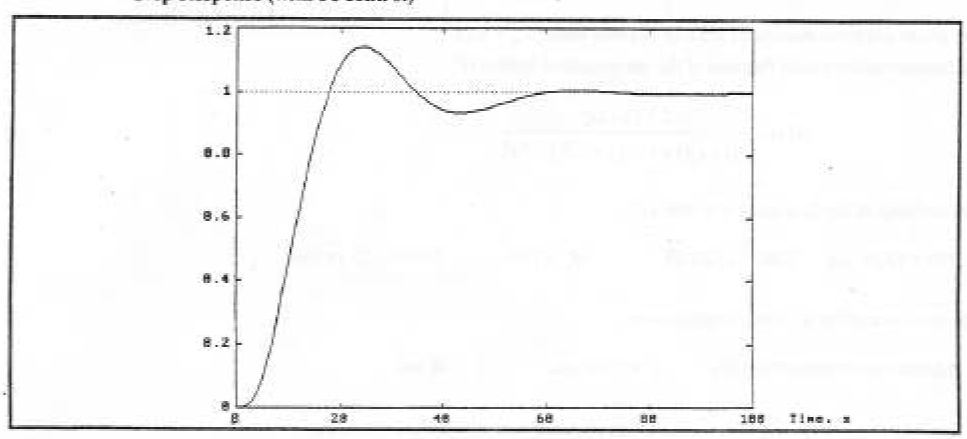
Maximum overshoot = 14.1% $t_r = 10.68$ sec $t_s = 48$ sec

Bode Plots

Bode Plots



Step Response (with PI control)



10-41 (c) Time-domain Design of PI Controller.

By setting $K_I = 0.01$ and varying K_P we found that the value of K_P that minimizes the maximum overshoot of the unit-step response is also 0.15. Thus, the unit-step response obtained in part (b) is still applicable for this case.

10-42 Closed-loop System Transfer Function.

$$\frac{Y(s)}{R(s)} = \frac{1}{s^3 + \left(4+k_3\right)s^2 + \left(3+k_2+k_3\right)s + k_1}$$

For zero steady-state error to a step input, $k_1 = 1$. For the complex roots to be located at $-1+j$ and $-1-j$,

we divide the characteristic polynomial by $s^2 + 2s + 2$ and solve for zero remainder.

$$\begin{array}{r|l}
 & s + \left(2+k_2\right) \\
\hline
s^2+2s+2 & s^3 + \left(4+k_3\right)s^2 + \left(3+k_2+k_3\right)s + 1 \\
 & s^3 + \quad 2s^2 \quad\quad + 2s \\
\hline
 & \left(2+k_3\right)s^2 + \left(1+k_2+k_3\right)s + 1 \\
 & \left(2+k_3\right)s^2 + \left(4+2k_3\right)s \quad + 4+2k_3 \\
\hline
 & \left(-3+k_2-k_3\right)s - 3 - 2k_3
\end{array}$$

For zero remainder, $-3-2k_3 = 0$ Thus $k_3 = -1.5$

$-3+k_2-k_3 = 0$ Thus $k_2 = 1.5$

The third root is at -0.5. Not all the roots can be arbitrarily assigned, due to the requirement on the steady-state error.

10-43 (a) Open-loop Transfer Function.

$$G(s) = \frac{X_1(s)}{E(s)} = \frac{k_3}{s\left[s^2 + \left(4+k_2\right)s + 3 + k_1 + k_2\right]}$$

Since the system is type 1, the steady-state error due to a step input is zero for all values of k_1, k_2, and k_3 that correspond to a stable system. The characteristic equation of the closed-loop system is

$$s^3 + \left(4+k_2\right)s^2 + \left(3+k_1+k_2\right)s + k_3 = 0$$

For the roots to be at $-1+j$, $-1-j$, and -10, the equation should be:

$$s^3 + 12s^2 + 22s + 20 = 0$$

Equating like coefficients of the last two equations, we have

$4 + k_2 = 12$ Thus $k_2 = 8$

$3 + k_1 + k_2 = 22$ Thus $k_1 = 11$

$k_3 = 20$ Thus $k_3 = 20$

(b) Open-loop Transfer Function.

$$\frac{Y(s)}{E(s)} = \frac{G_c(s)}{\left(s+1\right)\left(s+3\right)} = \frac{20}{s\left(s^2+12s+22\right)} \qquad \text{Thus} \qquad G_c(s) = \frac{20\left(s+1\right)\left(s+3\right)}{s\left(s^2+12s+22\right)}$$

10-44 (a)

$$\mathbf{A}^* = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 25.92 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -2.36 & 0 & 0 & 0 \end{bmatrix} \qquad \mathbf{B}^* = \begin{bmatrix} 0 \\ -0.0732 \\ 0 \\ 0.0976 \end{bmatrix}$$

The feedback gains, from k_1 to k_4:

−2.4071E+03	−4.3631E+02	−8.4852E+01	−1.0182E+02

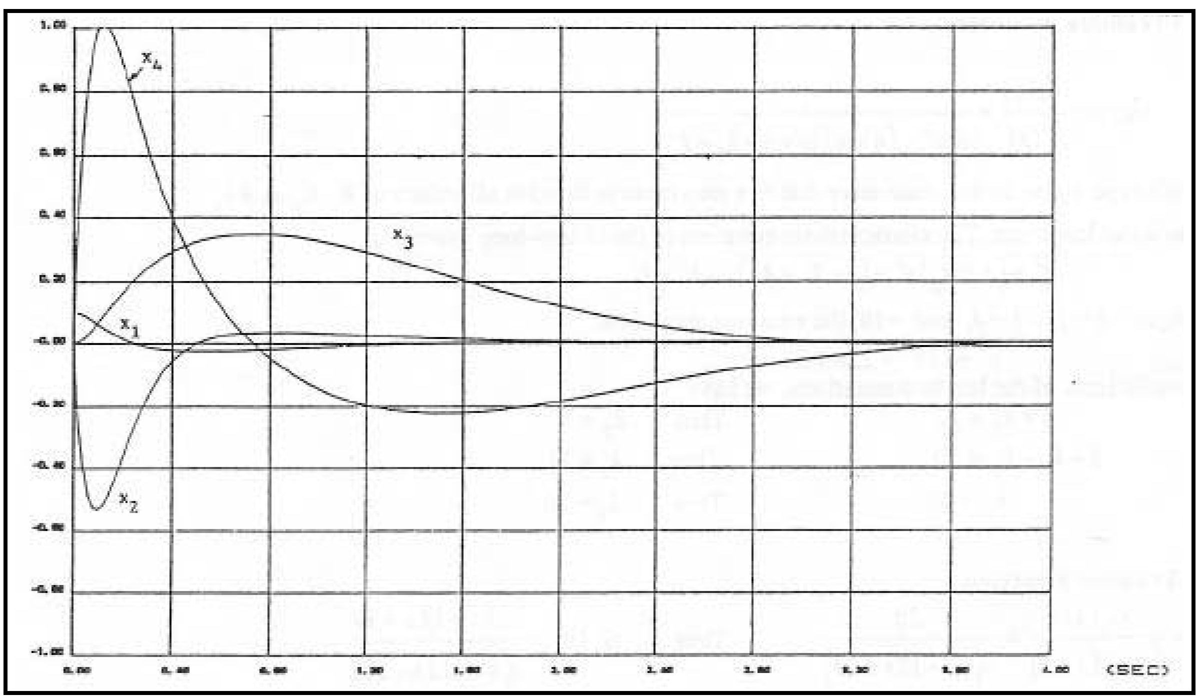
The $\mathbf{A}^* - \mathbf{B}^*\mathbf{K}$ matrix of the closed-loop system

0.0000E+00	1.0000E+00	0.0000E+00	0.0000E+00
−1.5028E+02	−3.1938E+01	−6.2112E+00	−7.4534E+00
0.0000E+00	0.0000E+00	0.0000E+00	1.0000E+00
2/3258E+02	4.2584E+01	8.2816E+00	9.9379E+00

The **B** vector

0.0000E+00
−7.3200E−02
0.0000E+00
9.7600E−02

Time Responses:



10-44 (b) The feedback gains, from k_1 to k_2:

−9.9238E+03	−1.6872E+03	−1.3576E+03	−8.1458E+02

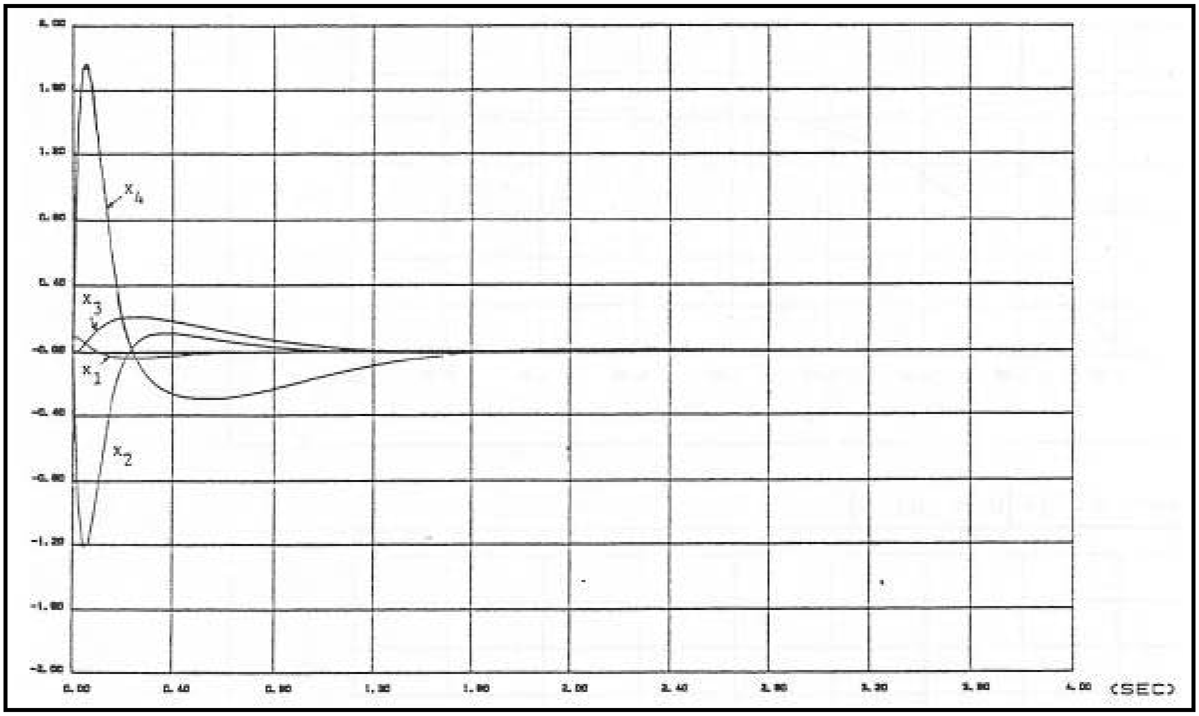
The $\mathbf{A}^* - \mathbf{B}^*\mathbf{K}$ matrix of the closed-loop system

0.0000E+00	1.0000E+00	0.0000E+00	0.0000E+00
−7.0051E+02	−1.2350E+02	−9.9379E+01	−5.9627E+01
0.0000E+00	0.0000E+00	0.0000E+00	1.0000E+00
9.6621E+02	1.6467E+02	1.3251E+02	7.9503E+01

The **B** vector

0.0000E+00
−7.3200E−02
0.0000E+00
9.7600E−02

Time Responses:



10-45 The solutions are obtained by using the pole-placement design option of the **linsys** program in the **ACSP** software package.

(a) The feedback gains, from k_1 to k_2:

−6.4840E+01	−5.6067E+00	2.0341E+01	2.2708E+00

The $\mathbf{A}^* - \mathbf{B}^*\mathbf{K}$ matrix of the closed-loop system

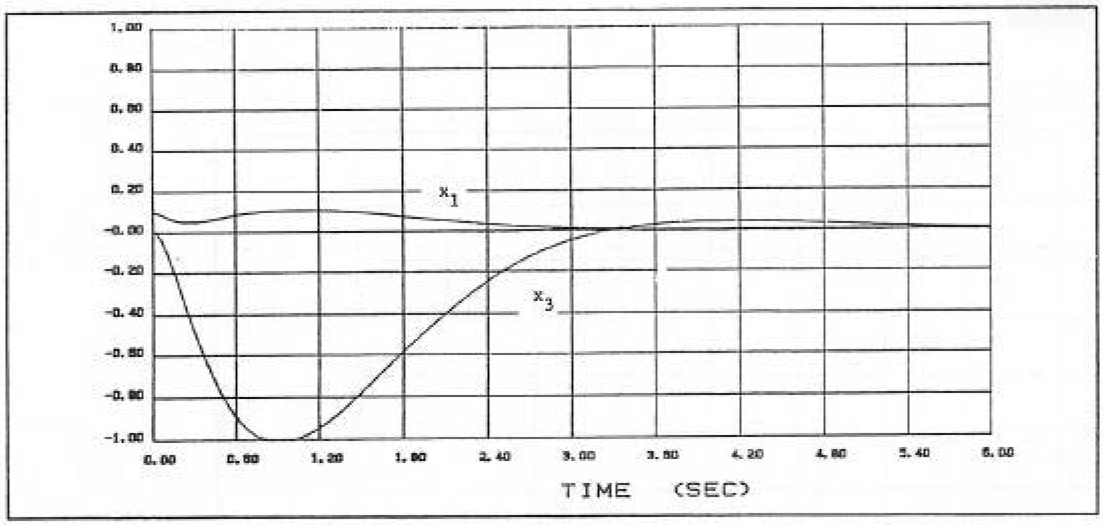
0.0000E+00	1.0000E+00	0.0000E+00	0.0000E+00
−3.0950E+02	−3.6774E+01	1.1463E+02	1.4874E+01
0.0000E+00	0.0000E+00	0.0000E+00	1.0000E+00
−4.6190E+02	−3.6724E+01	1.7043E+02	1.477eE+01

The **B** vector

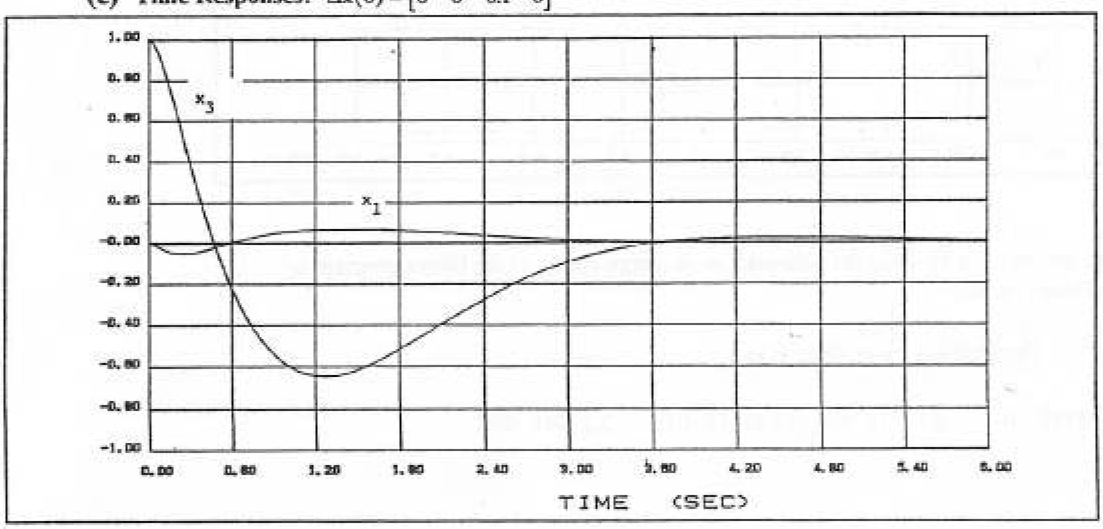
0.0000E+00
−6.5500E+00
0.0000E+00

$-6.5500E+00$

(b) Time Responses: $\Delta\mathbf{x}(0) = [0.1 \quad 0 \quad 0 \quad 0]'$



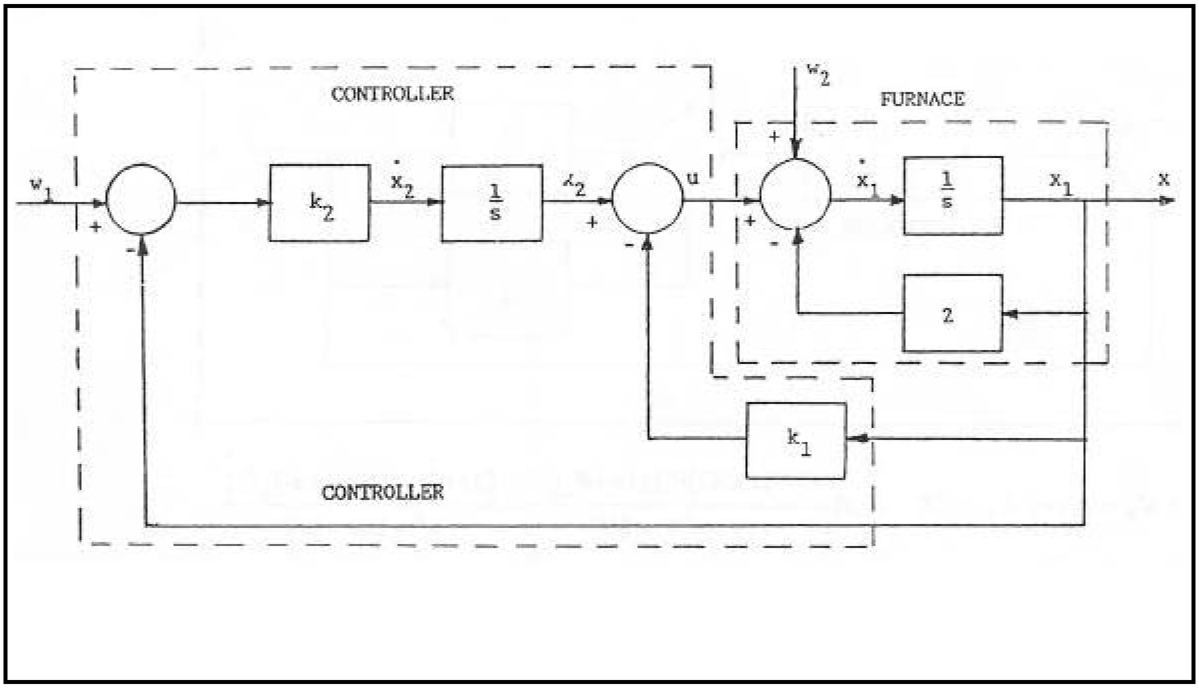
(c) Time Responses: $\Delta\mathbf{x}(0) = [0 \quad 0 \quad 0.1 \quad 0]'$



With the initial states $\Delta\mathbf{x}(0) = [0.1 \quad 0 \quad 0 \quad 0]'$, the initial position of Δx_1 or Δy_1 is preturbed downward from its stable equilibrium position. The steel ball is initially pulled toward the magnet, so $\Delta x_3 = \Delta y_2$ is negative at first. Finally, the feedback control pulls both bodies back to the equilibrium position.

With the initial states $\Delta\mathbf{x}(0) = [0 \quad 0 \quad 0.1 \quad 0]'$, the initial position of Δx_3 or Δy_2 is preturbed downward from its stable equilibrium. For $t > 0$, the ball is going to be attracted up by the magnet toward the equilibrium position. The magnet will initially be attracted toward the fixed iron plate, and then settles to the stable equilibrium position. Since the steel ball has a small mass, it will move more actively.

10-46 (a) Block Diagram of System.



$$u = -k_1 x_1 + k_2 \int \left(-x_1 + w_1\right) dt$$

State Equations of Closed-loop System:

$$\begin{bmatrix} \dfrac{dx_1}{dt} \\ \dfrac{dx_2}{dt} \end{bmatrix} = \begin{bmatrix} -2-k_1 & 1 \\ -k_2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ k_2 & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

Characteristic Equation:

$$\left| s\mathbf{I} - \mathbf{A} \right| = \begin{vmatrix} s+2+k_1 & -1 \\ k_2 & s \end{vmatrix} = s^2 + \left(2 + k_1\right) s + k_2 = 0$$

For $s = -10, -10$, $\left| s\mathbf{I} - \mathbf{A} \right| = s^2 + 20s + 200 = 0$ Thus $k_1 = 18$ and $k_2 = 200$

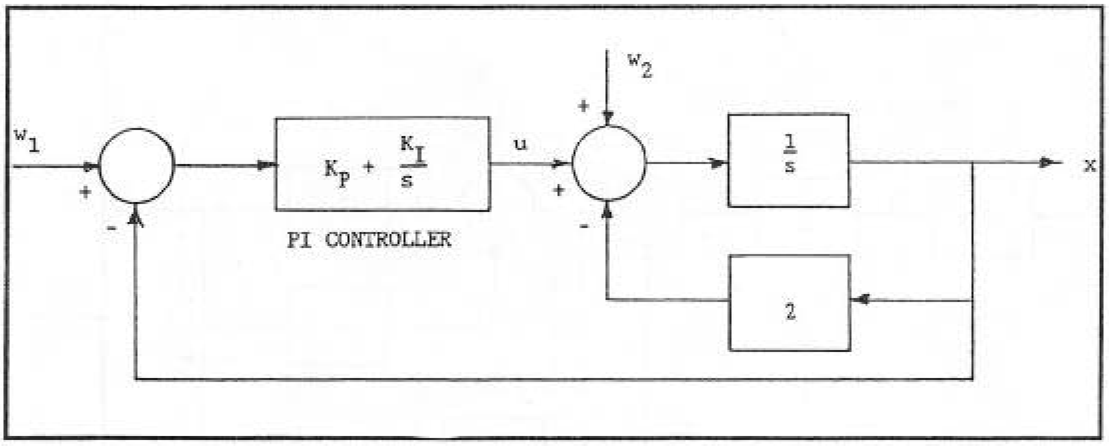
$$X(s) = X_1(s) = \frac{200\, W_1(s) s^{-2} + s^{-1} W_2(s)}{1 + 2s^{-1} + 18s^{-1} + 200\, s^{-2}} = \frac{200\, W_1(s) + sW_2(s)}{s^2 + 20s + 200}$$

$$W_1(s) = \frac{1}{s} \qquad W_2(s) = \frac{W_2}{s} \qquad W_2 = \text{const ant}$$

$$X(s) = \frac{200 + W_2 s}{s\left(s^2 + 20s + 200\right)} \qquad \lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s) = 1$$

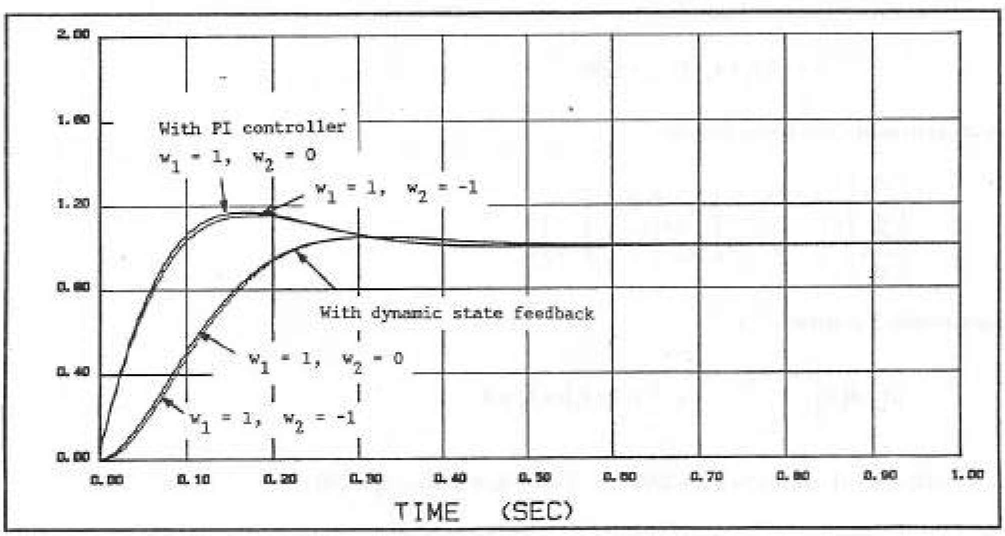
10-46 (b) With PI Controller:

Block Diagram of System:



Set $K_P = 2$ and $K_I = 200$. $X(s) = \dfrac{\left(K_P s + K_I\right) W_1(s) + sW_2(s)}{s^2 + 20s + 200} = \dfrac{\left(2s + 200\right) W_1(s) + sW_2(s)}{s^2 + 20s + 200}$

Time Responses:



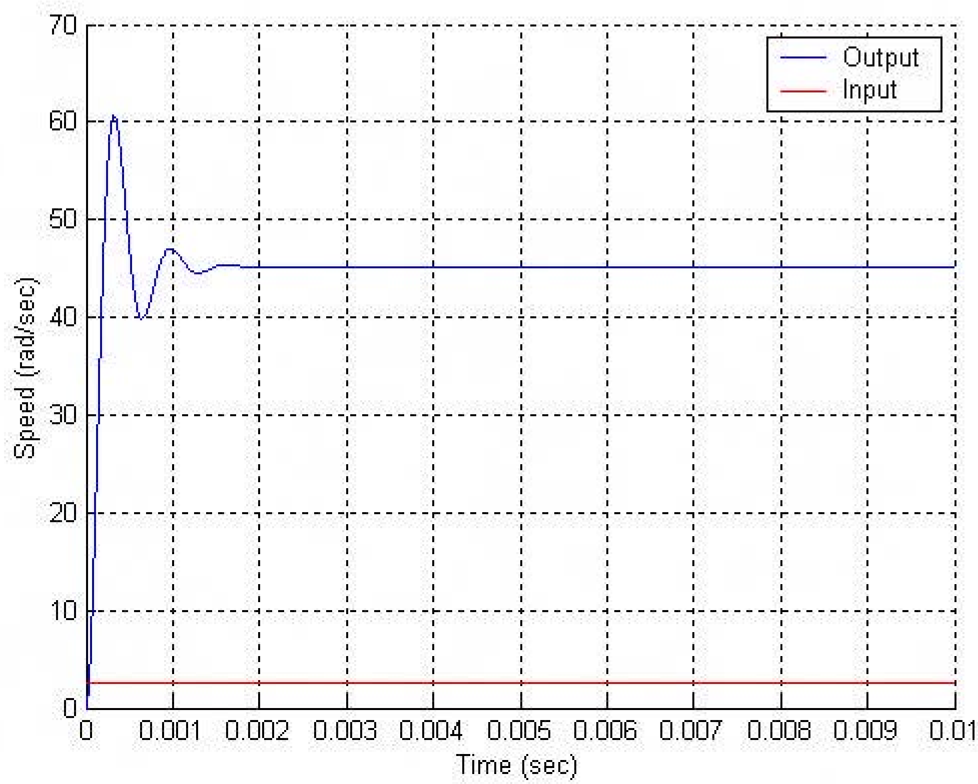
Chapter 11	THE VIRTUAL LAB

Part 1) Solution to Lab questions within Chapter 11

11-5-1 Open Loop Speed

1. Open loop speed response using SIMLab:

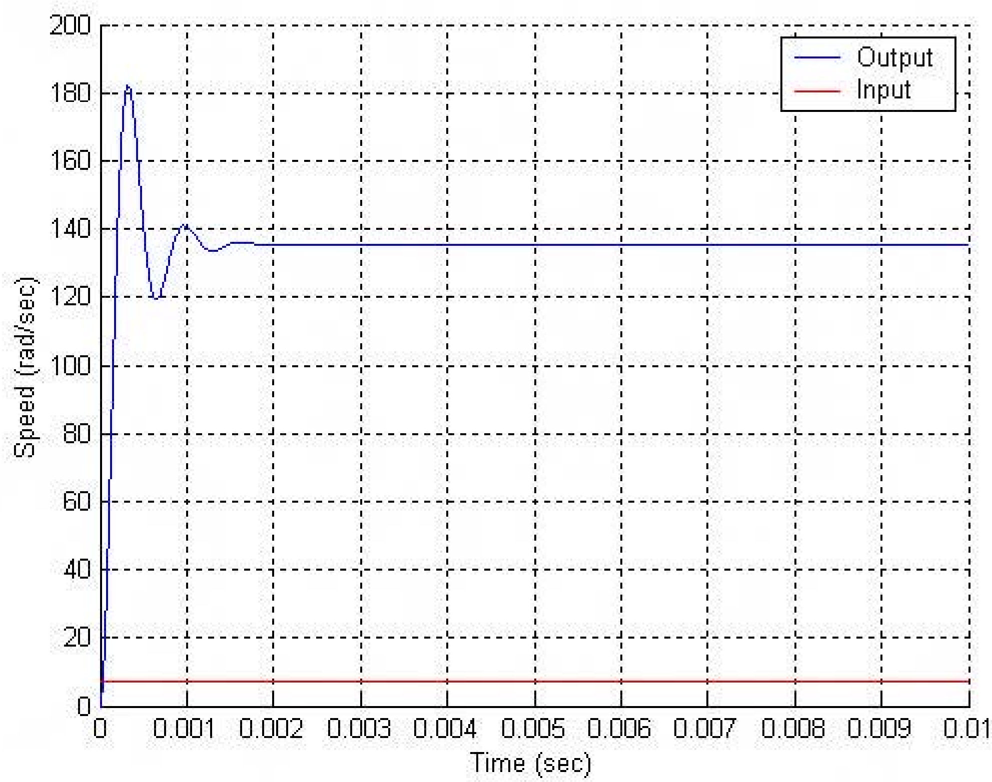
   a. +5 V input:



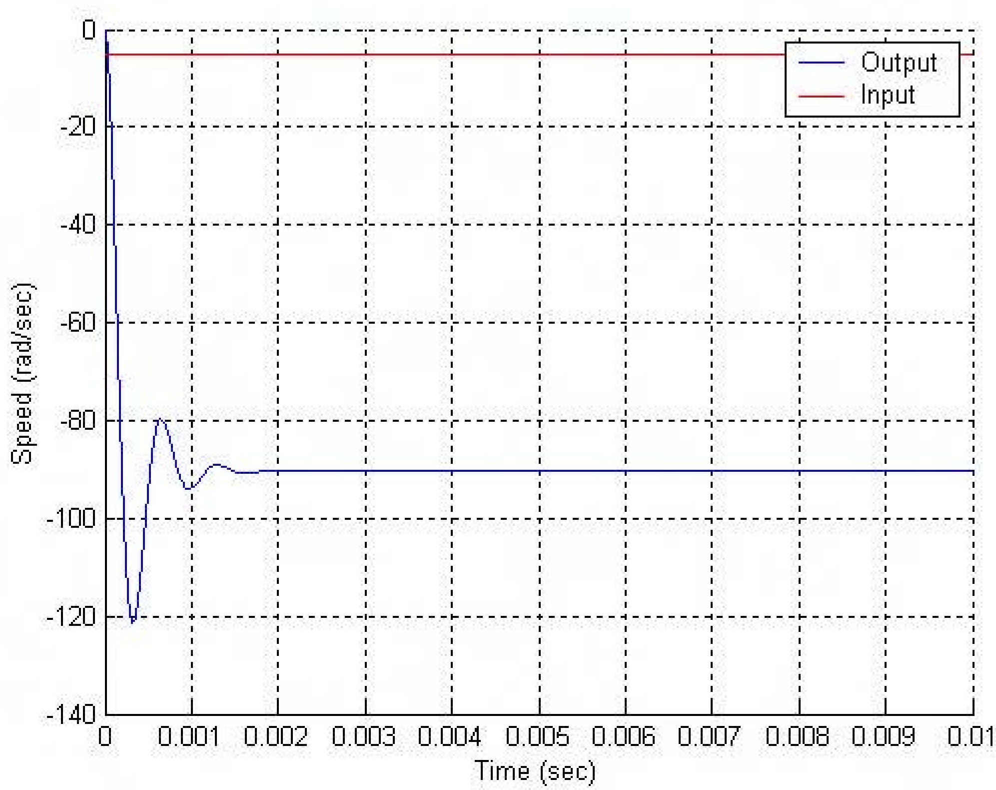
The form of response is like the one that we expected; a second order system response with overshoot and oscillation. Considering an amplifier gain of 2 and $K_b = 0.1$, the desired set point should be set to 2.5 and as seen in the figure, the final value is approximately 50 rad/sec which is armature voltage divided by K_b. To find the above response the systems parameters are extracted from 11-3-1 of the text and B is calculated from 11-3 by having t_m as:

$$t_m = \frac{R_a J_m}{R_a B + k_b k_m} \quad , \qquad B = \frac{R_a J_m - k_b k_m t_m}{R_a t_m} = 0.000792 kg \cdot m^2 / \sec$$

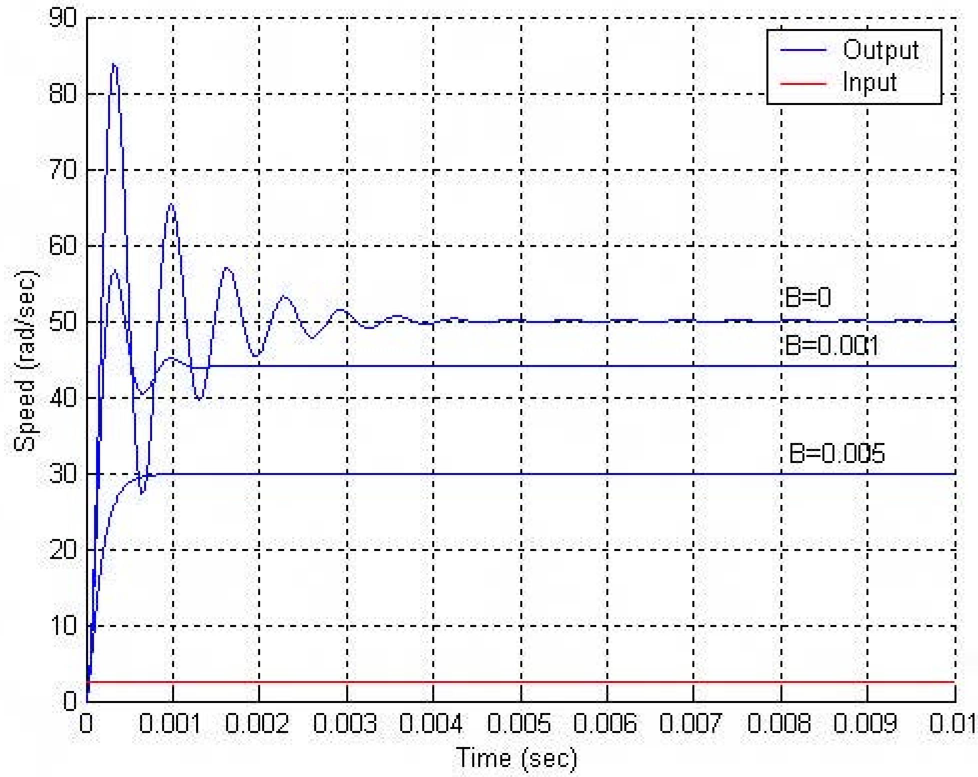
b. +15 V input:



c. –10 V input:

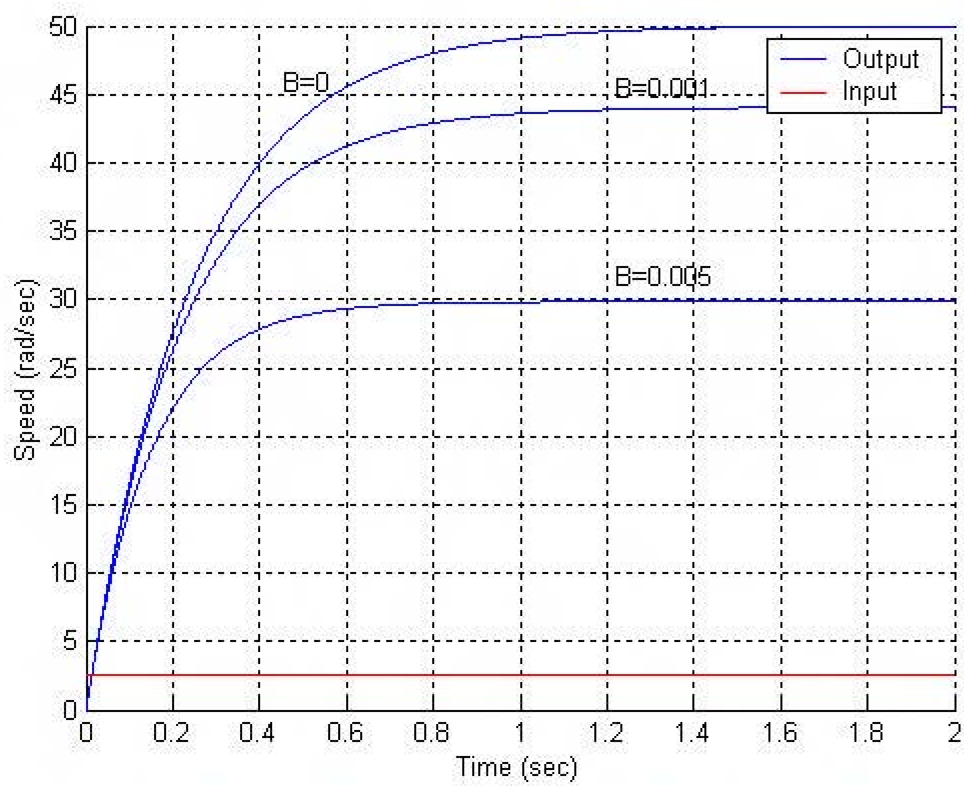


2. Study of the effect of viscous friction:



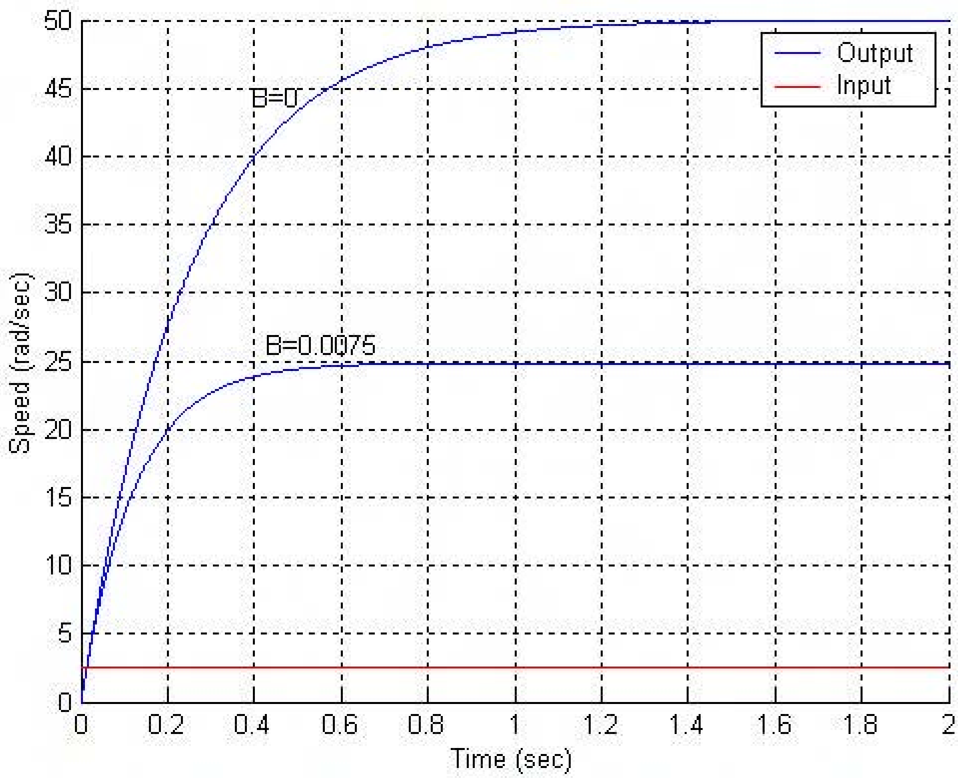
The above figure is plotted for three different friction coefficients (0, 0.001, 0.005) for 5 V armature input. As seen in figure, two important effects are observed as the viscous coefficient is increased. First, the final steady state velocity is decreased and second the response has less oscillation. Both of these effects could be predicted from Eq. (11.1) by increasing damping ratio ζ.

3. Additional load inertia effect:



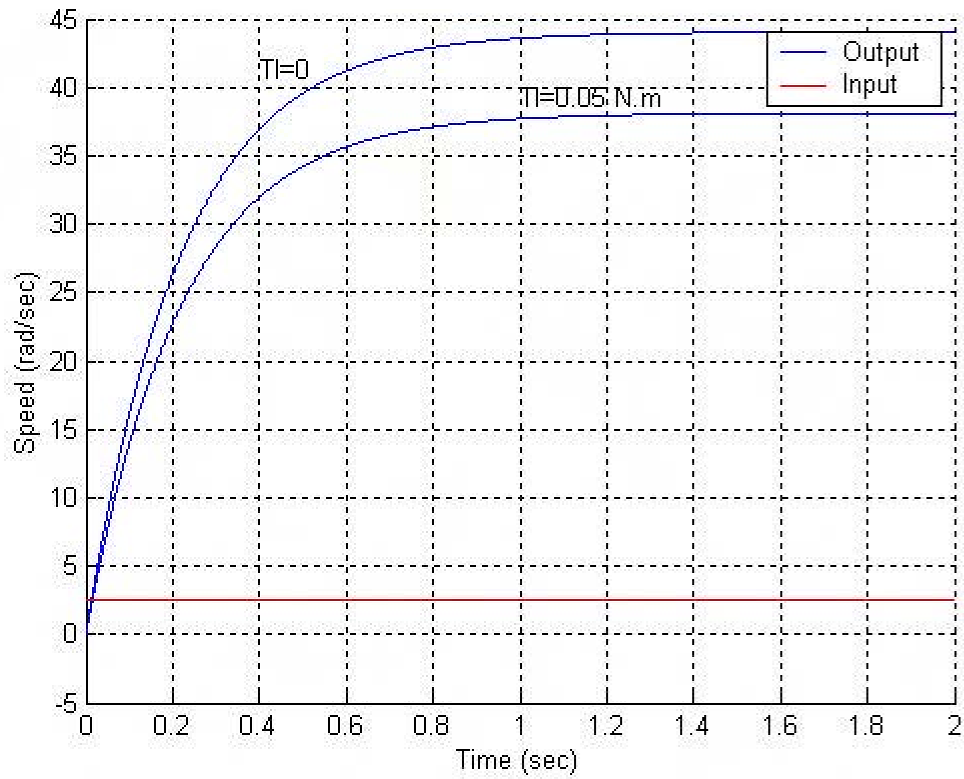
As the overall inertia of the system is increased by $0.005/5.2^2$ and becomes 1.8493×10^{-3} kg.m^2, the mechanical time constant is substantially increased and we can assume the first order model for the motor (ignoring the electrical sub-system) and as a result of this the response is more like an exponential form. The above results are plotted for 5 V armature input.

4. Reduce the speed by 50% by increasing viscous friction:



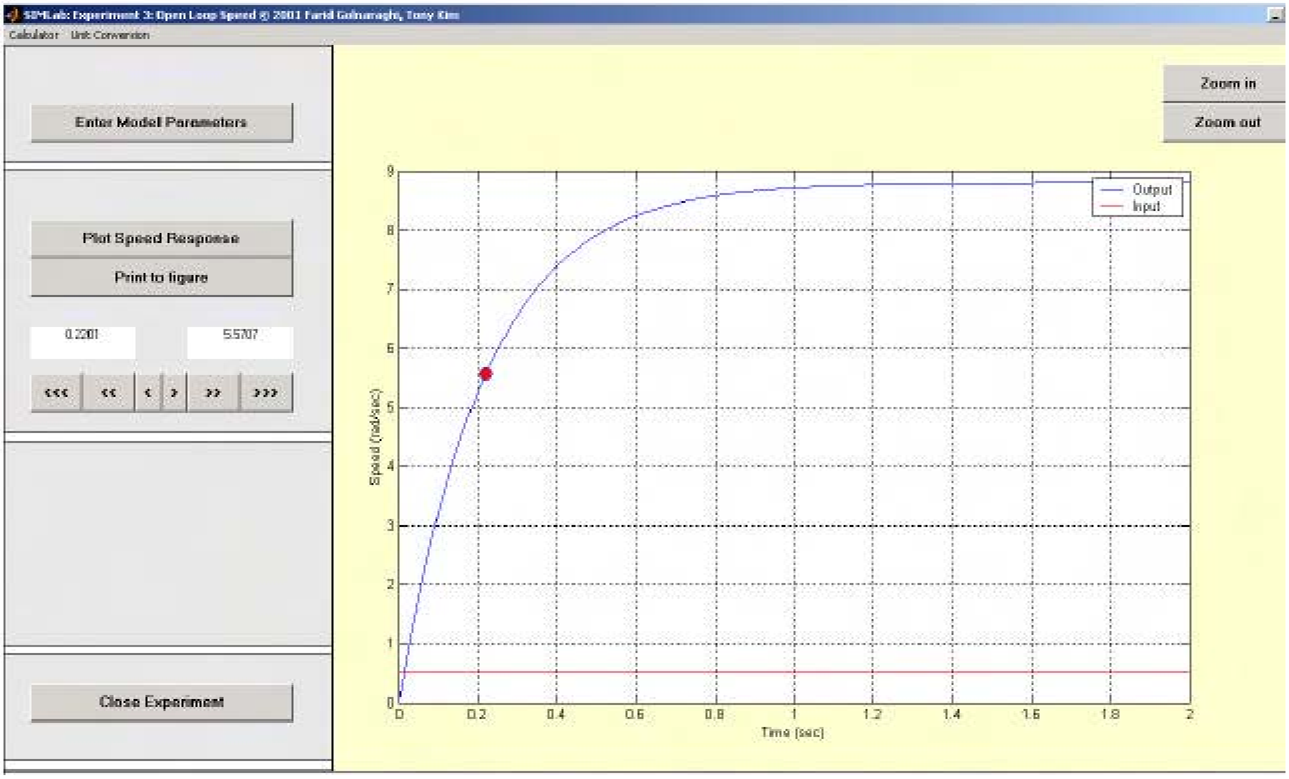
As seen in above figure, if we set B=0.0075 *N.s/m* the output speed drop by half comparing with the case that B=0 *N.s/m.* The above results are plotted for 5 V armature input.

5. Study of the effect of disturbance:



Repeating experiment 3 for B=0.001 N.s/m and T_L=0.05 N.m result in above figure. As seen, the effect of disturbance on the speed of open loop system is like the effect of higher viscous friction and caused to decrease the steady state value of speed.

6. Using speed response to estimate motor and load inertia:



Using first order model we are able to identify system parameters based on unit step response of the system. In above plot we repeated the experiments 3 with B=0.001 and set point voltage equal to 1 V. The final value of the speed can be read from the curve and it is 8.8, using the definition of system time constant and the cursor we read 63.2% of speed final value 5.57 occurs at 0.22 sec, which is the system time constant. Considering Eq. (11-3), and using the given value for the rest of parameters, the inertia of the motor and load can be calculated as:

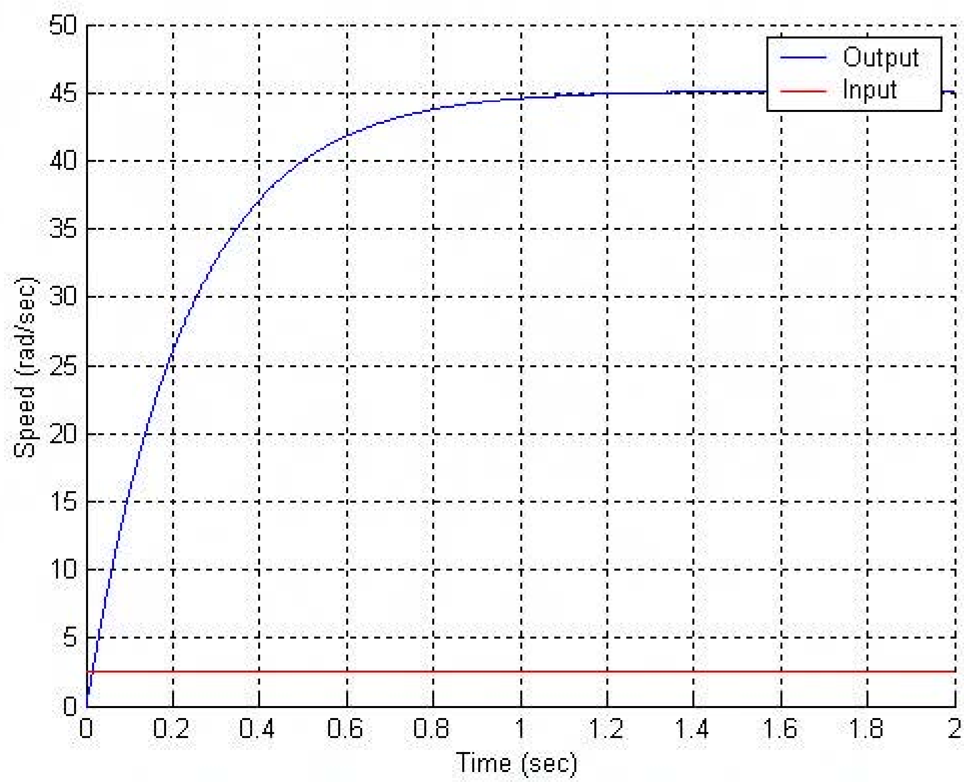
$$J = \frac{t_m (R_a B + K_m K_b)}{R_a} = \frac{0.22(1.35 \times 0.001 + 0.01)}{1.35} = 1.8496 \times 10^{-3} \text{ kg.m}^2$$

We also can use the open loop speed response to estimate B by letting the speed to coast down when it gets to the steady state situation and then measuring the required time to get to zero speed. Based on this time and energy conservation principle and knowing the rest of parameters we are able to calculate B. However, this method of identification gives us limited information about the system parameters and we need to measure some parameters directly from motor such as R_a, K_m, K_b and so on.

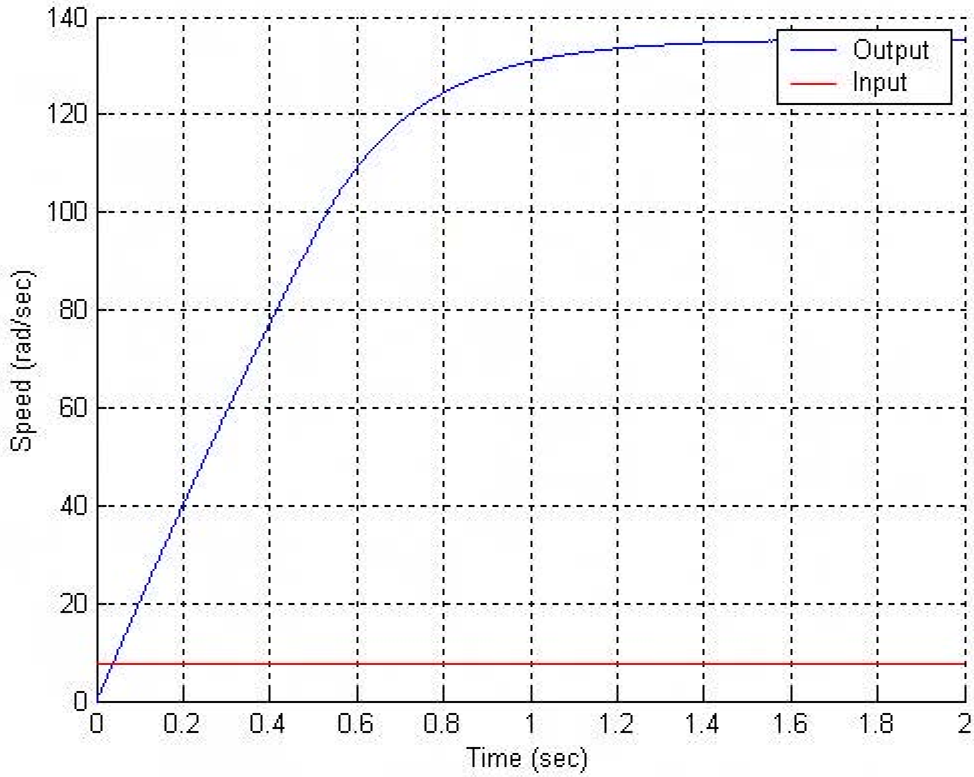
So far, no current or voltage saturation limit is considered for all simulations using SIMLab software.

7. Open loop speed response using Virtual Lab:

a. +5 V:

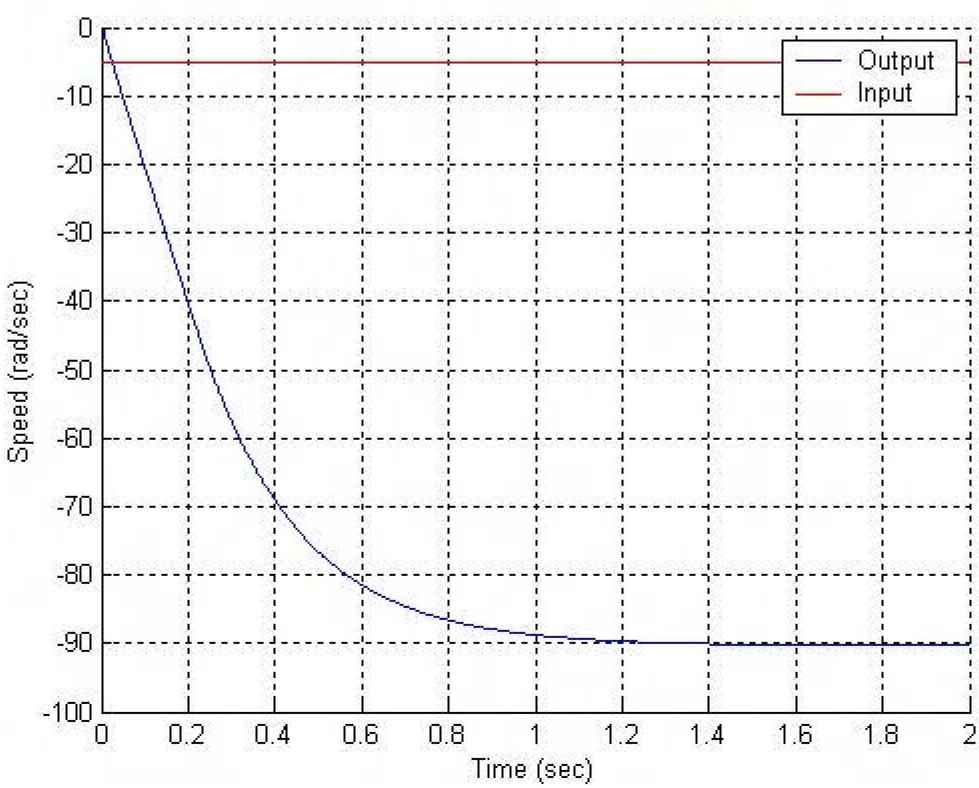


b. +15 V:



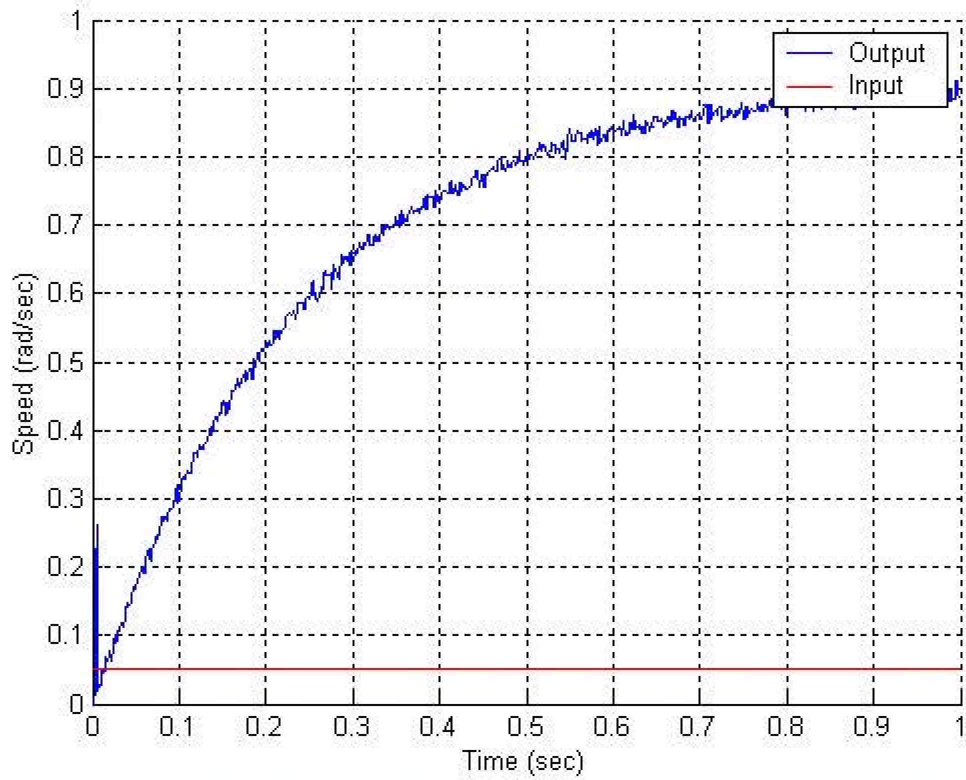
140
120
100
80
60
40
20
0
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
Time (sec)
Speed (rad/sec)
Output
Input

c. –10 V:

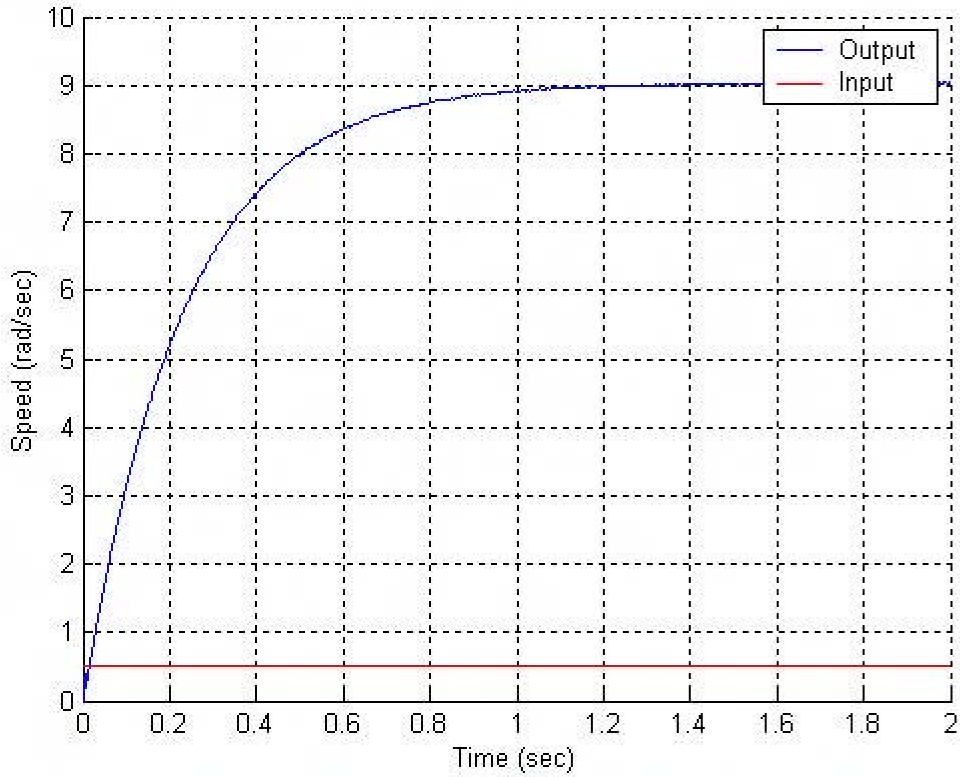


0
-10
-20
-30
-40
-50
-60
-70
-80
-90
-100
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
Time (sec)
Speed (rad/sec)
Output
Input

Comparing these results with the part 1, the final values are approximately the same but the shape of responses is closed to the first order system behavior. Then the system time constant is obviously different and it can be identified from open loop response. The effect of nonlinearities such as saturation can be seen in +15 V input with appearing a straight line at the beginning of the response and also the effects of noise and friction on the response can be observed in above curves by reducing input voltage for example, the following response is plotted for a 0.1 V step input:



8. Identifying the system based on open loop response:



Open loop response of the motor to a unit step input voltage is plotted in above figure. Using the definition of time constant and final value of the system, a first order model can be found as:

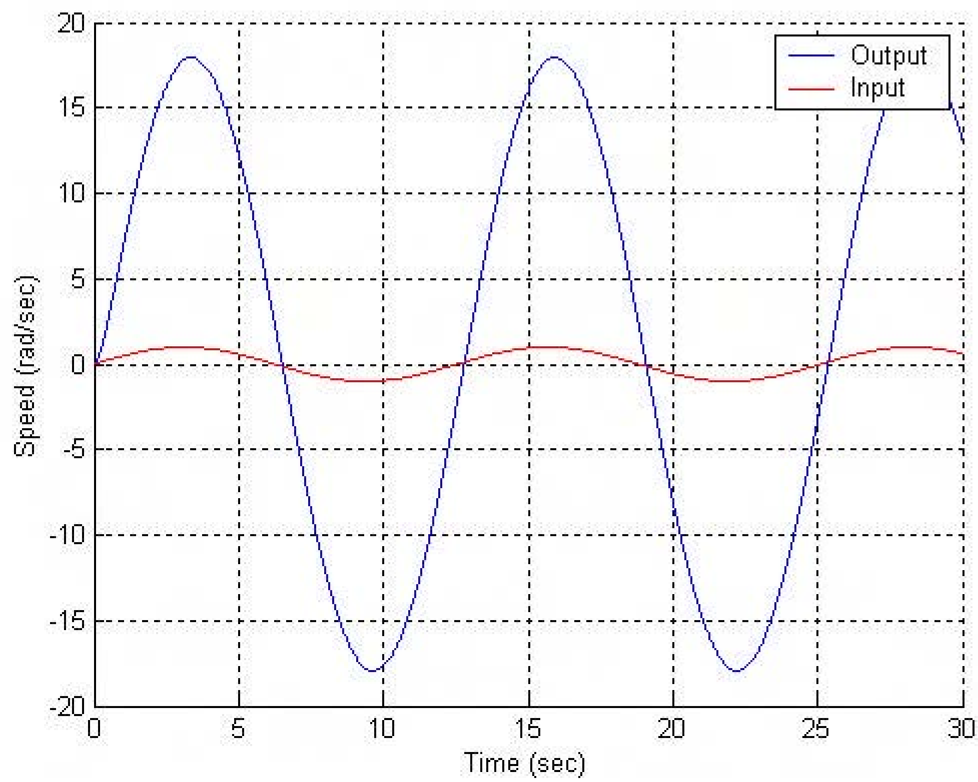
$$G(s) = \frac{9}{0.23s + 1},$$

where the time constant (0.23) is found at 5.68 rad/sec (63.2% of the final value).

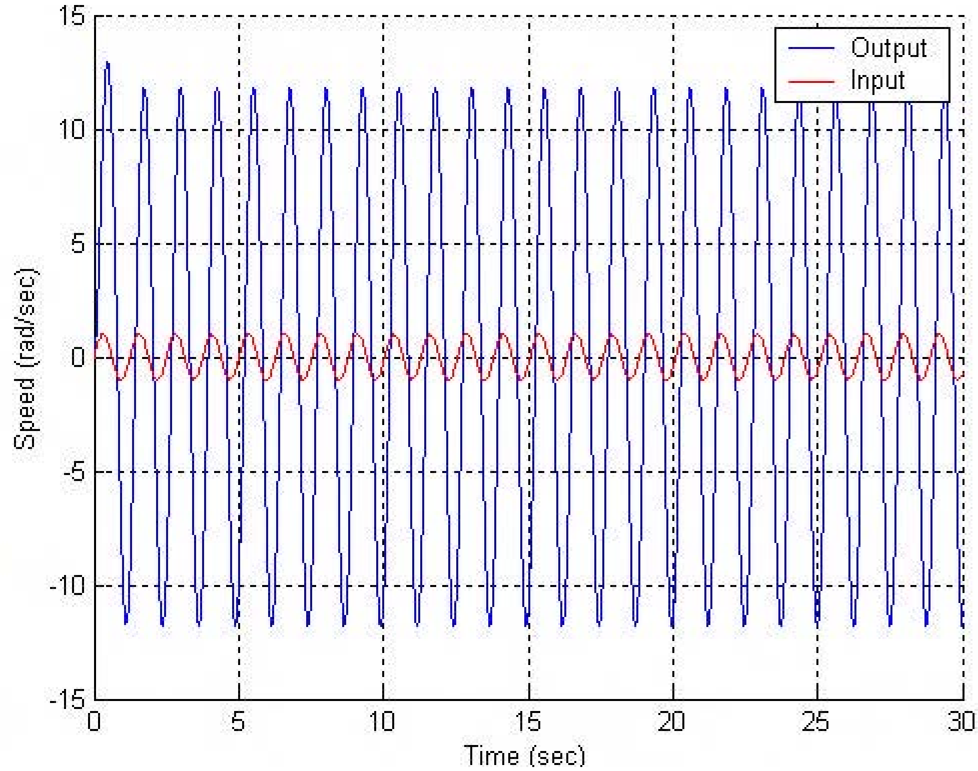
11-5-2 Open Loop Sine Input

9. Sine input to SIMLab and Virtual Lab (1 V. amplitude, and 0.5, 5, and 50 rad/sec frequencies)

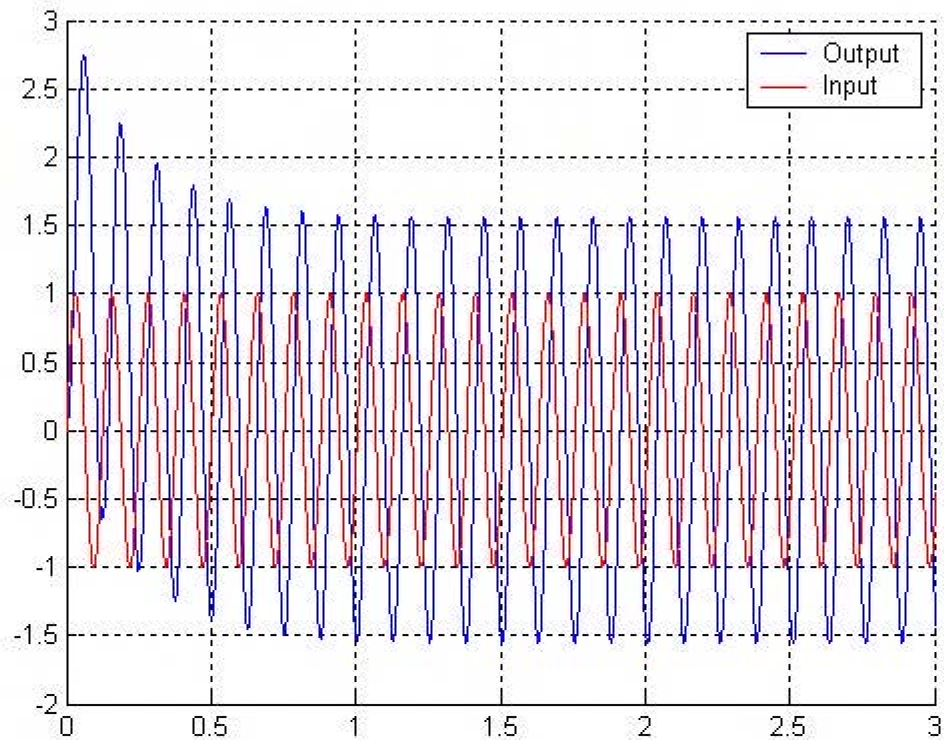
a. 0.5 rad/sec (SIMLab):



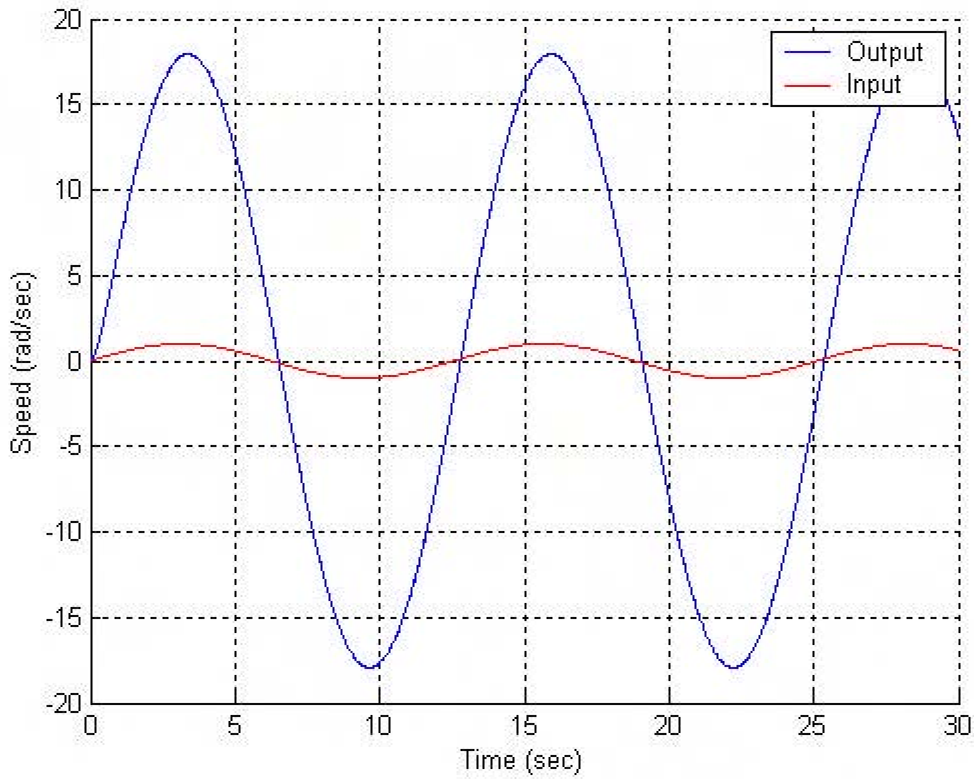
b. 5 rad/sec (SIMLab):



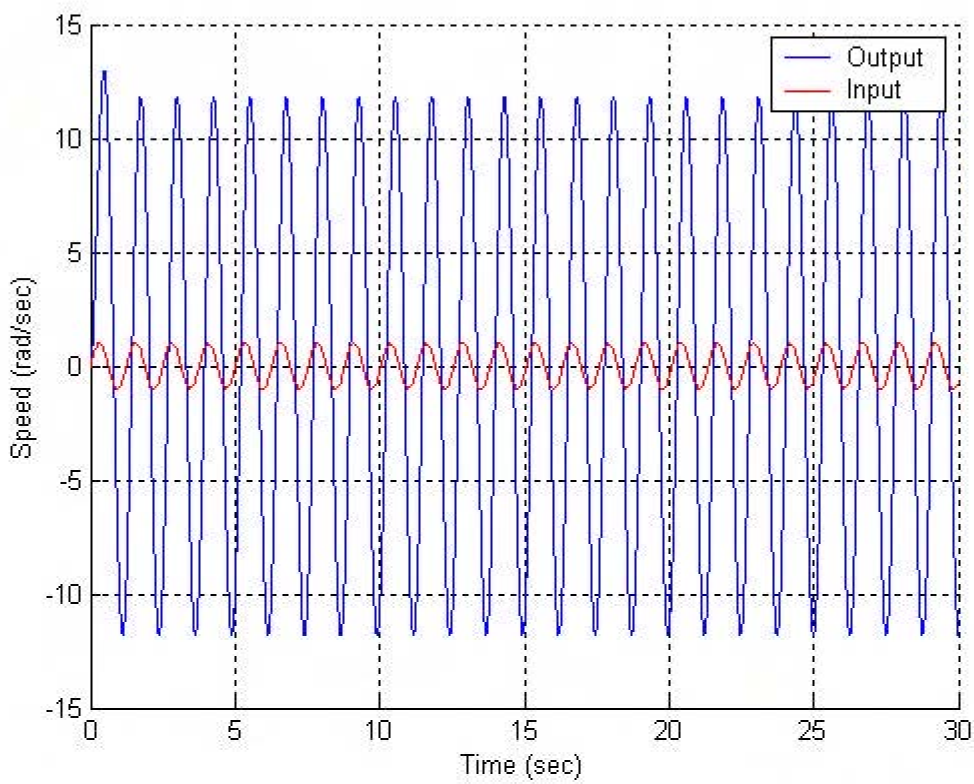
c. 50 rad/sec (SIMLab):



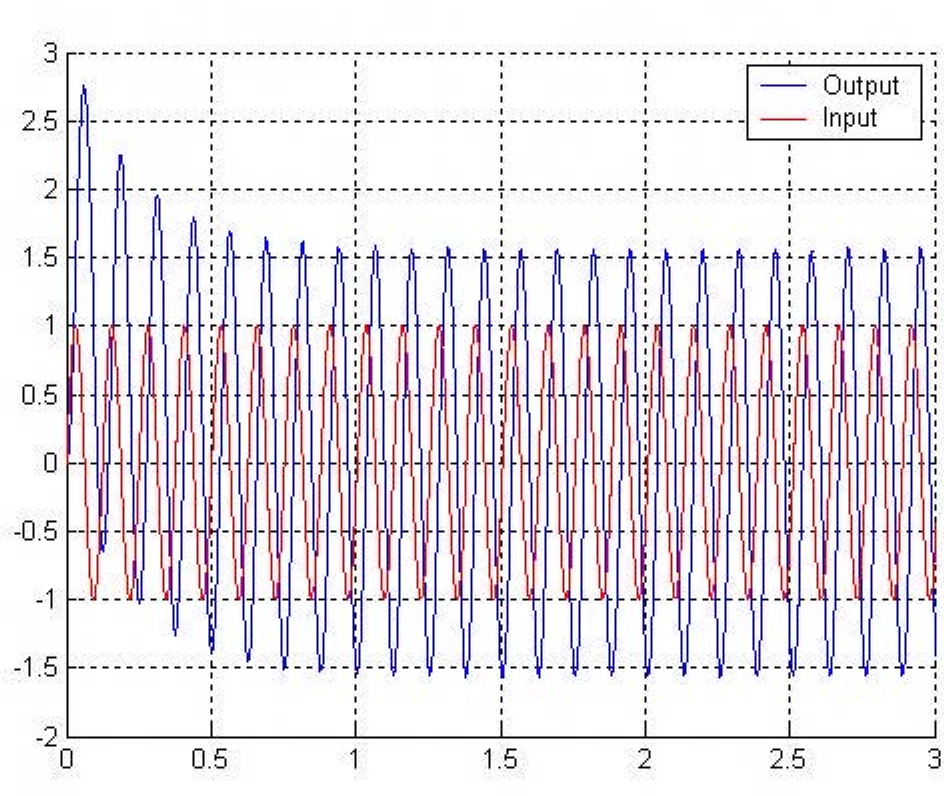
d. 0.5 rad/sec (Virtual Lab):



e. 5 rad/sec (Virtual Lab):

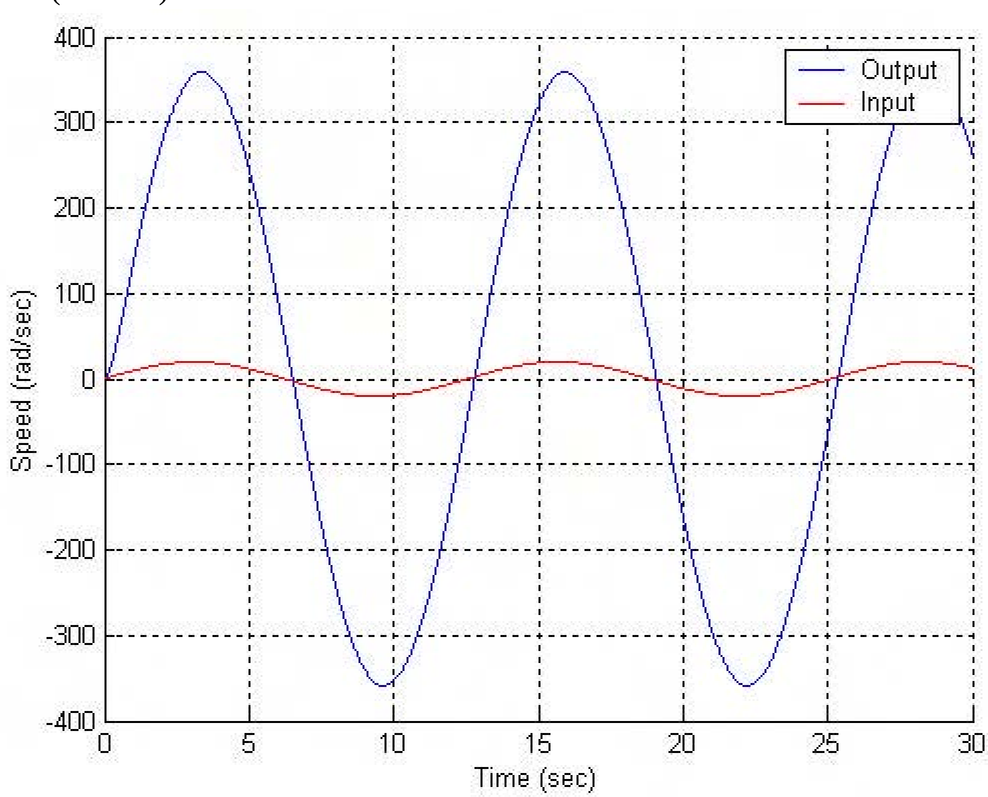


f. 50 rad/sec (Virtual Lab):

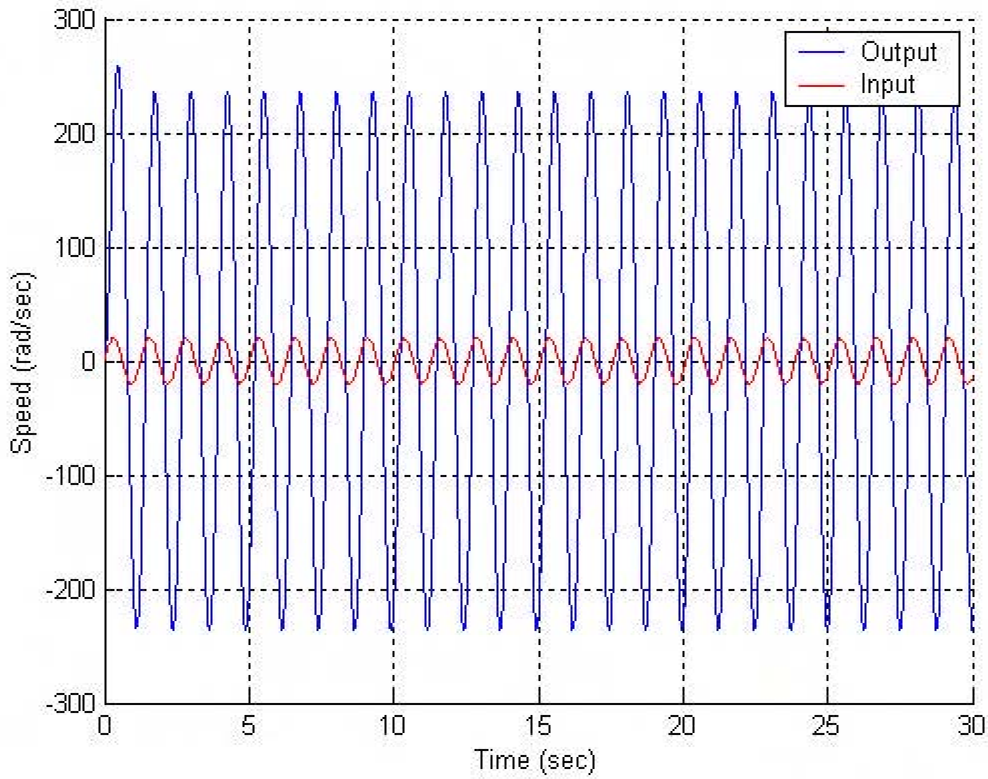


10. Sine input to SIMLab and Virtual Lab (20 V. amplitude)

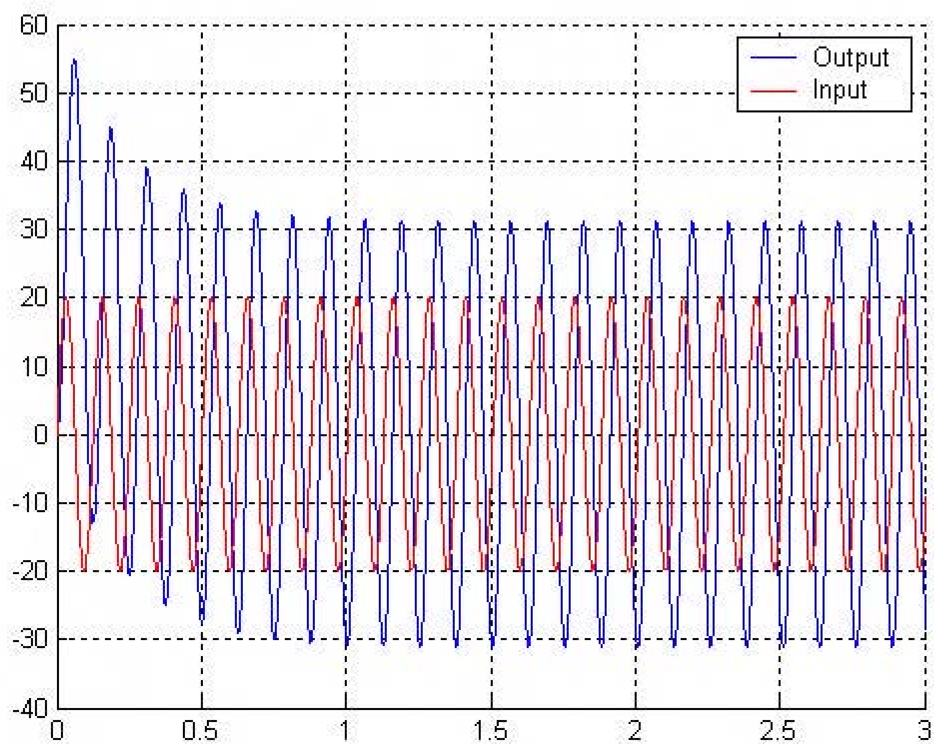
a. 0.5 rad/sec (SIMLab):



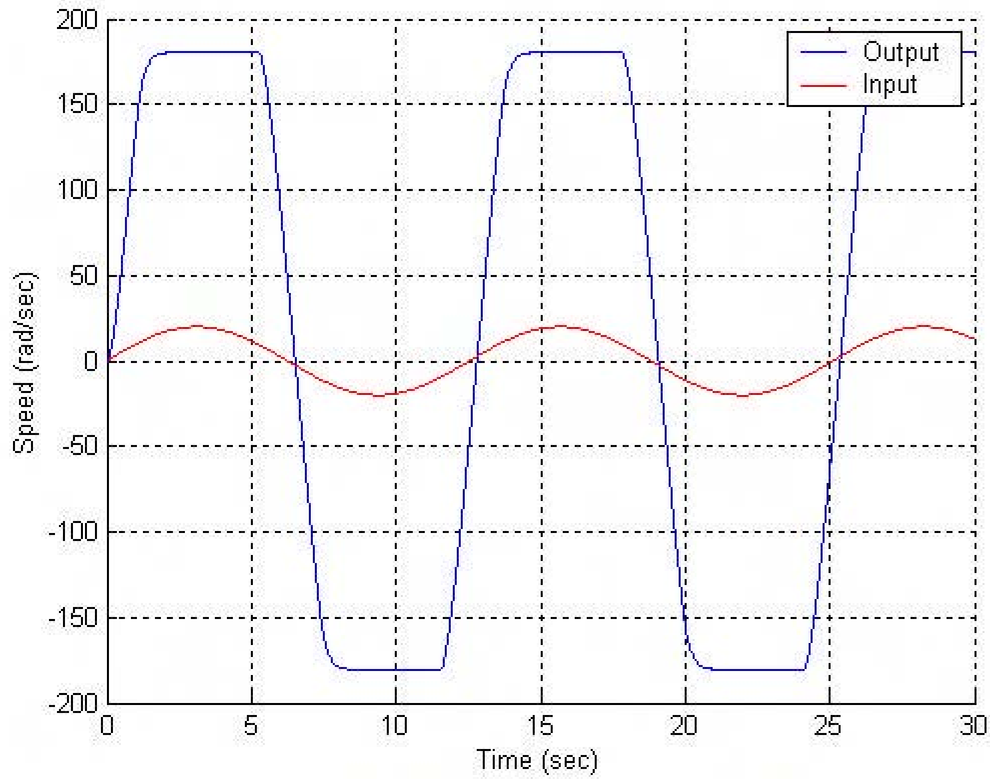
b. 5 rad/sec (SIMLab):



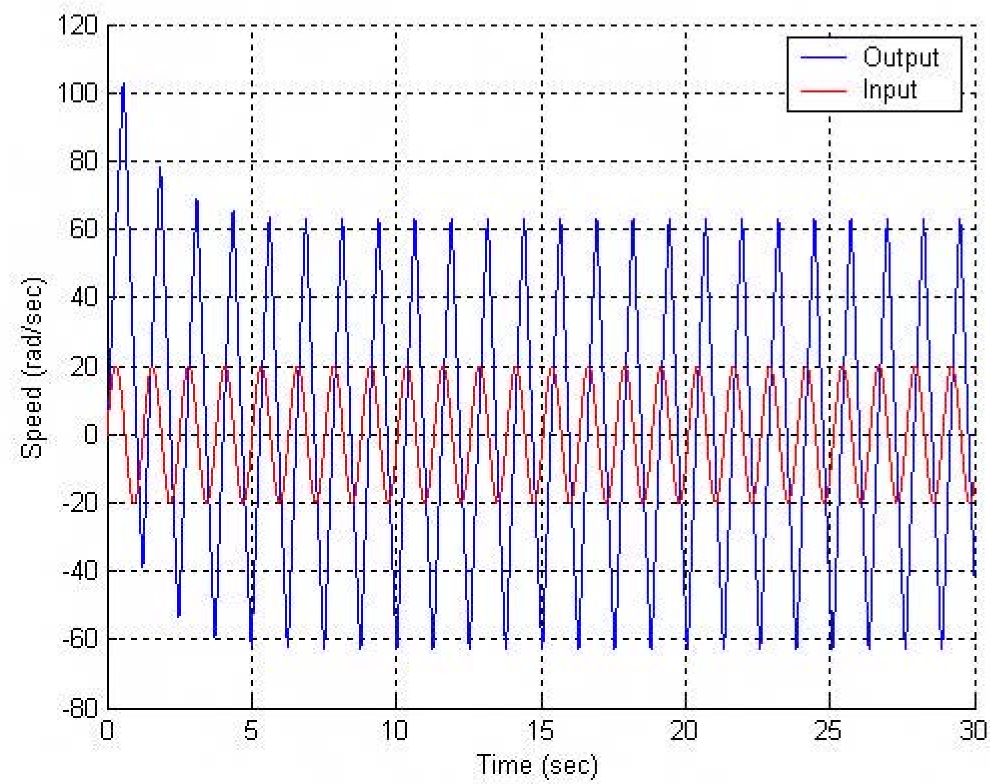
c. 50 rad/sec (SIMLab):



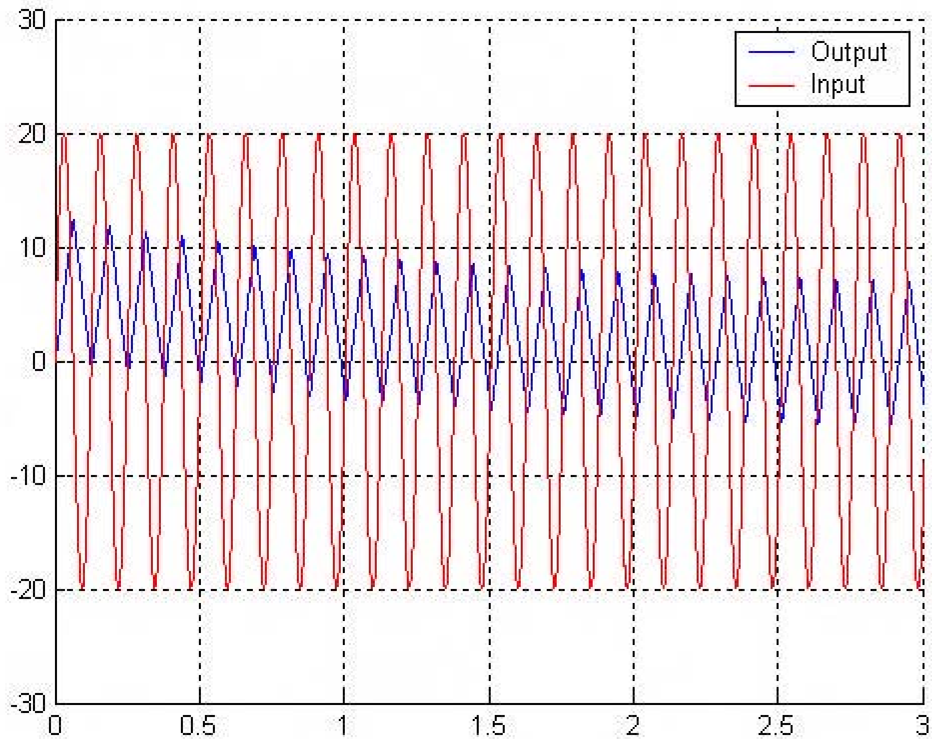
d. 0.5 rad/sec (Virtual Lab):



e. 5 rad/sec (Virtual Lab):



f. 5 rad/sec (Virtual Lab):



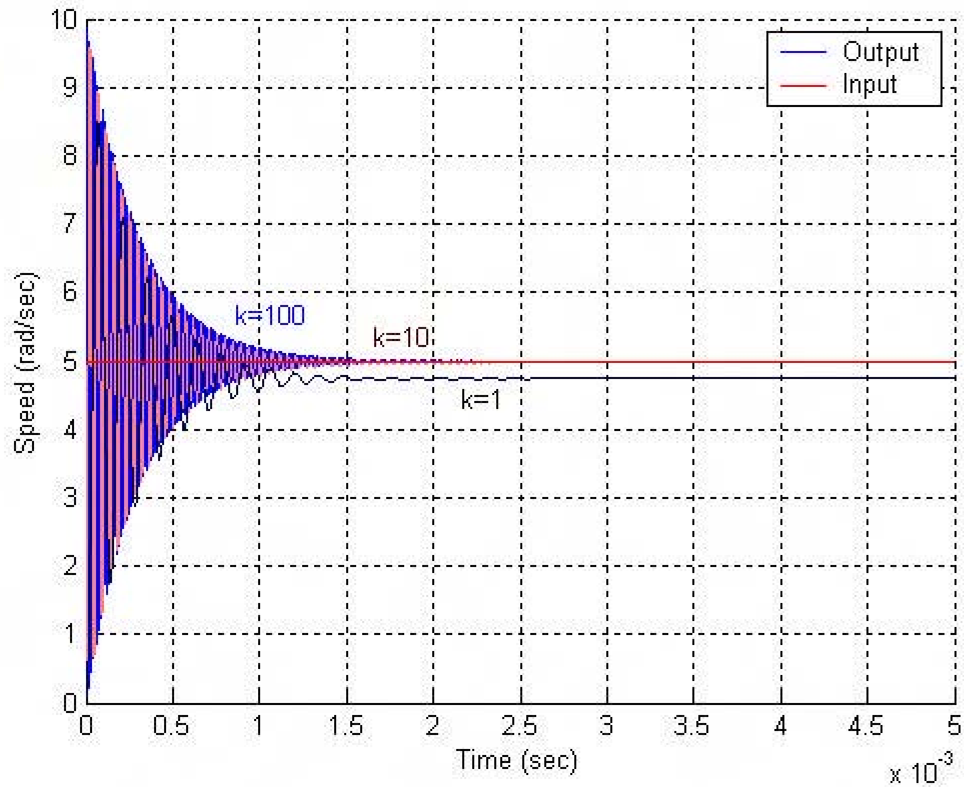
In both experiments 9 and 10, no saturation considered for voltage and current in SIMLab software. If we use the calculation of phase and magnitude in both SIMLab and Virtual Lab we will find that as input frequency increases the magnitude of the output decreases and phase lag increases. Because of existing saturations this phenomenon is more sever in the Virtual Lab experiment (10.f). In this experiments we observe that $M = 0.288$ and $j = -93.82^{\circ}$ for $w = 50$.

11-5-3 Speed Control

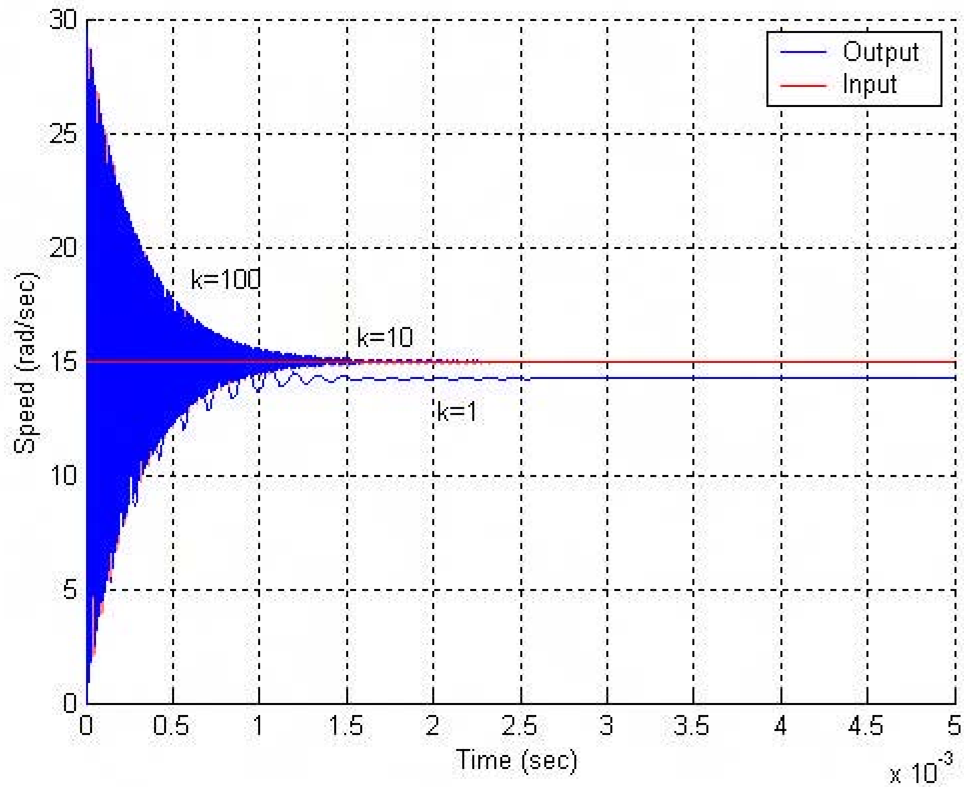
11. Apply step inputs (SIMLab)

In this section no saturation is considered either for current or for voltage.

a. +5 V:

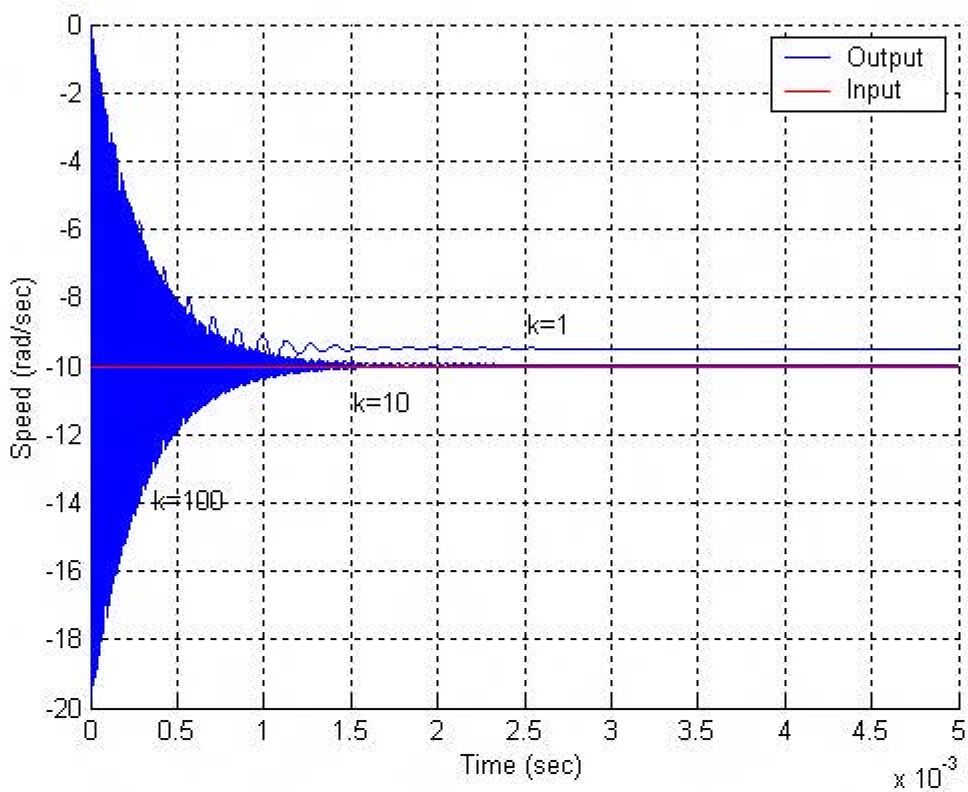


b. +15 V:



Output
Input
k=100
k=10
k=1
Speed (rad/sec)
Time (sec)
x 10^-3

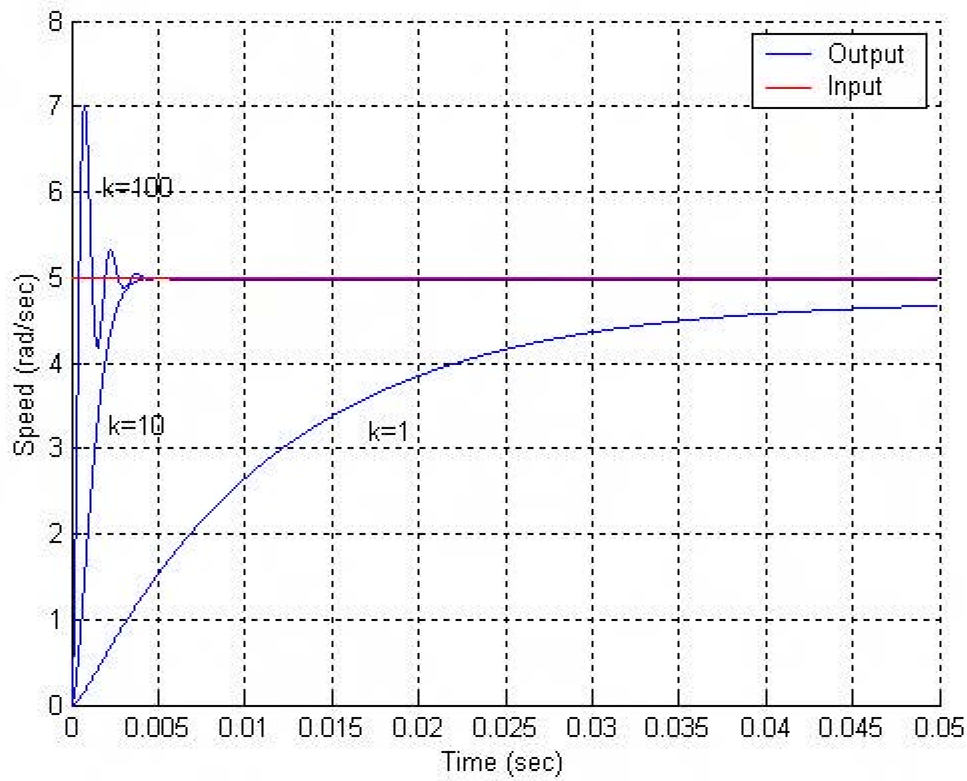
c. -10 V:



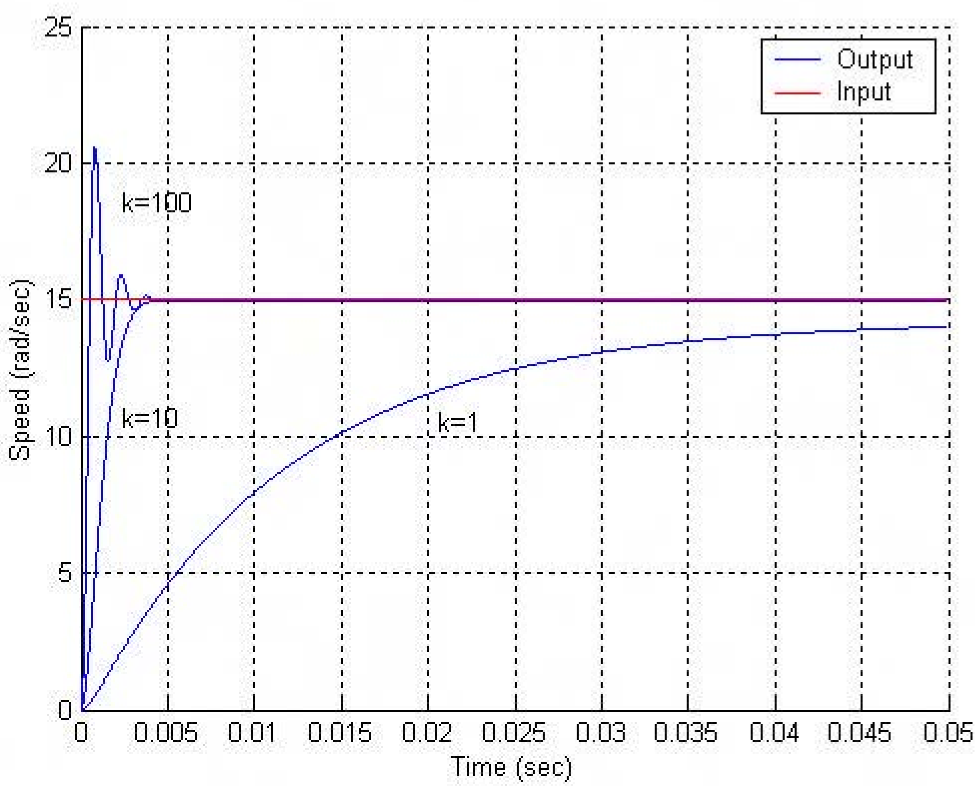
Output
Input
k=1
k=10
k=100
Speed (rad/sec)
Time (sec)
x 10^-3

12. Additional load inertia effect:

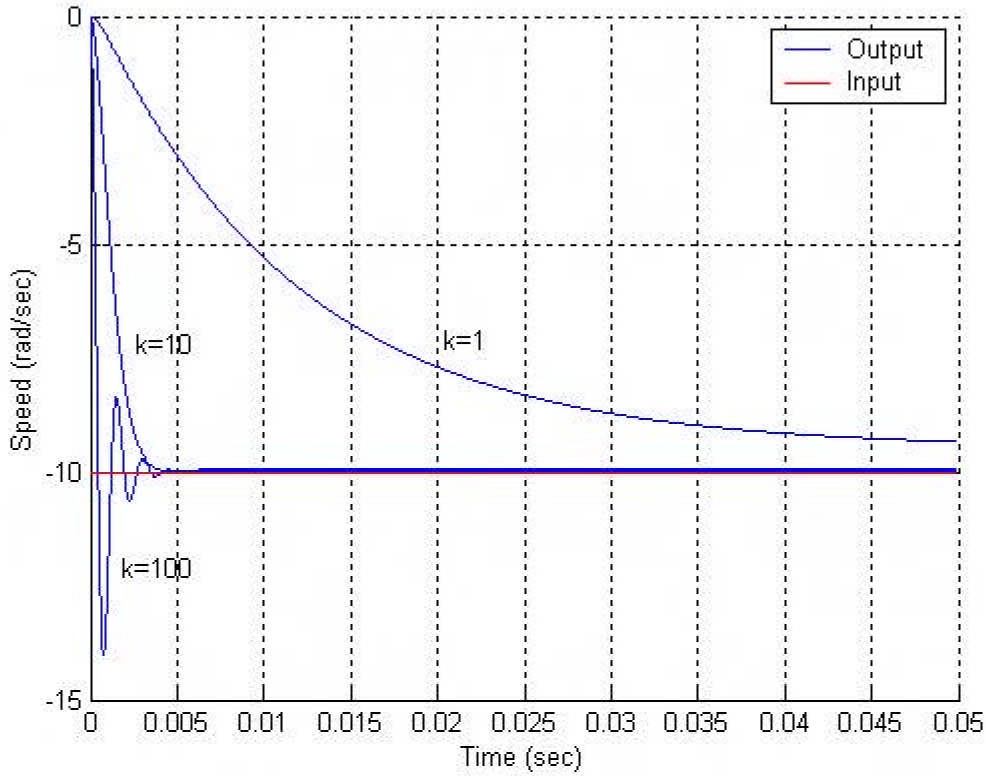
a. +5 V:



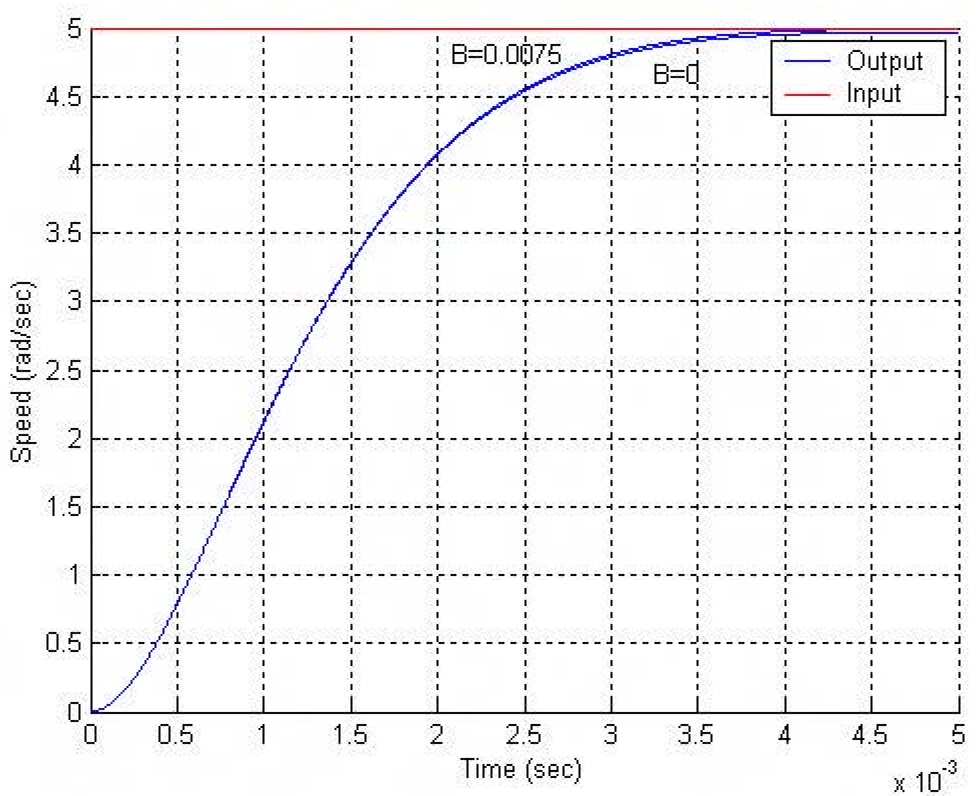
b. +15 V:



c. -10 V:

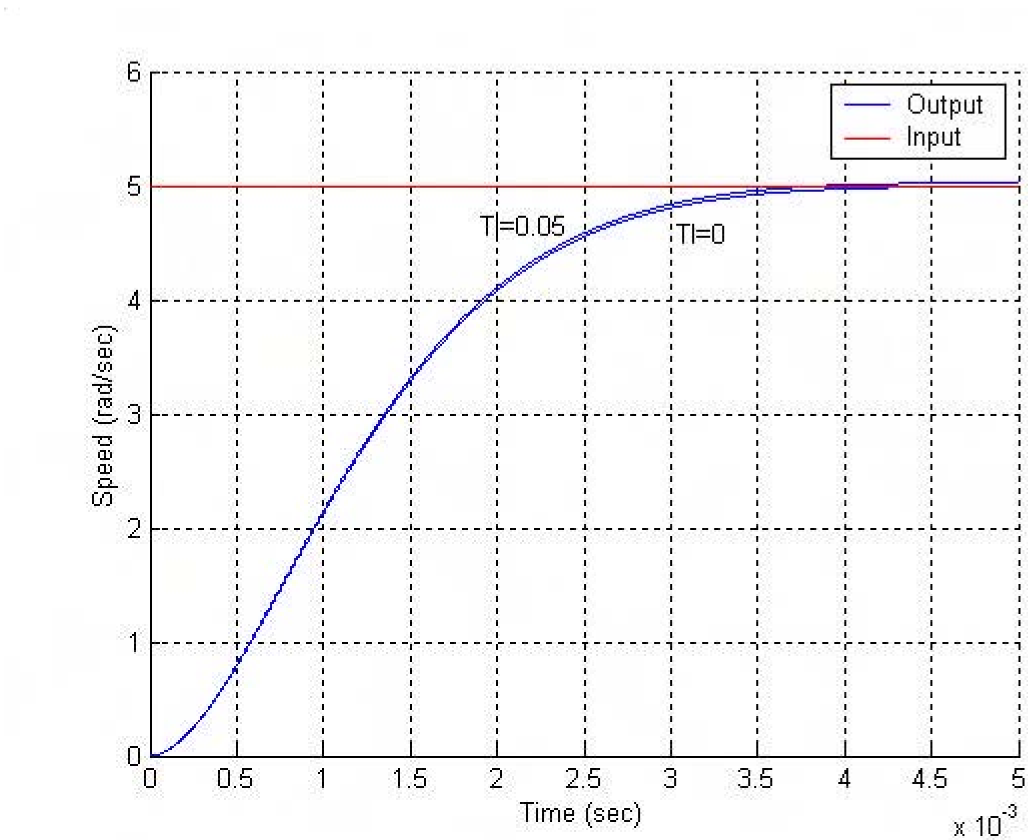


13. Study of the effect of viscous friction:

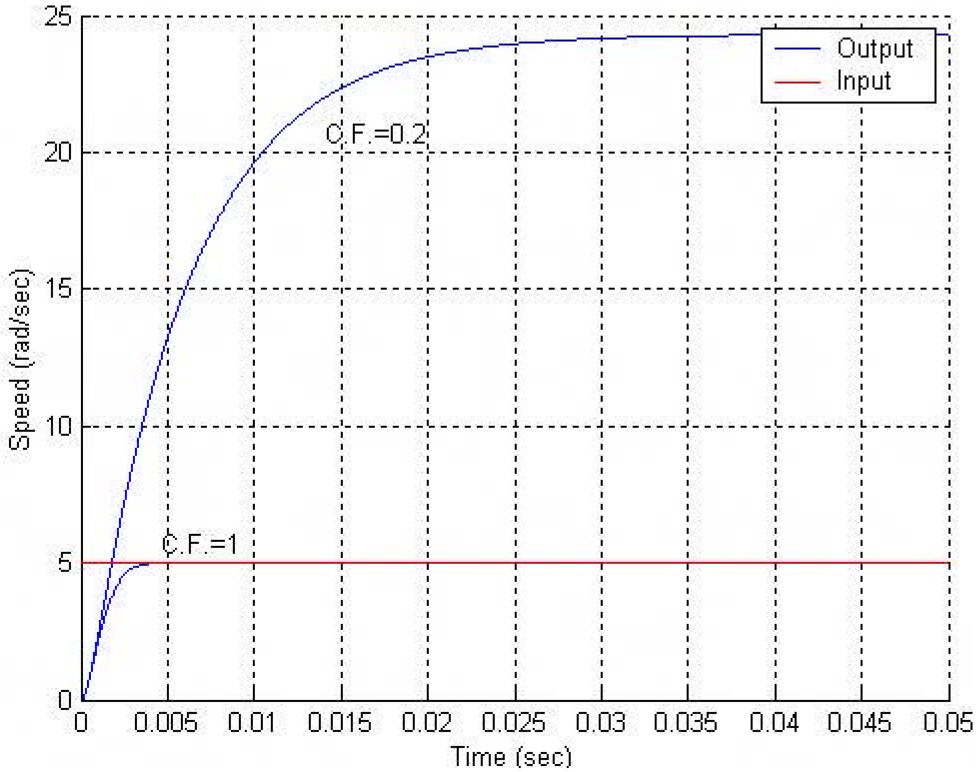


As seen in above figure, two different values for B are selected, zero and 0.0075. We could change the final speed by 50% in open loop system. The same values selected for closed loop speed control but as seen in the figure the final value of speeds stayed the same for both cases. It means that closed loop system is robust against changing in system's parameters. For this case, the gain of proportional controller and speed set point are 10 and 5 rad/sec, respectively.

14. Study of the effect of disturbance:

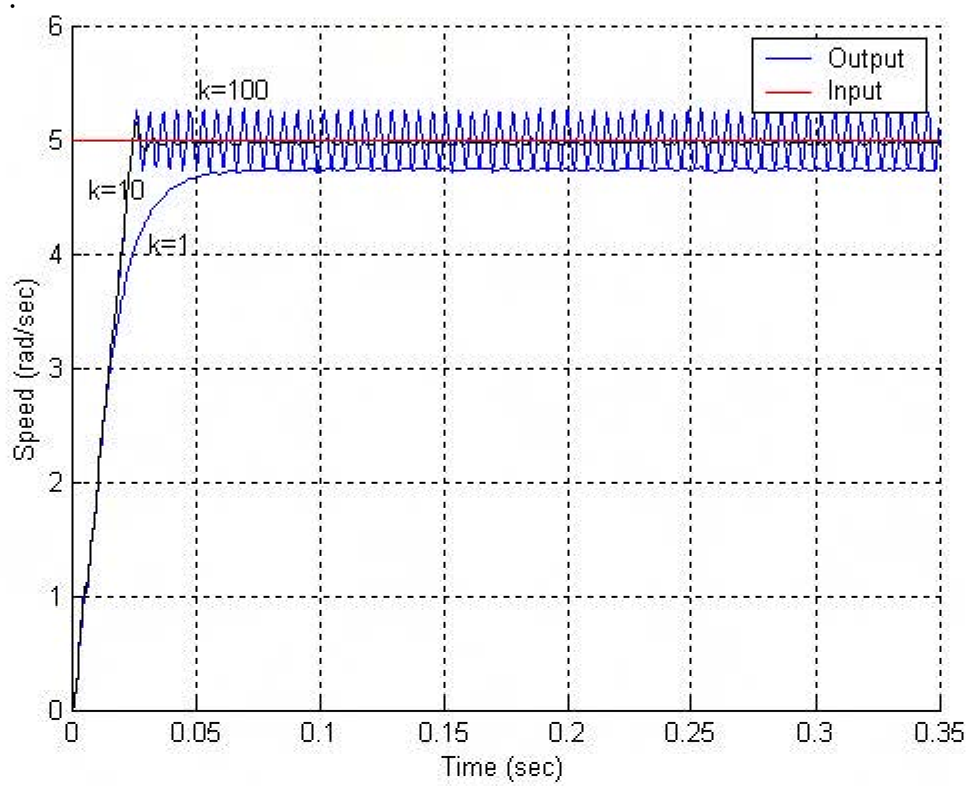


Repeating part 5 in section 11-5-1 for B=0.001 and T_L =0.05 N.m result in above figure. As seen, the effect of disturbance on the speed of closed loop system is not substantial like the one on the open loop system in part 5, and again it is shown the robustness of closed loop system against disturbance. Also, to study the effects of conversion factor see below figure, which is plotted for two different C.F. and the set point is 5 V.

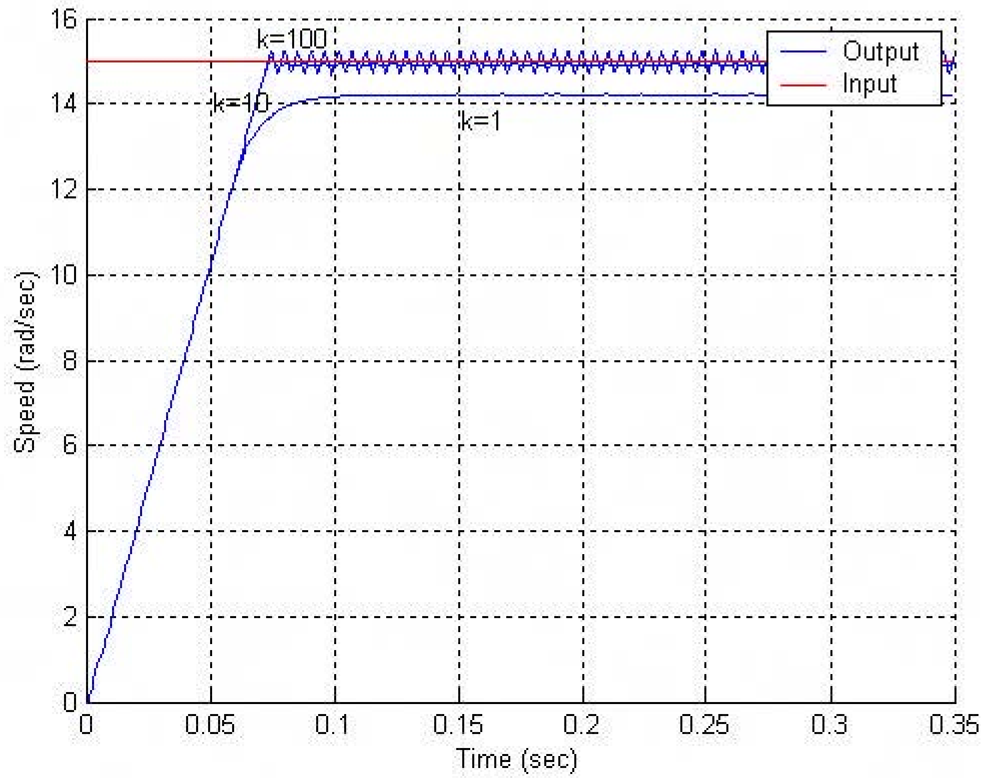


By decreasing the C.F. from 1 to 0.2, the final value of the speed increases by a factor of 5.

15. Apply step inputs (Virtual Lab)
   a. +5 V:

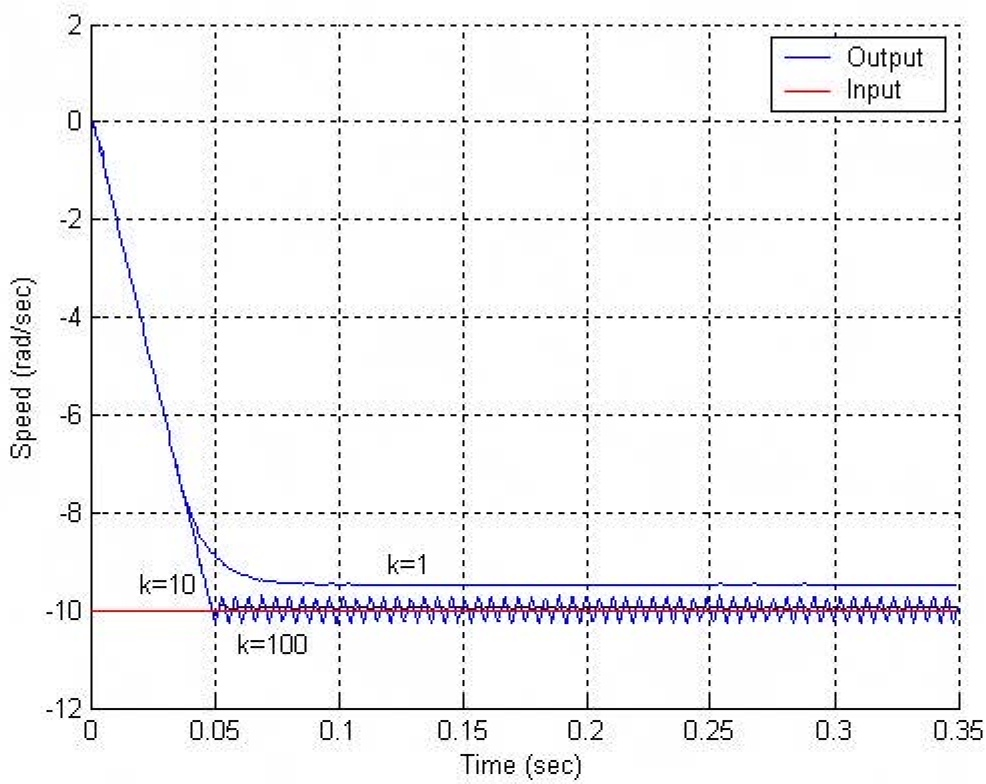


b. +15 V:



k=100
Output
Input
k=10
k=1
16
14
12
10
8
6
4
2
0
Speed (rad/sec)
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
Time (sec)

c. –10 V:

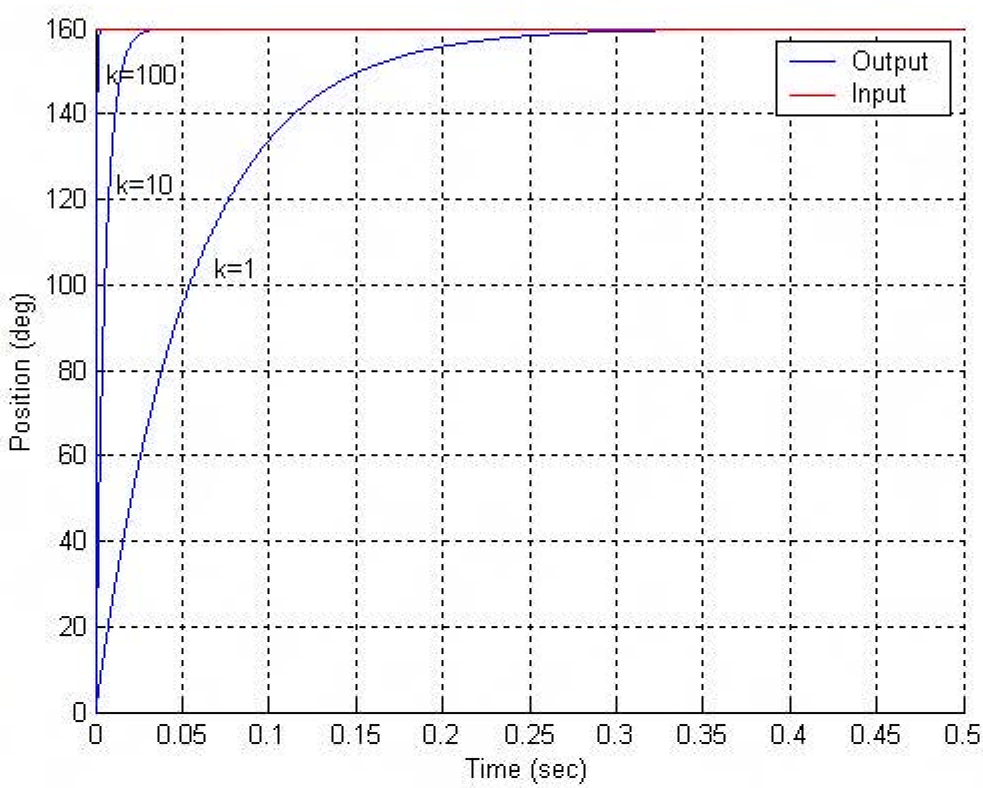


Output
Input
k=1
k=10
k=100
2
0
-2
-4
-6
-8
-10
-12
Speed (rad/sec)
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
Time (sec)

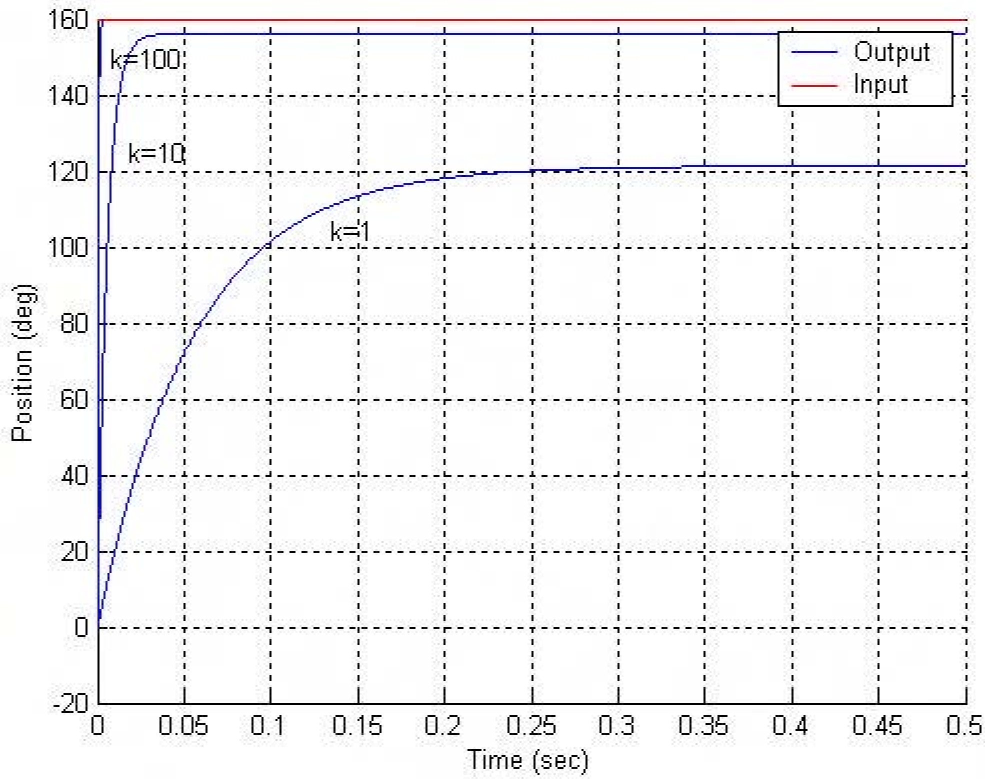
As seen the responses of Virtual Lab software, they are clearly different from the same results of SIMLab software. The nonlinearities such as friction and saturation cause these differences. For example, the chattering phenomenon and flatness of the response at the beginning can be considered as some results of nonlinear elements in Virtual Lab software.

11-5-4 Position Control

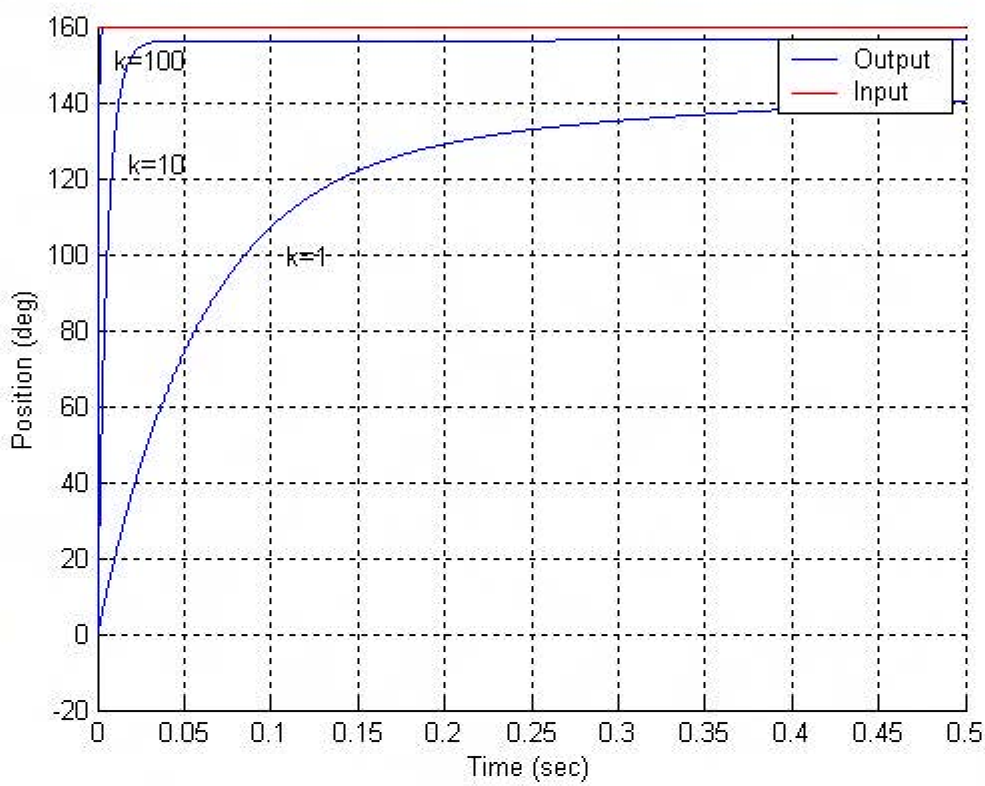
16. 160° step input (SIMLab)



17. –0.1 N.m step disturbance

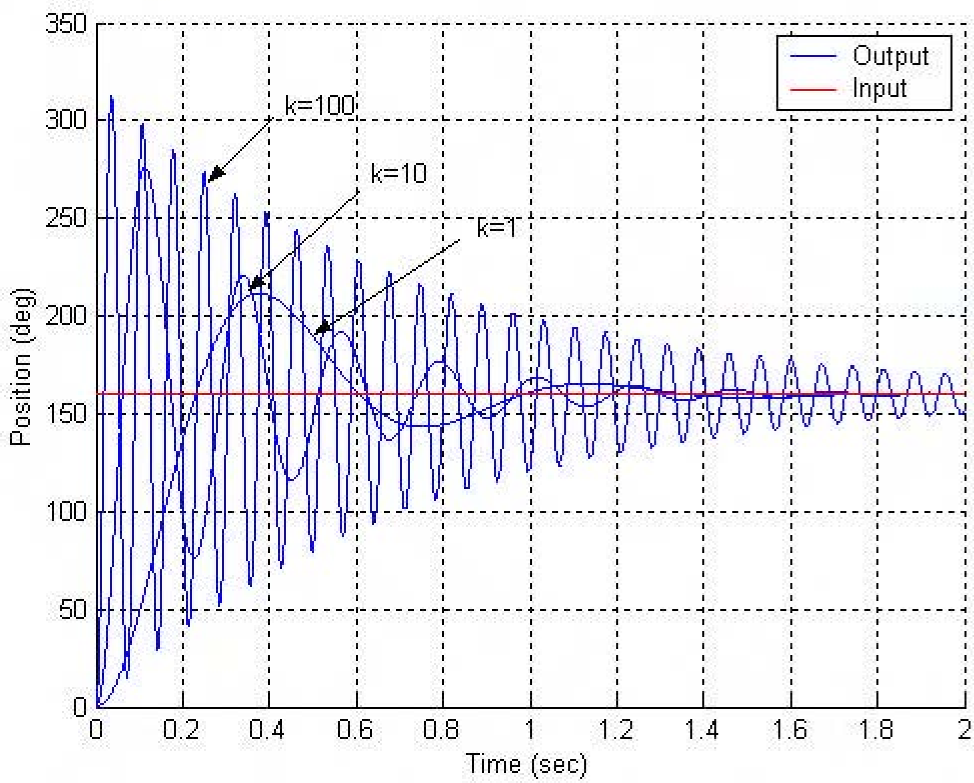


18. Examine the effect of integral control

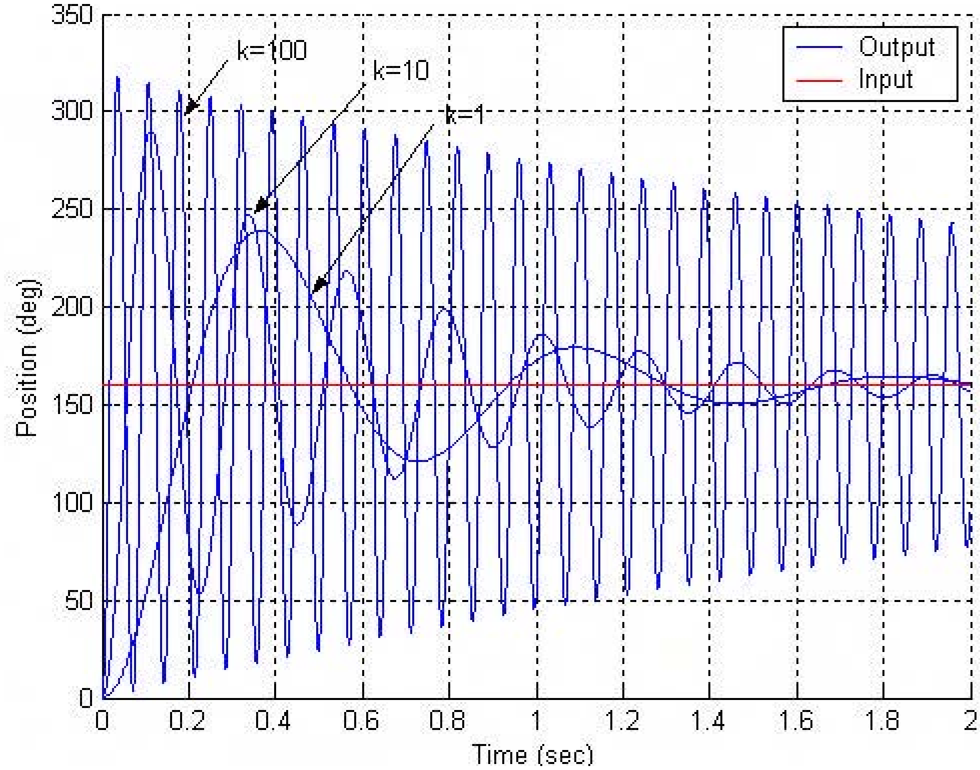


In above figure, an integral gain of 1 is considered for all curves. Comparing this plot with the previous one without integral gain, results in less steady state error for the case of controller with integral part.

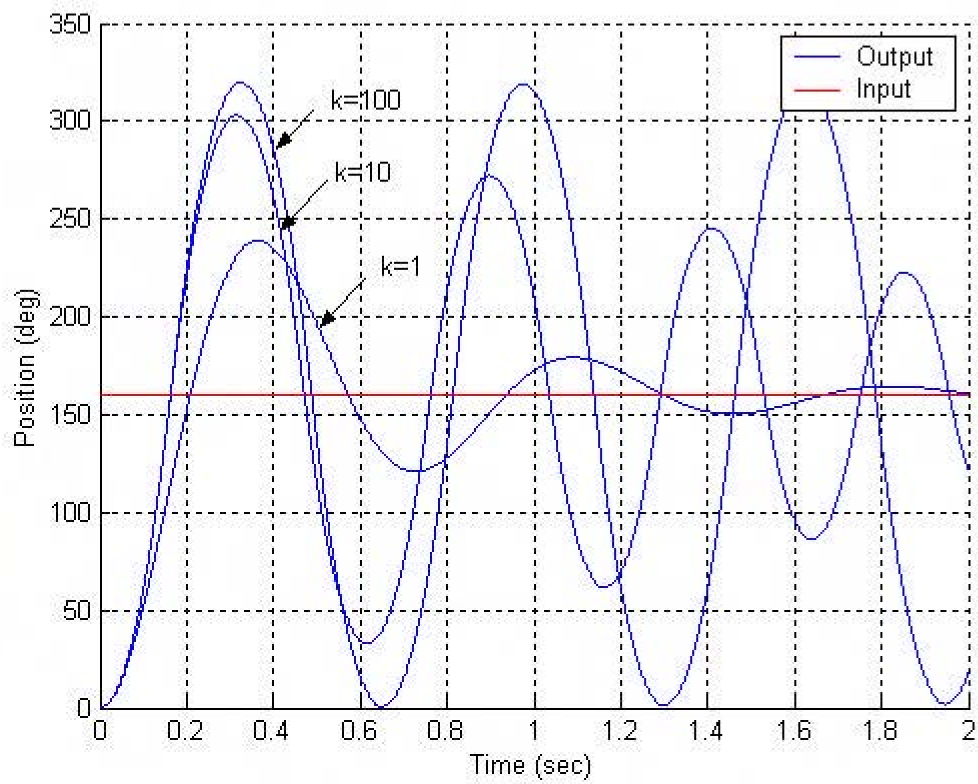
19. Additional load inertia effect (J=0.0019, B=0.004):



20. Set B=0:



21. Study the effect of saturation

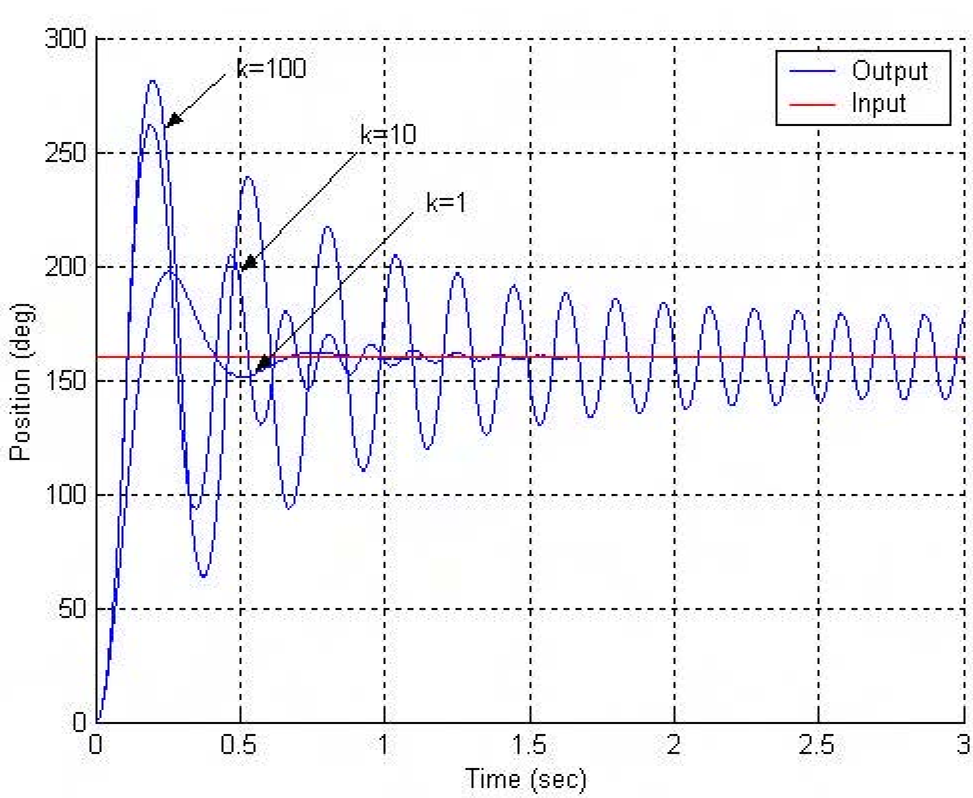


The above figure is obtained in the same conditions of part 20 but in this case we considered ± 10 V. and ± 4 A. as the saturation values for voltage and current, respectively. As seen in the figure, for higher proportional gains the effect of saturations appears by reducing the frequency and damping property of the system.

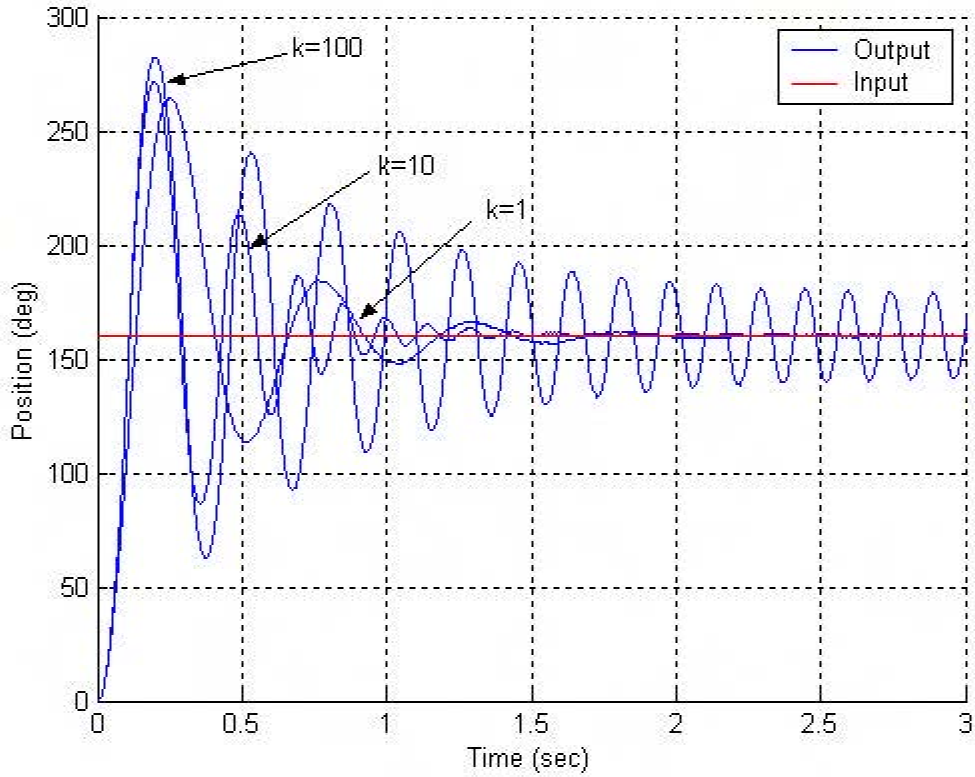
22. Comments on Eq. 11-13

After neglecting of electrical time constant, the second order closed loop transfer function of position control obtained in Eq. 11-13. In experiments 19 through 21 we observe an under damp response of a second order system. According to the equation, as the proportional gain increases, the damped frequency must be increased and this fact is verified in experiments 19 through 21. Experiments16 through 18 exhibits an over damped second order system responses.

23. In following, we repeat parts 16 and 18 using Virtual Lab:



Study the effect of integral gain of 5:

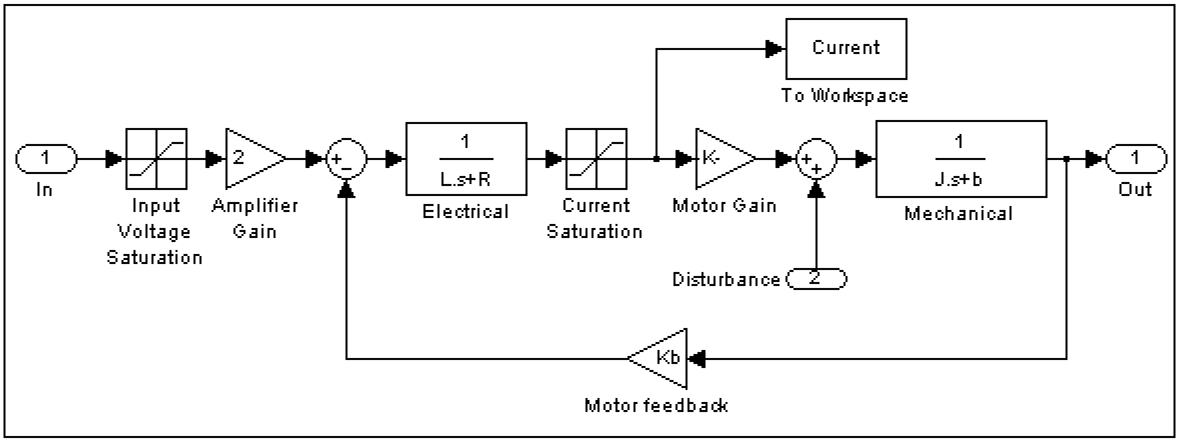


Output
Input
k=100
k=10
k=1
Position (deg)
Time (sec)
300
250
200
150
100
50
0
0
0.5
1
1.5
2
2.5
3

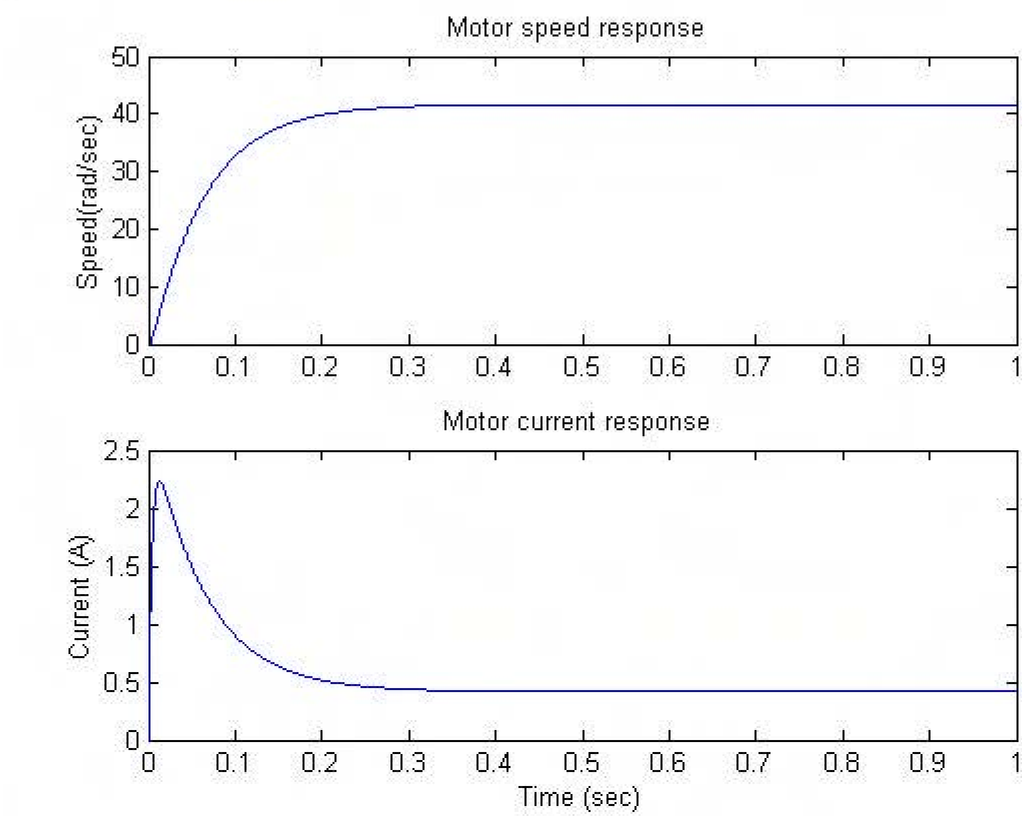
<u>**Ch. 11 Problem Solutions**</u>

Part 2) Solution to Problems in Chapter 11

11-1. In order to find the current of the motor, the motor constant has to be separated from the electrical component of the motor.

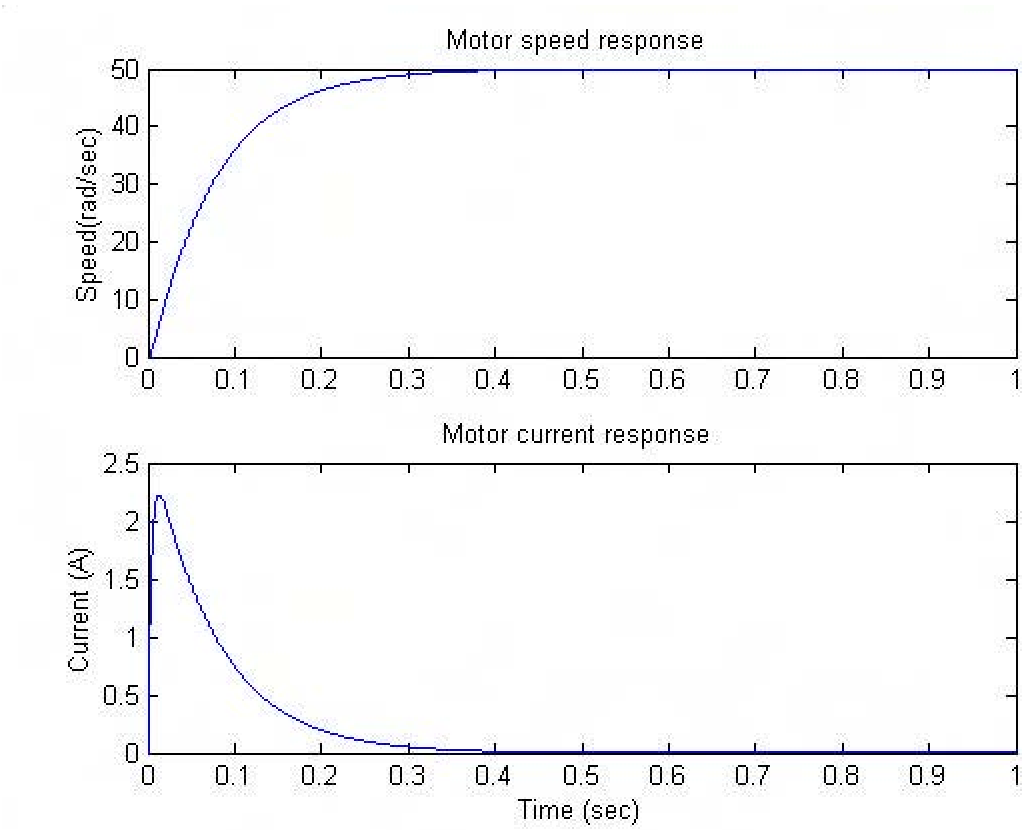


The response of the motor when 5V of step input is applied is:



a) The steady state speed: 41.67rad/sec
b) It takes 0.0678 second to reach 63% of the steady state speed (26.25rad/sec). This is the time constant of the motor.
c) The maximum current: 2.228A

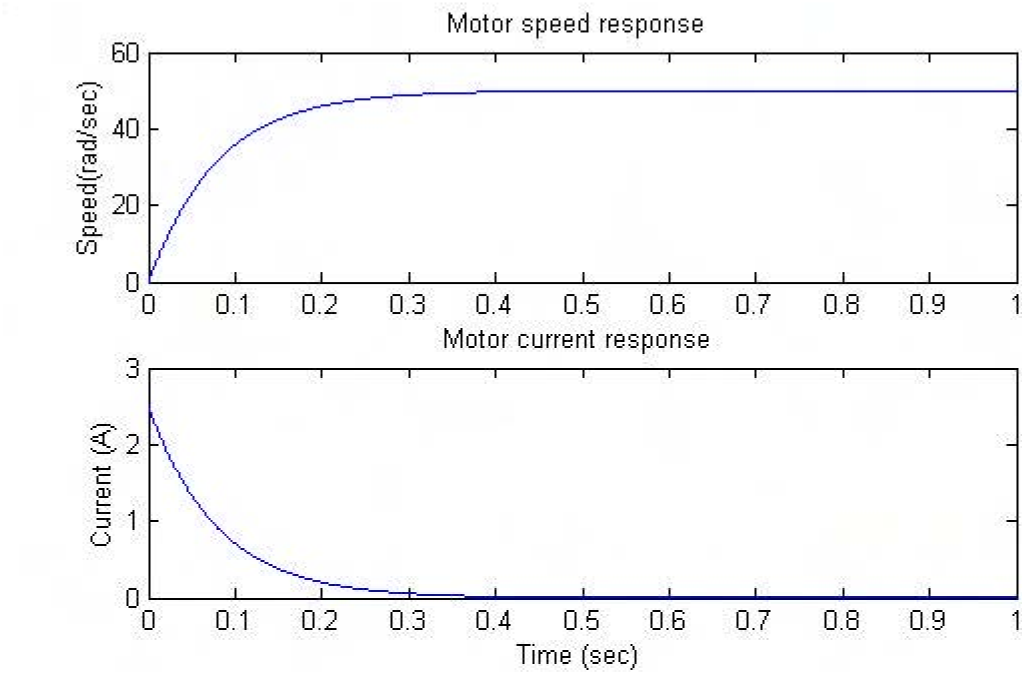
11.2



The steady state speed at 5V step input is 50rad/sec.

a) It takes 0.0797 seconds to reach 63% of the steady state speed (31.5rad/sec).
b) The maximum current: 2.226A
c) 100rad/sec

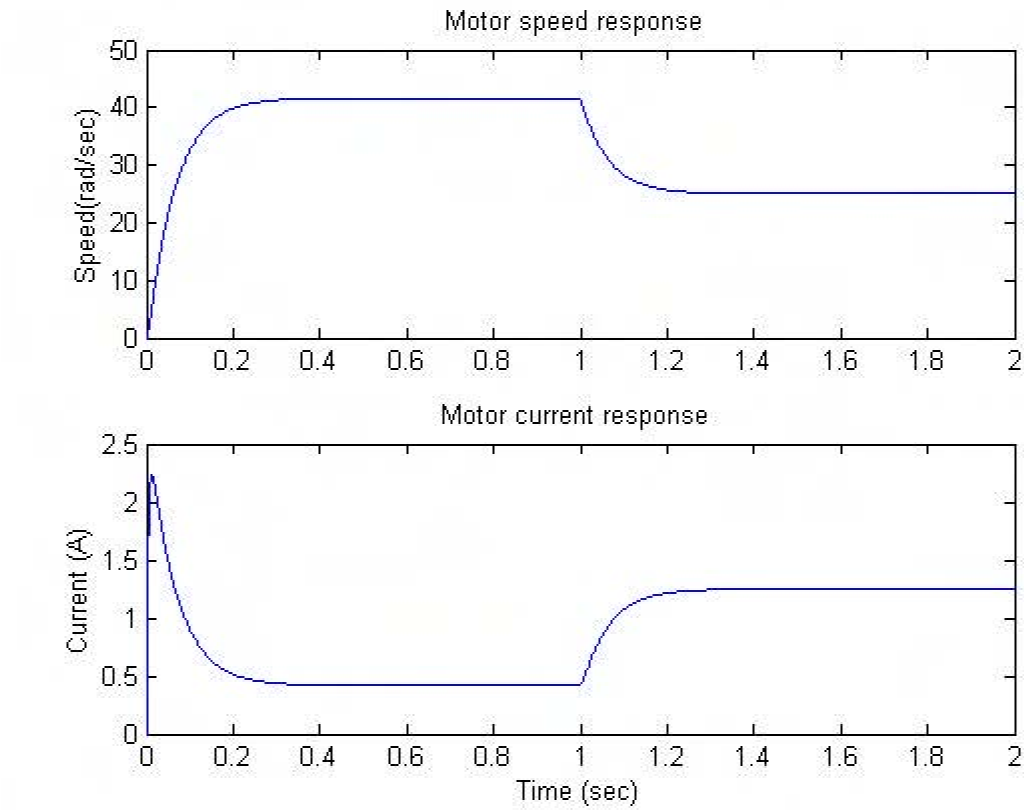
11-3



a) 50rad/sec
b) 0.0795 seconds
c) 2.5A. The current
d) When J_m is increased by a factor of 2, it takes 0.159 seconds to reach 63% of its steady state speed, which is exactly twice the original time period . This means that the time constant has been doubled.

11-4

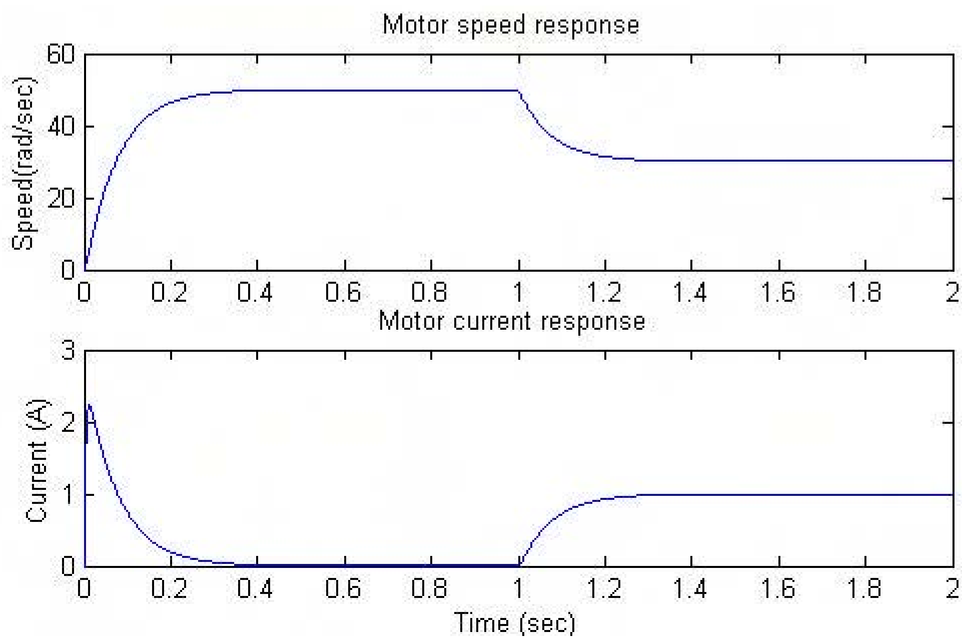
Part 1: Repeat problem 11-1 with T_L = -0.1Nm



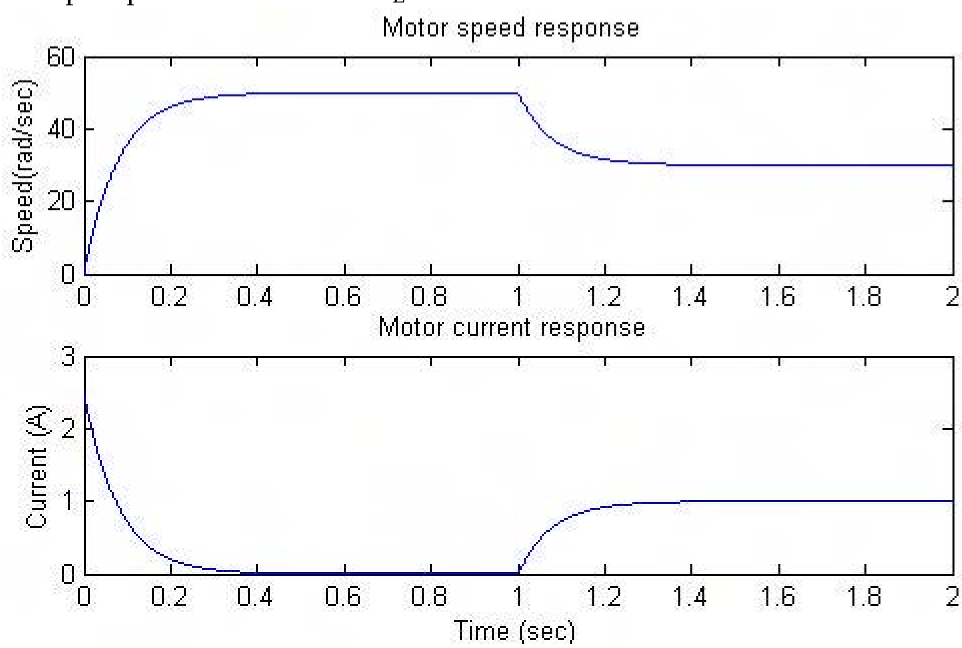
a) It changes from 41.67 rad/sec to 25 rad/sec.
b) First, the speed of 63% of the steady state has to be calculated.
$41.67 - (41.67 - 25) \times 0.63 = 31.17$ rad/sec.
The motor achieves this speed 0.0629 seconds after the load torque is applied
c) 2.228A. It does not change

Part 2: Repeat problem 11-2 with T_L = -0.1Nm

a) It changes from 50 rad/sec to 30 rad/sec.
b) The speed of 63% of the steady state becomes
$50 - (50 - 30) \times 0.63 = 37.4$ rad/sec.
The motor achieves this speed 0.0756 seconds after the load torque is applied
c) 2.226A. It does not change.



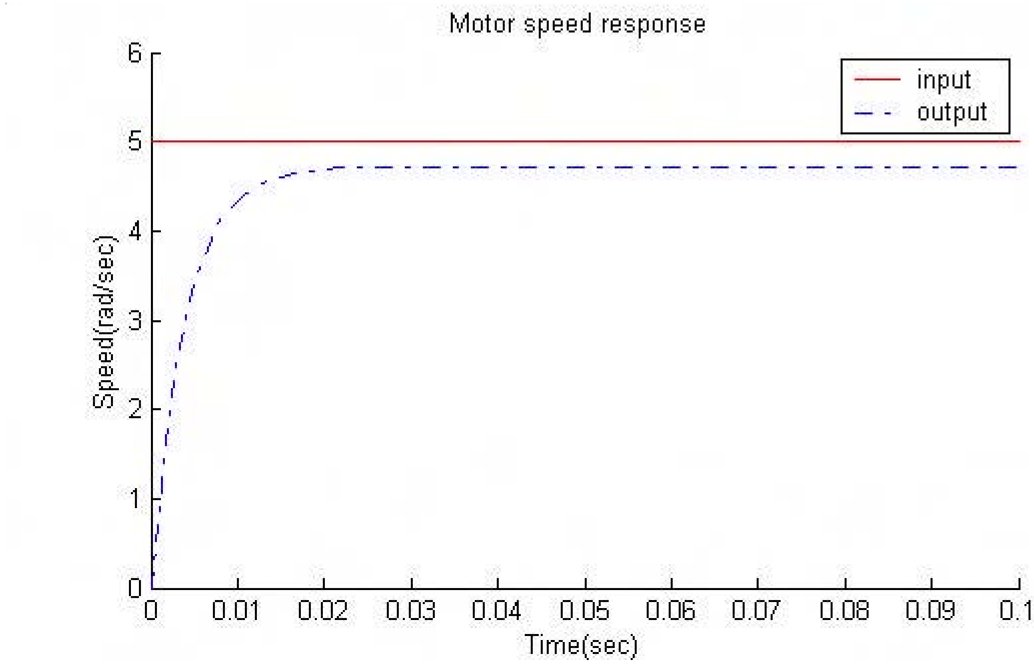
Part 3: Repeat problem 11-3 with T_L = -0.1Nm



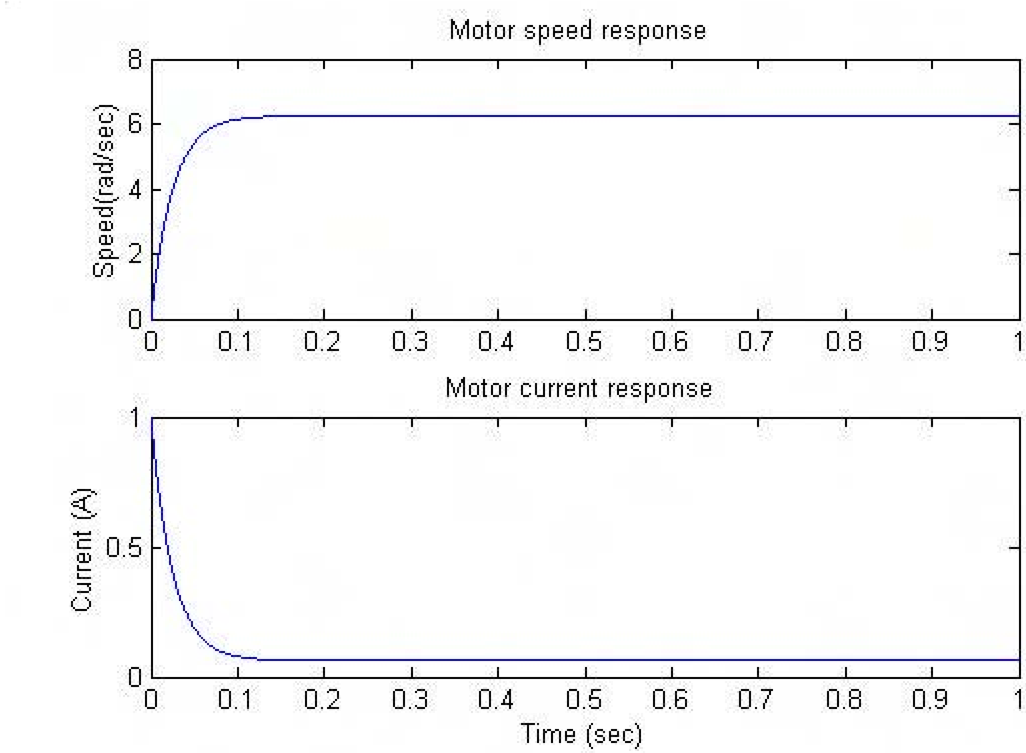
a) It changes from 50 rad/sec to 30 rad/sec.
b) $50 - (50 - 30) \times 0.63 = 37.4$ rad/s

   The motor achieves this speed 0.0795 seconds after the load torque is applied. This is the same as problem 11-3.
c) 2.5A. It does not change
d) As T_L increases in magnitude, the steady state velocity decreases and steady state current increases; however, the time constant does not change in all three cases.

11-5 The steady state speed is 4.716 rad/sec when the amplifier input voltage is 5V:



11-6

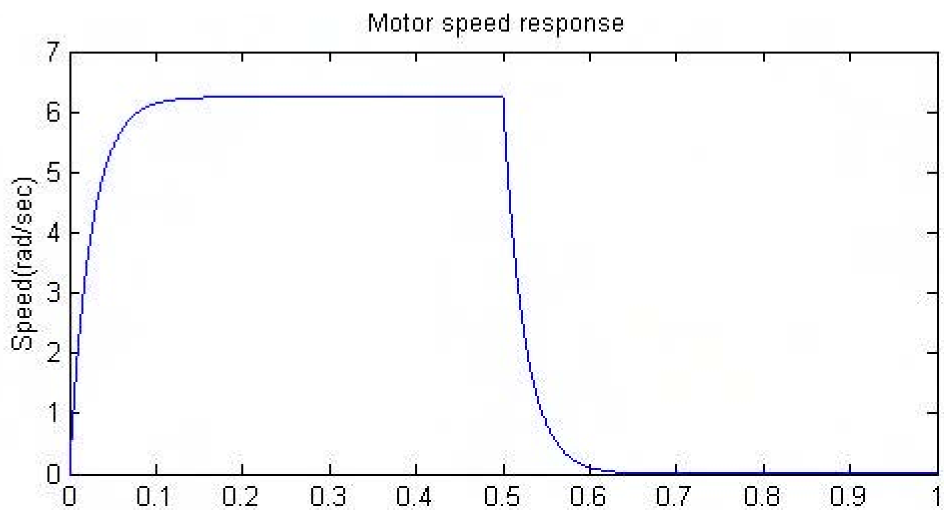


a) 6.25 rad/sec.
b) 63% of the steady state speed: $6.25 \times 0.63 = 3.938$ rad/sec
   It takes 0.0249 seconds to reach 63% of its steady state speed.
c) The maximum current drawn by the motor is 1 Ampere.

11-7

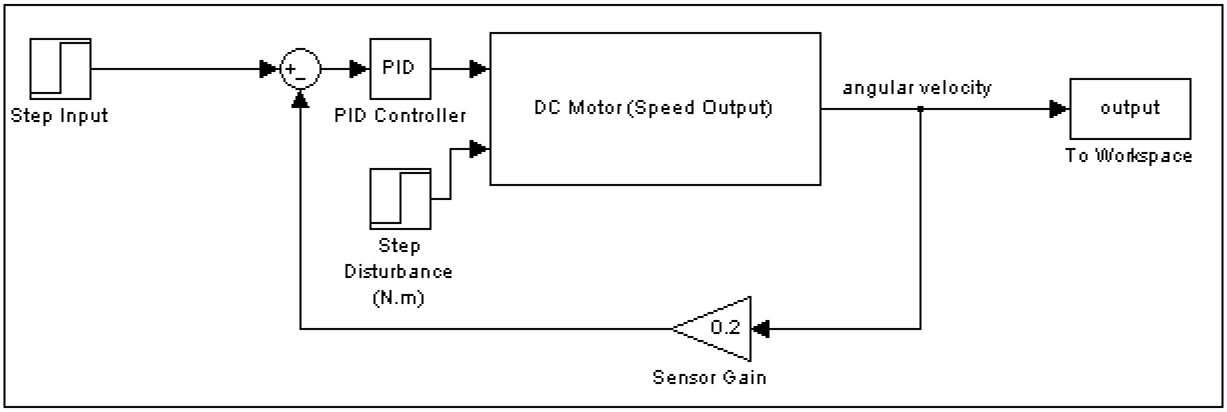
a) 9.434 rad/sec.
b) 63% of the steady state speed: $9.434 \times 0.63 = 5.943$ rad/sec
   It takes 0.00375 seconds to reach 63% of its steady state speed.
c) The maximum current drawn by the motor is 10 Amperes.
d) When there is no saturation, higher K_p value reduces the steady state error and decreases the rise time. If there is saturation, the rise time does not decrease as much as it without saturation. Also, if there is saturation and K_p value is too high, chattering phenomenon may appear.

11-8

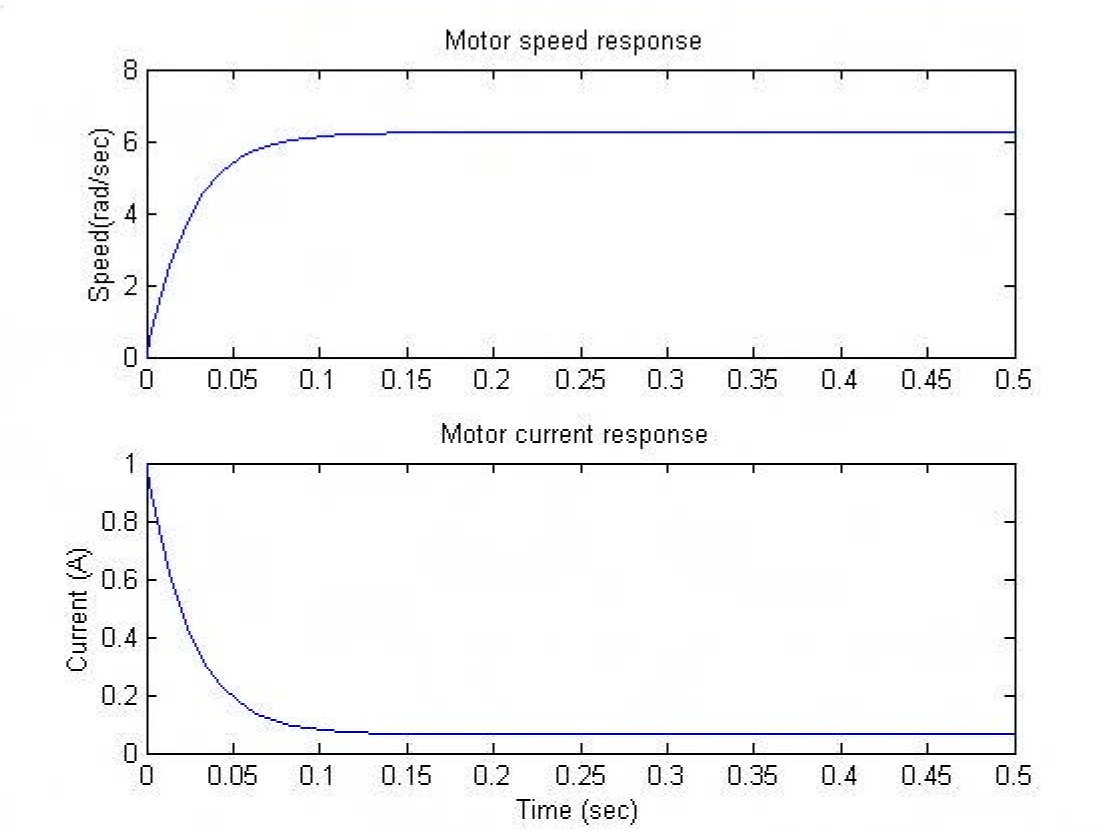


a) The steady state becomes zero. The torque generated by the motor is 0.1 Nm.
b) $6.25 - (6.25 - 0) \times 0.63 = 2.31$ rad/sec. It takes 0.0249 seconds to reach 63% of its new steady state speed. It is the same time period to reach 63% of its steady state speed without the load torque (compare with the answer for the Problem 11-6 b).

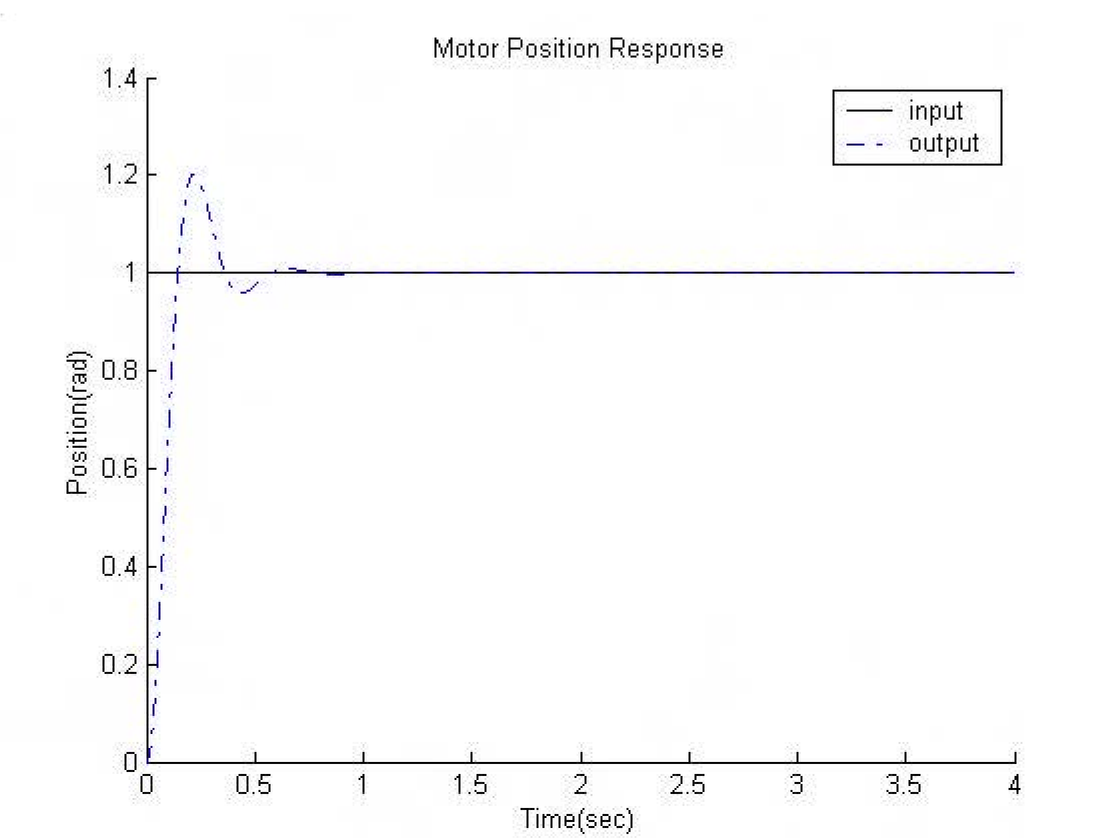
11-9 The SIMLab model becomes



The sensor gain and the speed input are reduced by a factor of 5. In order to get the same result as Problem 11-6, the K_p value has to increase by a factor of 5. Therefore, $K_p = 0.5$.
The following graphs illustrate the speed and current when the input is 2 rad/sec and $K_p = 0.5$.

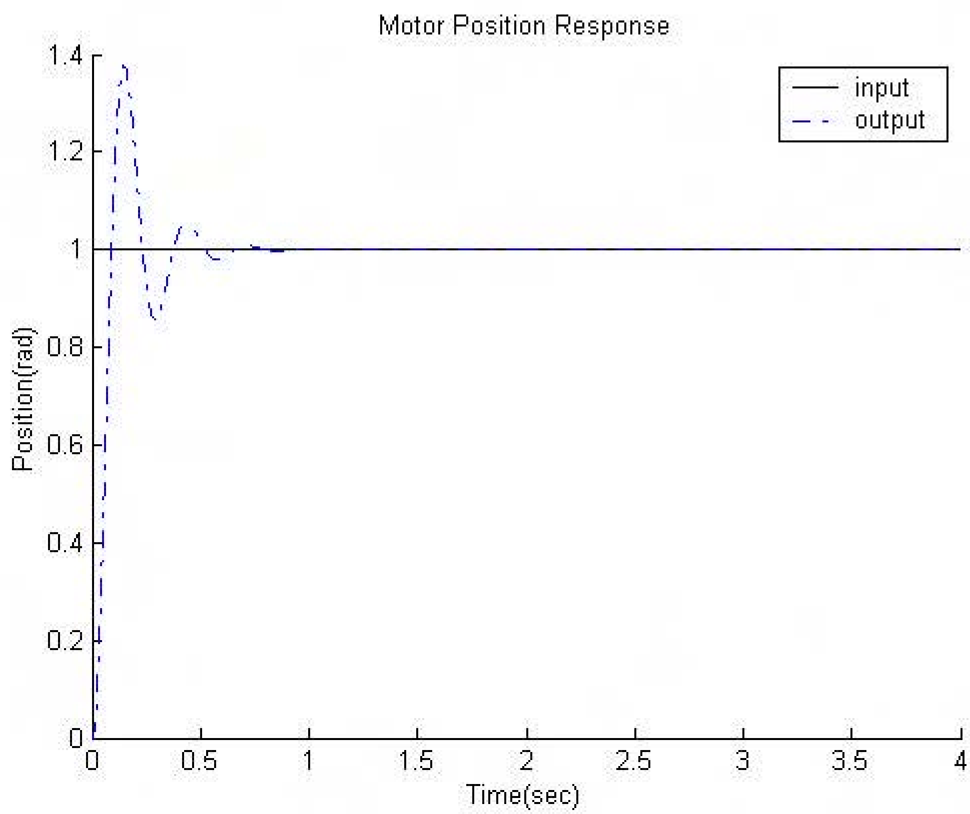

Motor speed response
Speed(rad/sec)
8
6
4
2
0
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5
Motor current response
Current (A)
1
0.8
0.6
0.4
0.2
0
Time (sec)

11-10


Motor Position Response
input
output
Position(rad)
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0
0.5
1
1.5
2
2.5
3
3.5
4
Time(sec)

a) 1 radian.
b) 1.203 radians.
c) 0.2215 seconds.

11-11

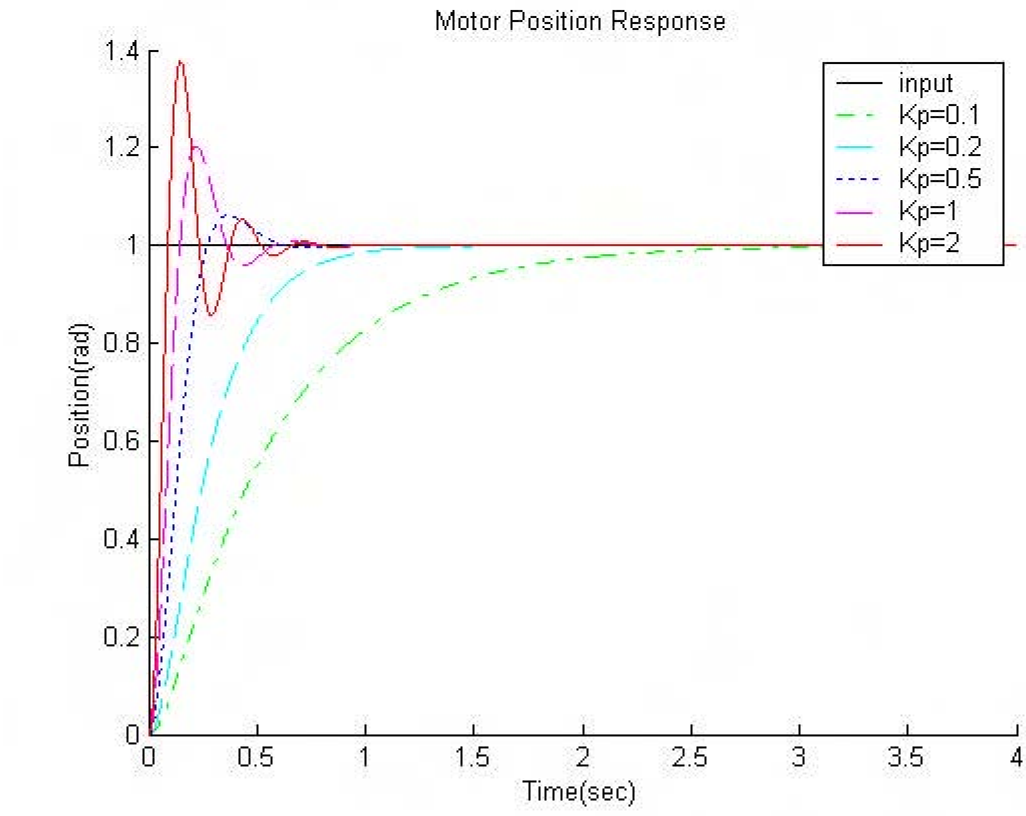


a) The steady state position is very close to 1 radian.
b) 1.377 radians.
c) 0.148 seconds.

It has less steady state error and a faster rise time than Problem 11-10, but has larger overshoot.

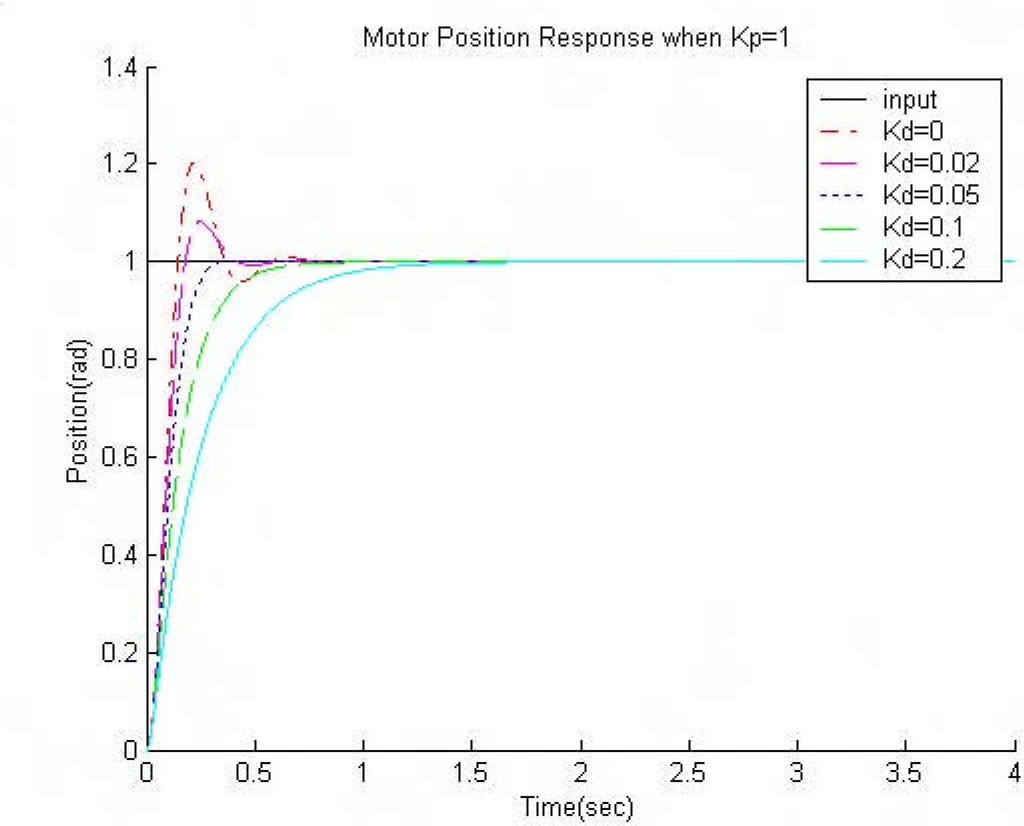
11-12

Different proportional gains and their corresponding responses are shown on the following graph.



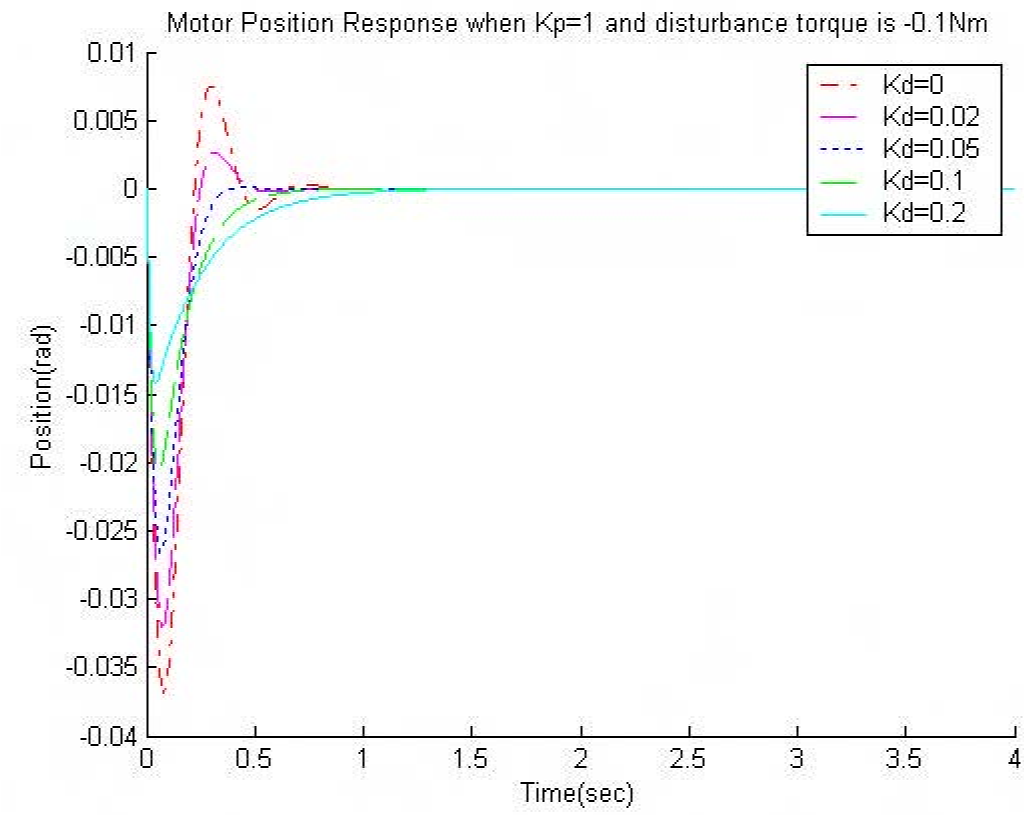
As the proportional gain gets higher, the motor has a faster response time and lower steady state error, but if it the gain is too high, the motor overshoot increases. If the system requires that there be no overshoot, $K_p = 0.2$ is the best value. If the system allows for overshoot, the best proportional gain is dependant on how much overshoot the system can have. For instance, if the system allows for a 30% overshoot, $K_p = 1$ is the best value.

11-13 Let $K_p = 1$ is the best value.



As the derivative gain increases, overshoot decreases, but rise time increases.

11-14



Motor Position Response when Kp=1 and disturbance torque is -0.1Nm
Kd=0
Kd=0.02
Kd=0.05
Kd=0.1
Kd=0.2
Position(rad)
Time(sec)
0.01
0.005
0
-0.005
-0.01
-0.015
-0.02
-0.025
-0.03
-0.035
-0.04
0
0.5
1
1.5
2
2.5
3
3.5
4

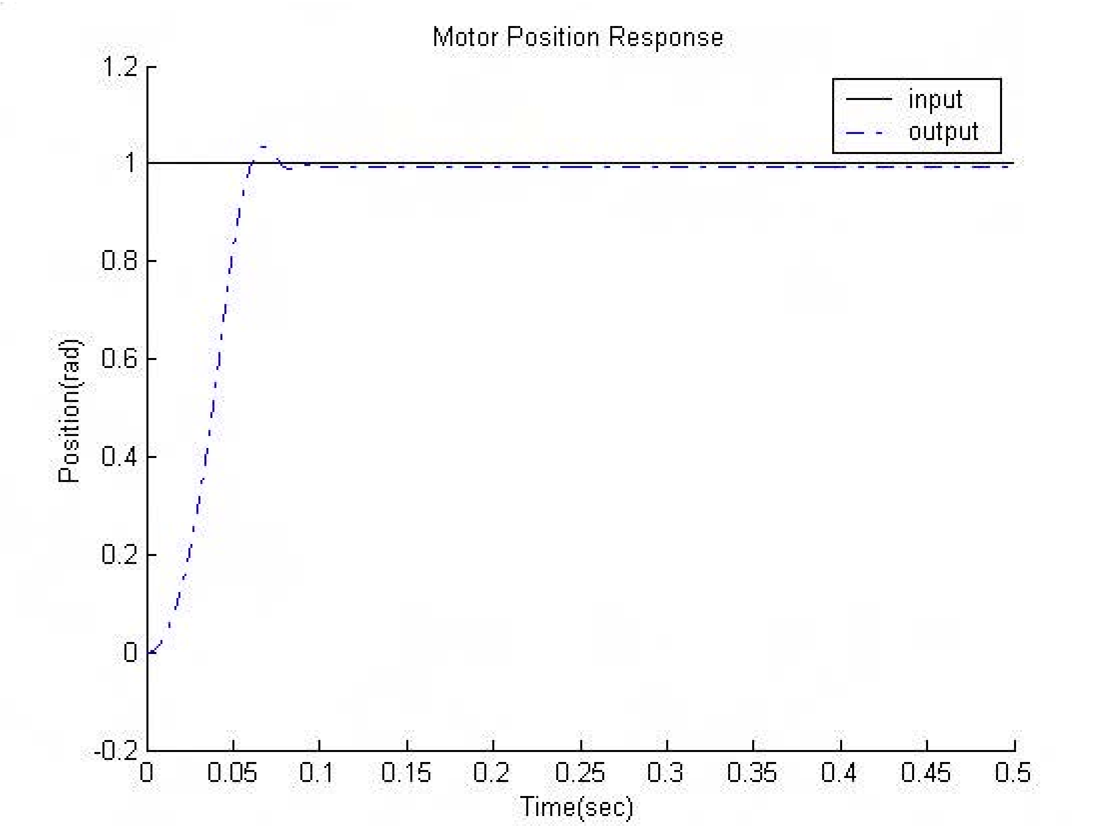
11-15

There could be many possible answers for this problem. One possible answer would be

K_p= 100

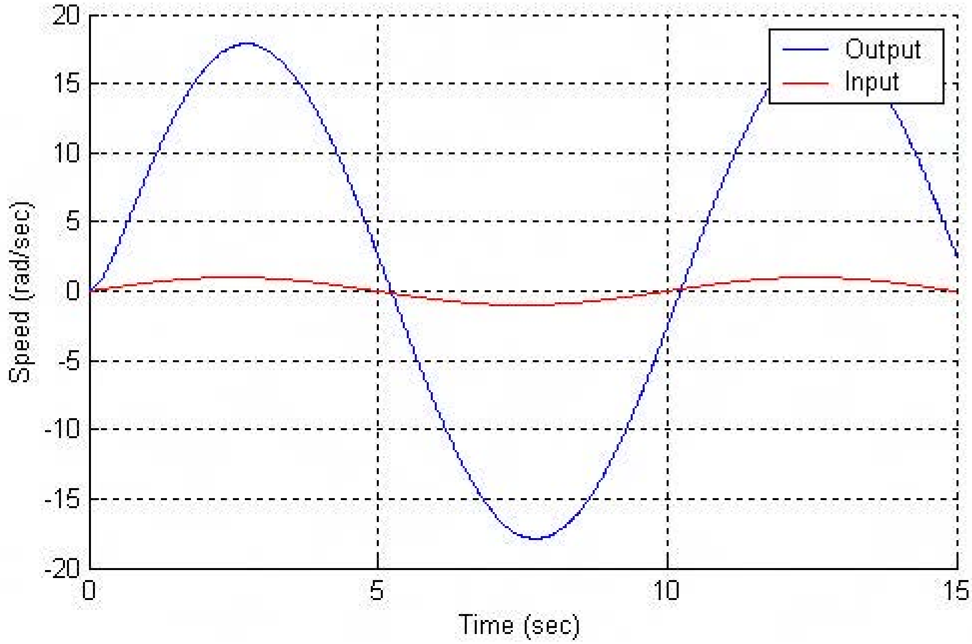
K_i= 10

K_d= 1.4

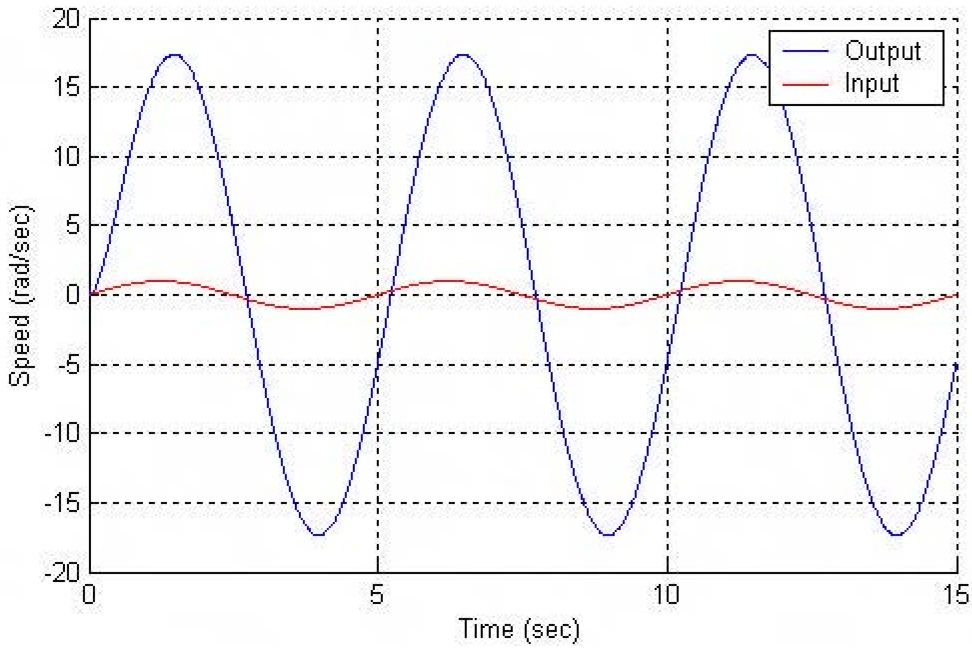


The Percent Overshoot in this case is 3.8%.

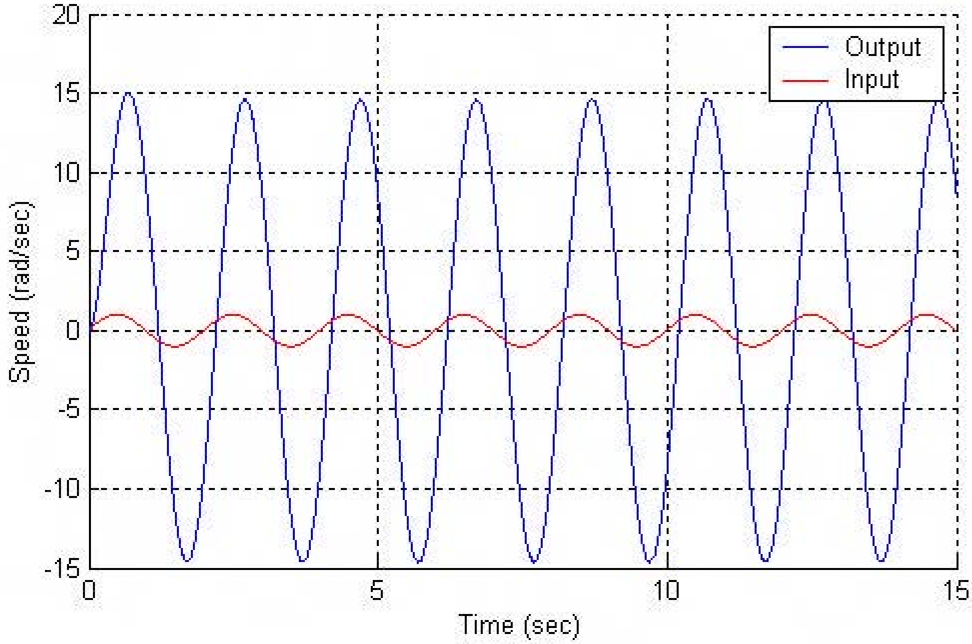
11-16
0.1 Hz



0.2 Hz

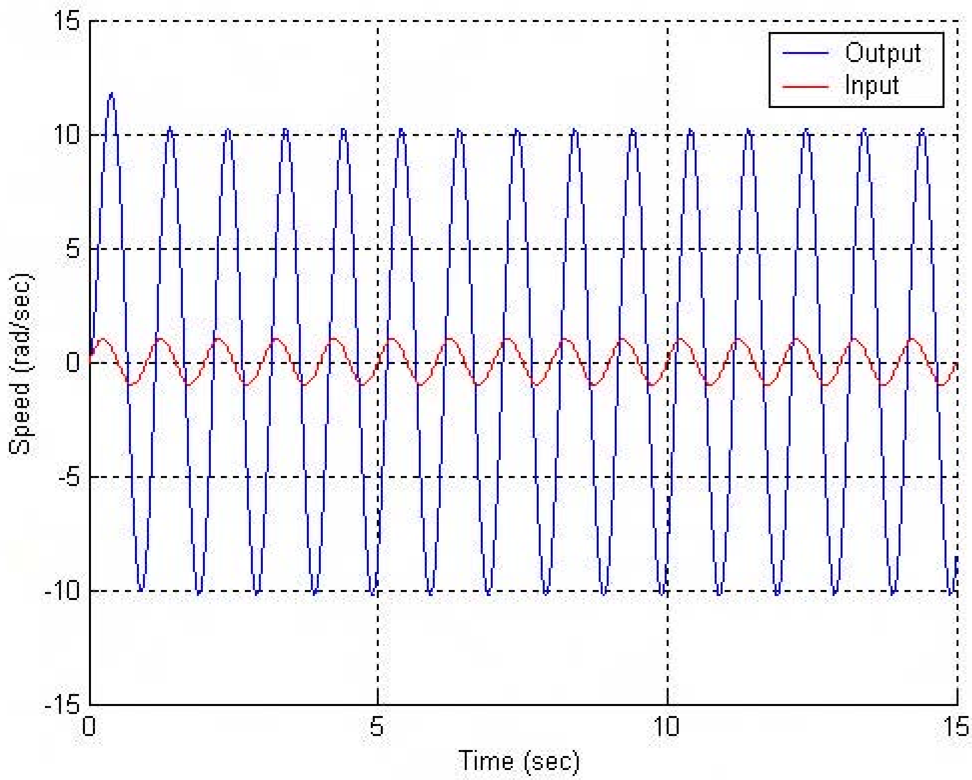


0.5 Hz



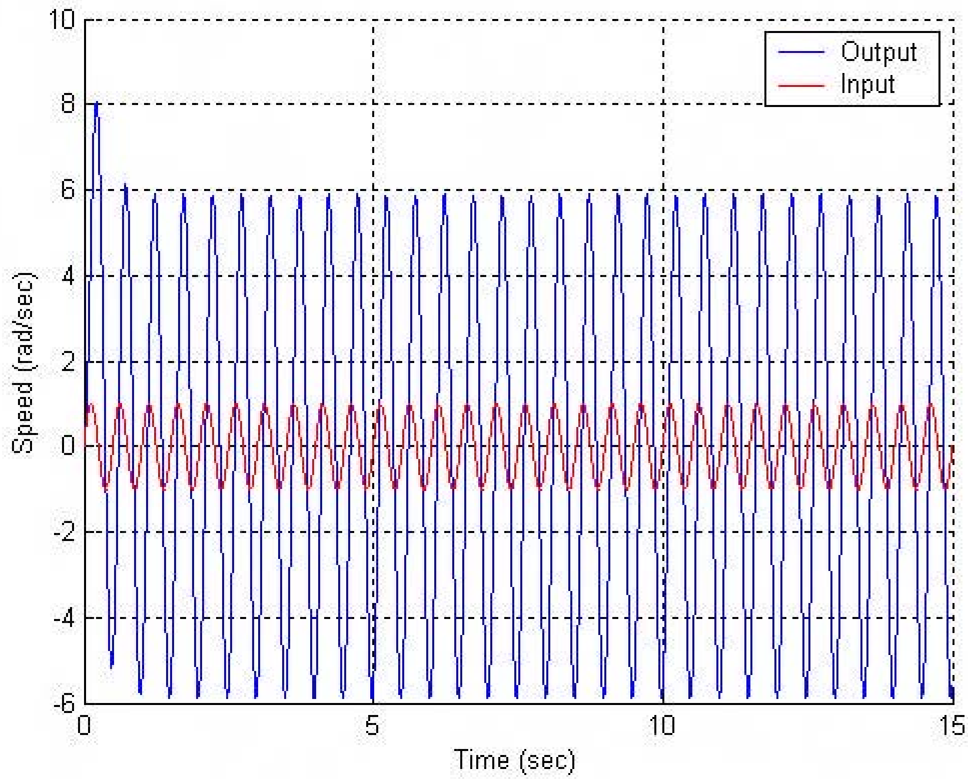
Output
Input
Speed (rad/sec)
Time (sec)
20
15
10
5
0
-5
-10
-15
0
5
10
15

1 Hz

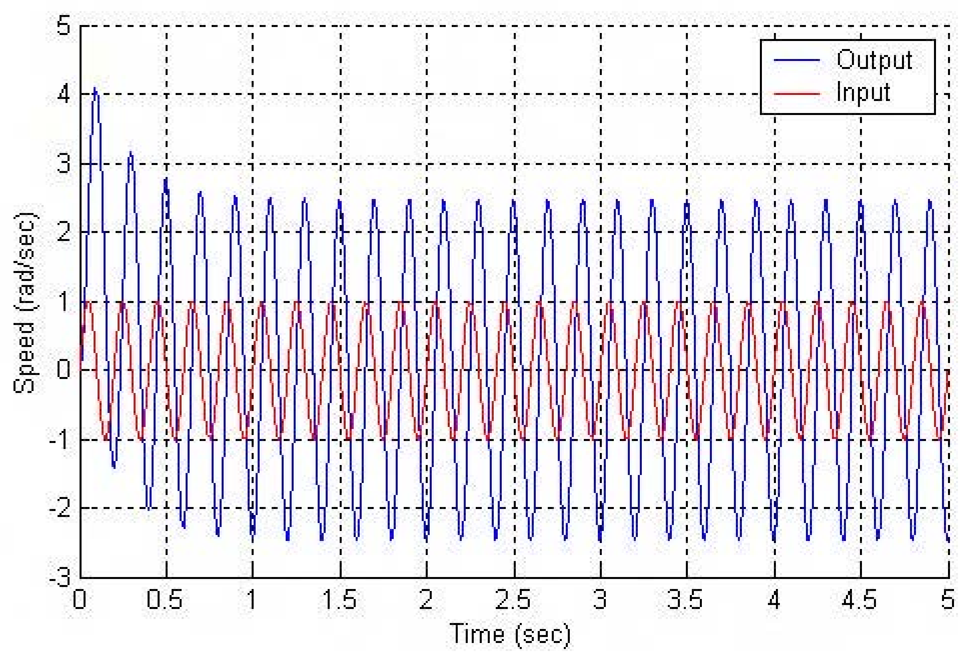


Output
Input
Speed (rad/sec)
Time (sec)
15
10
5
0
-5
-10
-15
0
5
10
15

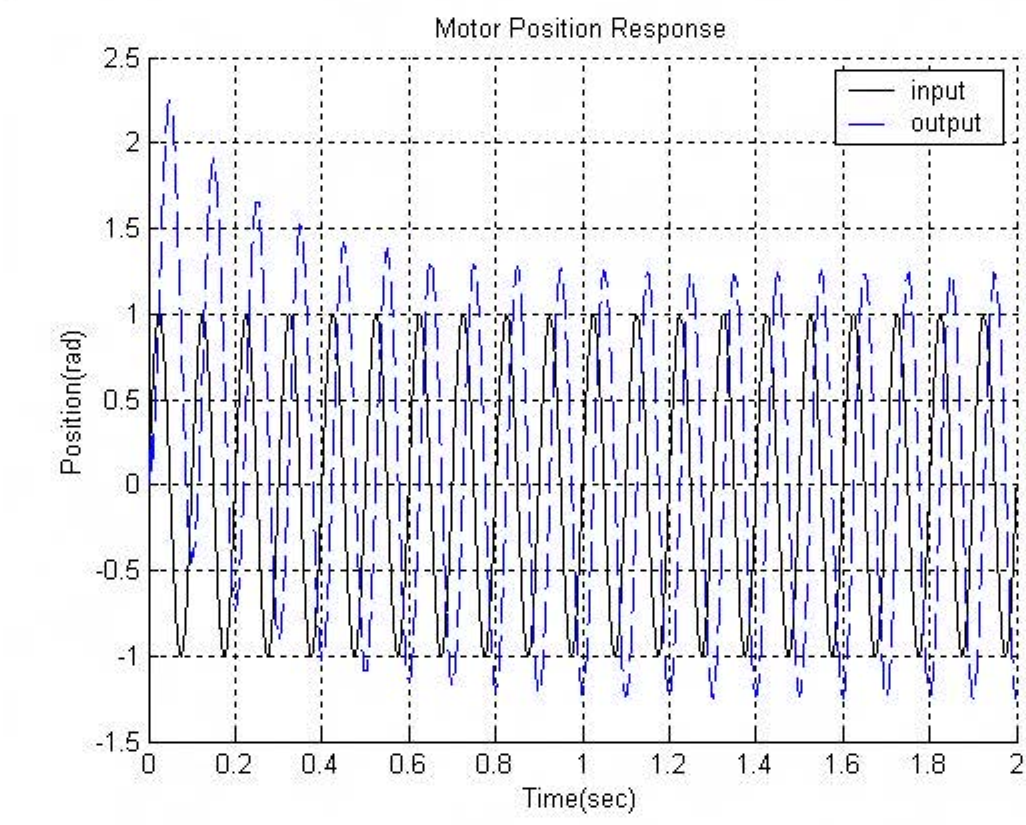
2Hz


Output
Input
Speed (rad/sec)
Time (sec)
10
8
6
4
2
0
-2
-4
-6
0
5
10
15

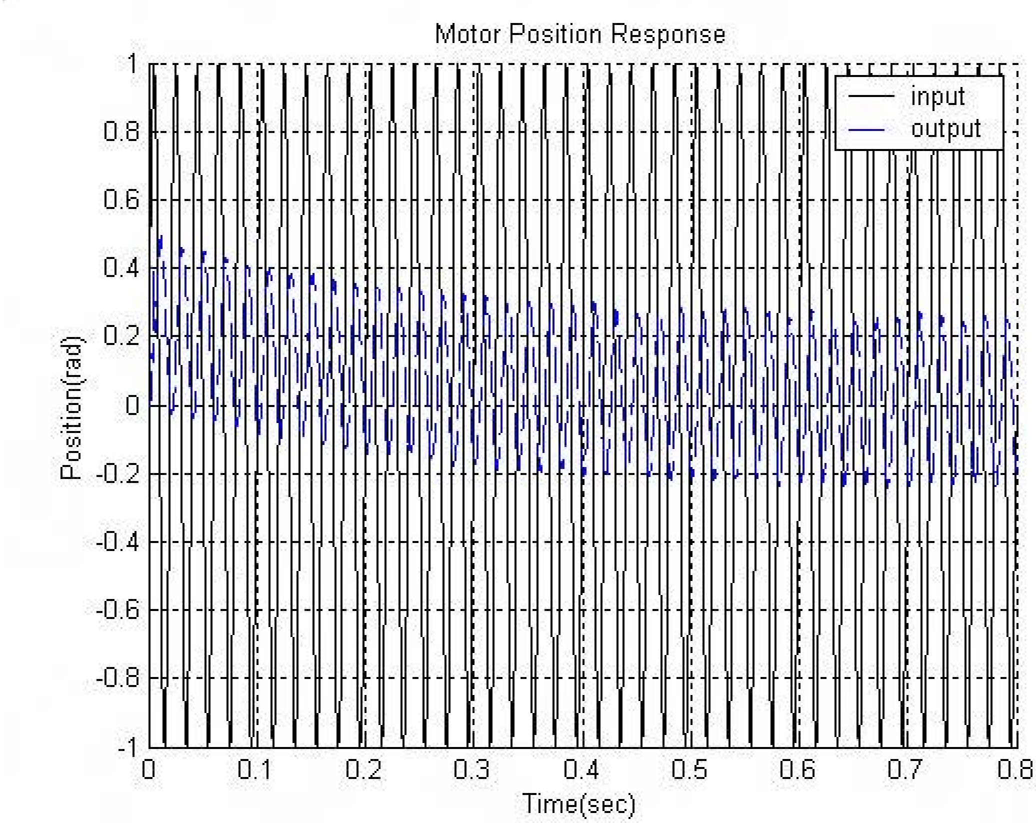
5Hz


Output
Input
Speed (rad/sec)
Time (sec)
5
4
3
2
1
0
-1
-2
-3
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5

10Hz

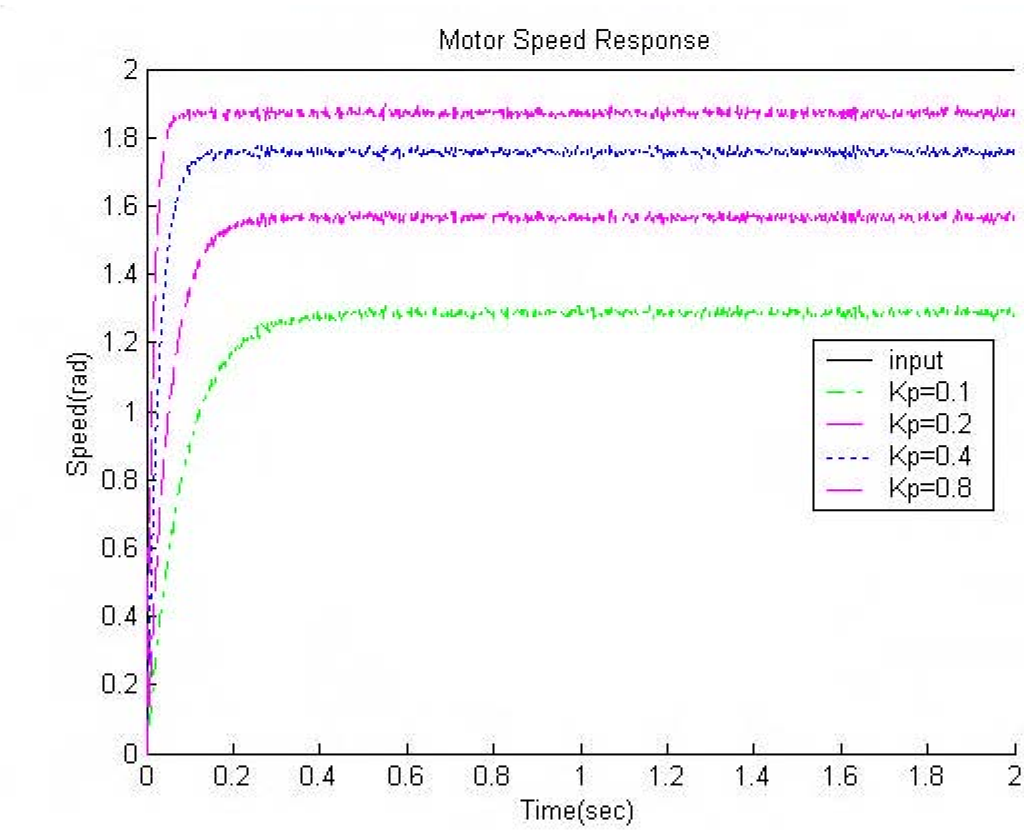

Motor Position Response
input
output
Position(rad)
Time(sec)
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2

50Hz


Motor Position Response
input
output
Position(rad)
Time(sec)
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8

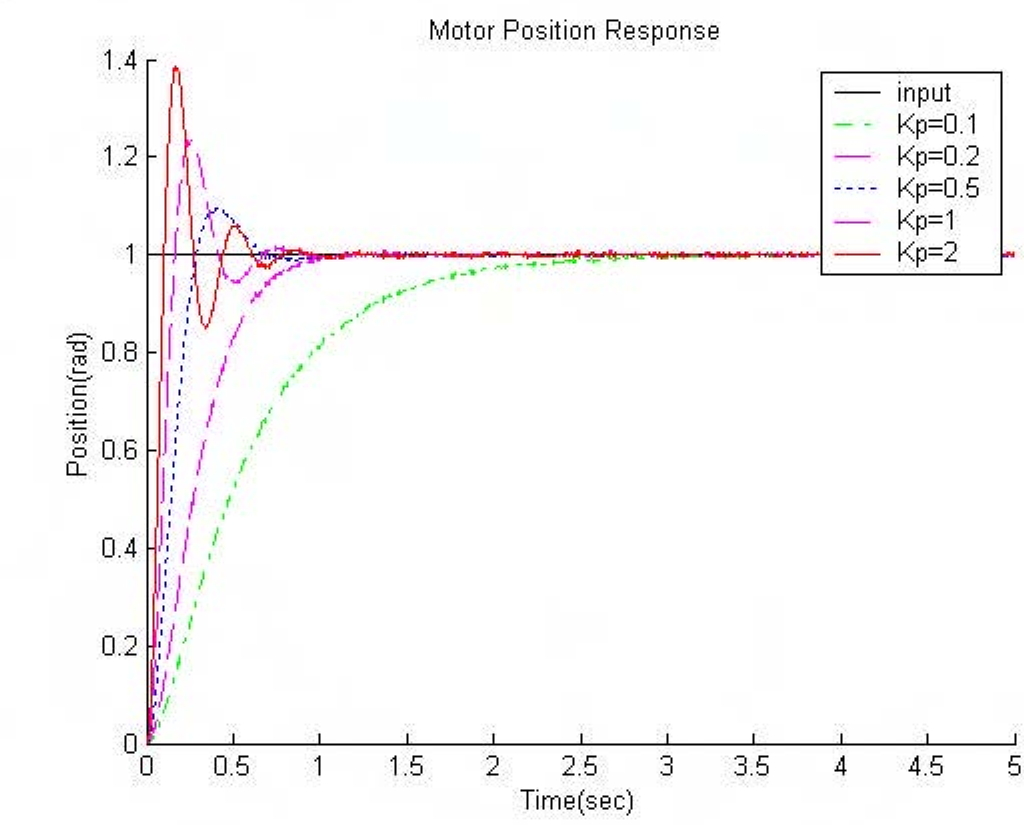
As frequency increases, the phase shift of the input and output also increase. Also, the amplitude of the output starts to decrease when the frequency increases above 0.5Hz.

11-17



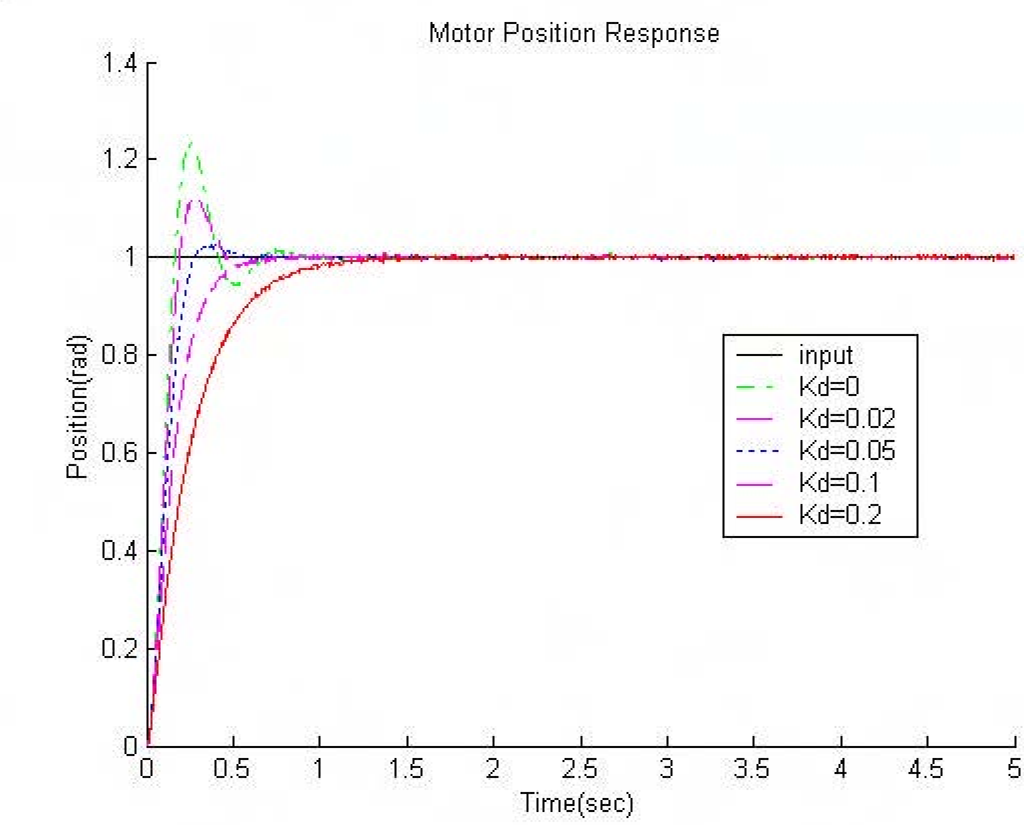
As proportional gain increases, the steady state error decreases.

11-18



Considering fast response time and low overshoot, K_p=1 is considered to be the best value.

11-19 It was found that the best $K_p = 1$



As K_d value increases, the overshoot decreases and the rise time increases.

Appendix H GENERAL NYQUIST CRITERION

H-1 (a)

$$L(s) = \frac{5(s-2)}{s(s+1)(s-1)} \qquad P_w = 1 \qquad P = 1$$

When $w = 0$: $\angle L(j0) = -90^o$ $\quad |L(j0)| = \infty$

When $w = \infty$: $\angle L(j\infty) = -180^o$ $\quad |L(j\infty)| = 0$

$$L(jw) = \frac{5(jw-2)}{-jw\left(1+w^2\right)} = \frac{-5(w+2j)}{w\left(1+w^2\right)}$$

When $L(jw) = 0$, $\quad w = \infty$.

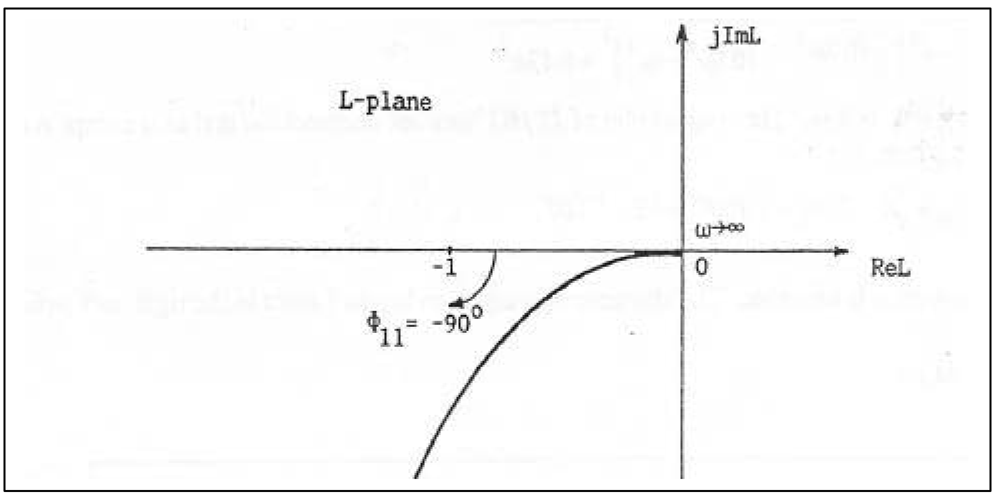
The Nyquist plot of $L(jw)$ does not intersect the real axis except at the origin.

$$\Phi_{11} = \left(Z - 0.5P_w - P\right)180^\circ = \left(Z - 1.5\right)180^\circ = -90^\circ$$

Thus, Z=1.

The closed-loop system is unstable. The characteristic equation has 1 root in the right-half *s*-plane.

Nyquist Plot of $L(jw)$:



(b) $L(jw) = \dfrac{50}{s(s+5)(s-1)} \qquad P_w = 1 \qquad P = 1$

When $w = 0$: $\angle L(j0) = 90^o$ $\quad |L(j0)| = \infty$

When $w = \infty$: $\angle L(j\infty) = -270^o$ $\quad |L(j\infty)| = 0$

$$L(jw)=\frac{50}{-4w^2-jw\left(5+w^2\right)}=\frac{50\left[-4w^2+jw\left(5+w^2\right)\right]}{16w^4+w^2\left(5+w^2\right)^2}$$

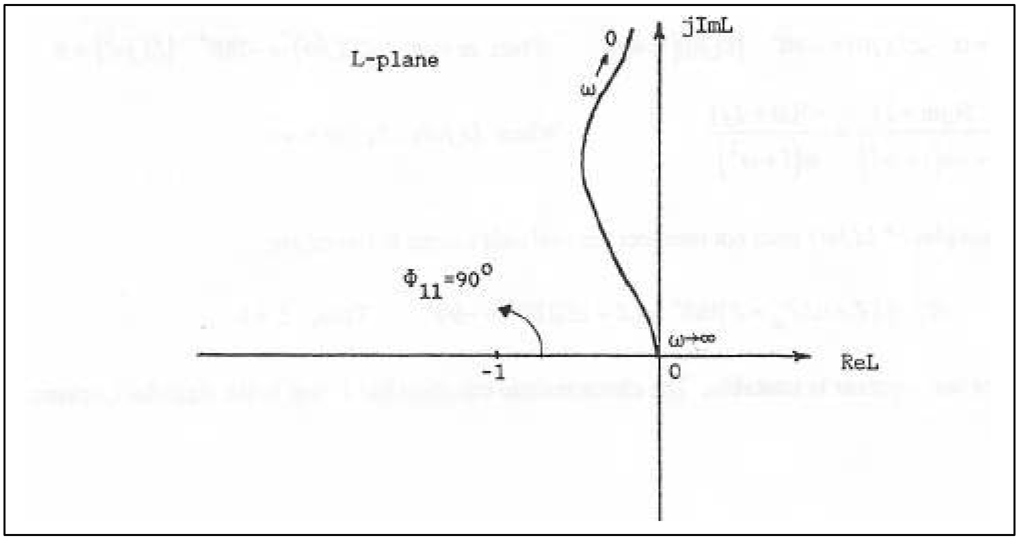
For $\text{Im}\left[L(jw)\right]=0$, $\quad w=\infty$.

Thus, the Nyquist plot of $L(s)$ intersects the real axis only at the origin where $w=\infty$.

$$\Phi_{11}=\left(Z-0.5P_w-P\right)180^o=\left(Z-1.5\right)180^o=90^o \qquad \text{Thus, } Z=2$$

The closed-loop system is unstable. The characteristic equation has two roots in the right-half *s*-plane.

The Nyquist plot of $L(jw)$:



(c) $L(s)=\dfrac{3(s+2)}{s\left(s^3+3s+1\right)} \qquad P_w=1 \qquad P=2$

When $w=0$: $\quad \angle L(j0)=-90^o \qquad \left|L(j0)\right|=\infty$

When $w=\infty$: $\quad \angle L(j\infty)=-270^o \qquad \left|L(j\infty)\right|=0$

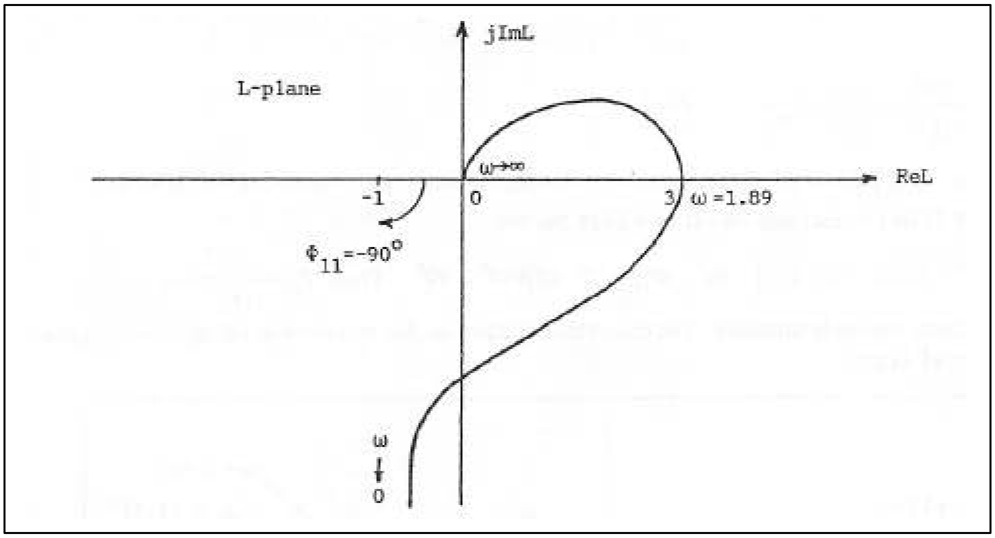
$$L(jw)=\frac{3(jw+2)}{\left(w^4-3w^2\right)+jw}=\frac{3(jw+2)\left[\left(w^4-3w^2\right)-jw\right]}{\left(4^4-3w^2\right)^2+w^2} \qquad \text{Setting } \text{Im}\left[L(jw)\right]=0,$$

$w^4-3w^2-2=0 \qquad$ or $\qquad w^2=3.56 \qquad w=\pm 1.89$ rad/sec. $\qquad L(j1.89)=3$

$$\Phi_{11}=\left(Z-0.5P_w-P\right)180^o=\left(Z-2.5\right)180^o=-90^o \qquad \text{Thus, } Z=2$$

The closed-loop system is unstable. The characteristic equation has two roots in the right-half *s*-plane.

Nyquist Plot of $L(jw)$:



(d) $L(s) = \dfrac{100}{s(s+1)(s^2+2)}$ $\qquad P_w = 3 \qquad P = 0$

When $w = 0$: $\angle L(j0) = -90^o$ $\quad |L(j0)| = \infty$

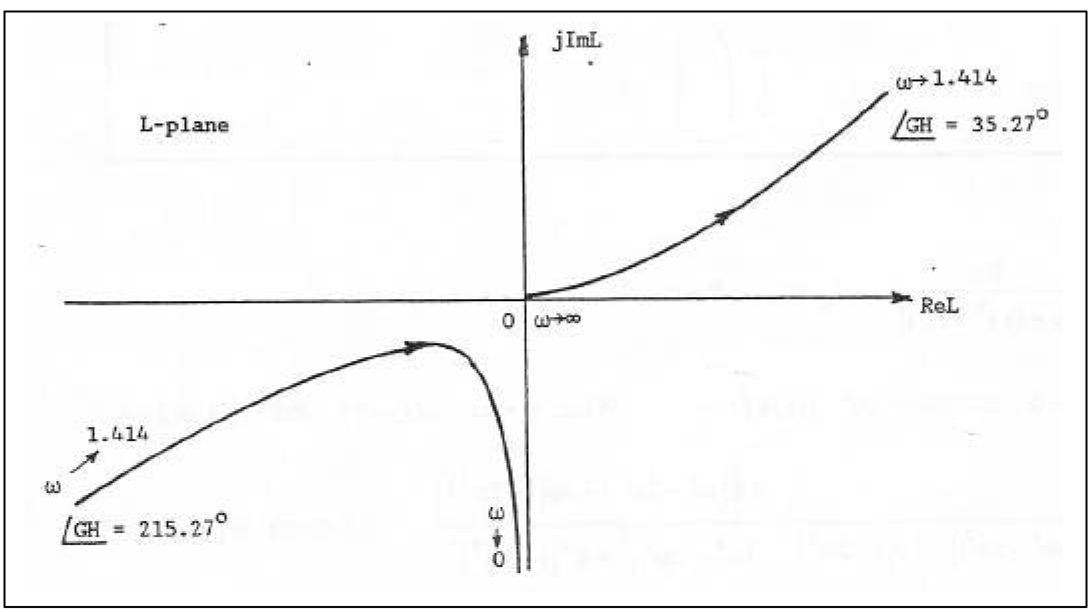
When $w = \infty$: $\angle L(j\infty) = -360^o$ $\quad |L(j\infty)| = 0$

The phase of $L(jw)$ is discontinuous at $w = 1.414$ rad/sec.

$\Phi_{11} = 35.27^o + (270^o - 215.27^o) = 90^o \quad \Phi_{11} = (Z - 1.5)180^o = 90^o$ $\quad$ Thus, $P_{11} = \dfrac{360^o}{180^o} = 2$

The closed-loop system is unstable. The characteristic equation has two roots in the right-half *s*-plane.

Nyquist Plot of $L(jw)$:



(e) $L(s) = \dfrac{s^2 - 5s + 2}{s\left(s^3 + 2s^2 + 2s + 10\right)}$ $\qquad P_w = 1 \qquad P = 2$

When $w = 0$: $\angle L(j0) = -90^o$ $\quad |L(j0)| = \infty$
When $w = \infty$: $\angle L(j\infty) = -180^o$ $\quad |L(j\infty)| = 0$

$$L(jw) = \frac{\left(2 - w^2\right) - j5w}{\left(w^4 - 2w^2\right) + jw\left(10 - 2w^2\right)} = \frac{\left[\left(2 - w^2\right) - j5w\right]\left[\left(w^4 - 2w^2\right) - jw\left(10 - 2w^2\right)\right]}{\left(w^4 - 2w^2\right)^2 + w^2\left(10 - 2w^2\right)^2}$$

Setting $L(jw) = 0$, $\quad \left(2 - w^2\right)\left(10 - 2w^2\right) + 5\left(w^4 - 2w^2\right) = 0$

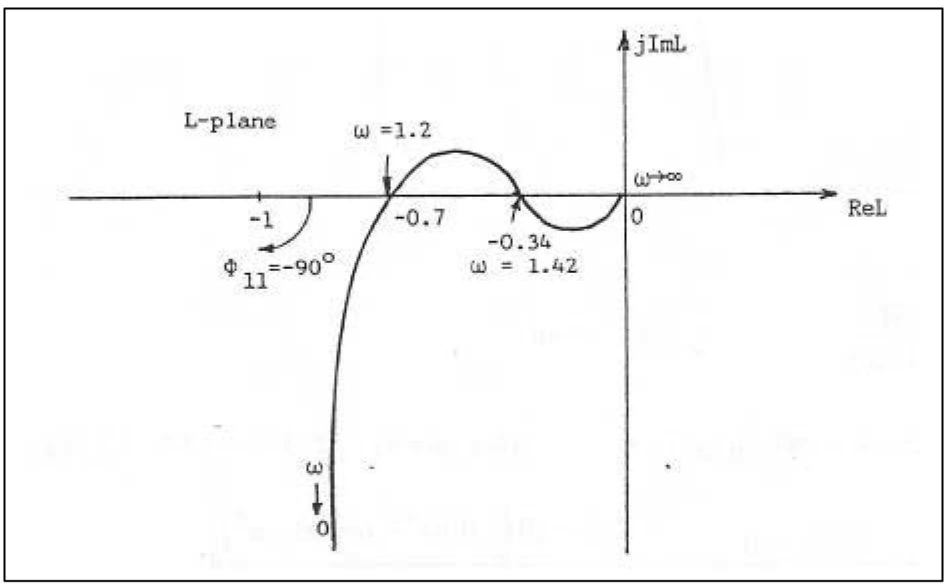
or $\quad w^4 - 3.43w^2 + 2.86 = 0$

Thus, $w^2 = 1.43$ or 2.01. $\quad w = \pm 1.2$ rad / sec or ± 1.42 rad / sec
$L(j1.2) = -0.7 \quad L(j1.42) = -0.34$

$$\Phi_{11} = \left(Z - 0.5P_w - P\right)180^\circ = \left(Z - 2.5\right)180^\circ = -90^\circ \qquad \text{Thus, } Z = 2.$$

The closed-loop system is unstable. The characteristic equation has two roots in the right-half *s*-plane.

Nyquist Plot of $L(jw)$:



(f) $L(s) = \dfrac{-\left(s^2 - 1\right)\left(s + 2\right)}{s\left(s^2 + s + 1\right)} = \dfrac{-0.1s^3 - 0.2s^2 + 0.1s + 0.2}{s\left(s^2 + s + 1\right)}$ $\qquad P_w = 1 \qquad P = 0$

When $w = 0$: $\angle L(j0) = -90^o$ $\quad |L(j0)| = \infty$
When $w = \infty$: $\angle L(j\infty) = 180^o$ $\quad |L(j\infty)| = 0.1$

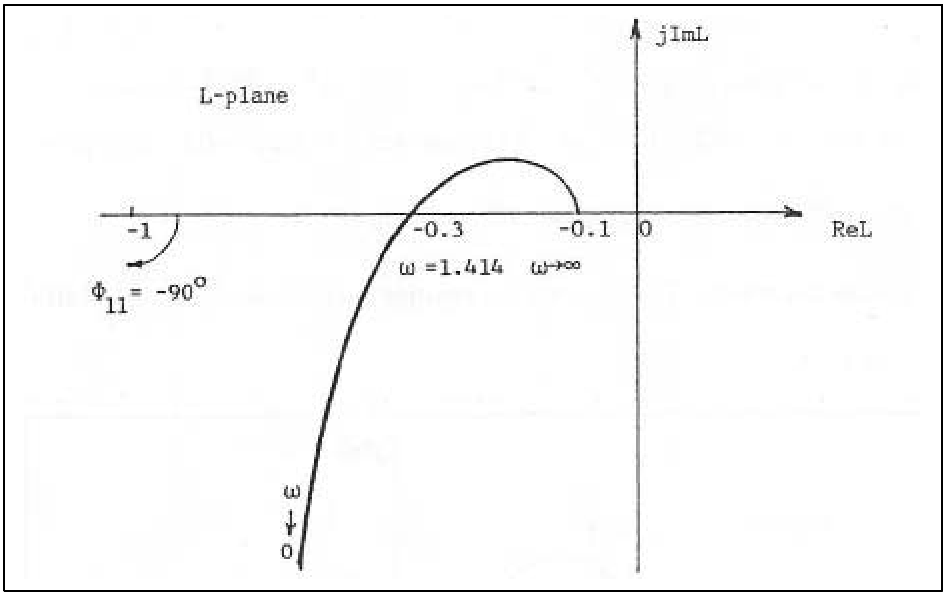
$$L(jw)=\frac{\left(0.2+0.2w^2\right)+0.1jw\left(1+w^2\right)}{-w^2+jw\left(1-w^2\right)}=\frac{\left(1+w^2\right)\left(0.2+j0.1w\right)\left[-w^2-jw\left(1-w^2\right)\right]}{w^4+w^2\left(1-w^2\right)^2}$$

Setting $\text{Im}\left[L(jw)\right]=0$, $w^2=2$. Thus, $w=\pm 1.414$ rad/sec $L(j1.414)=-0.3$

$\Phi_{11}=\left(Z-0.5P_w-P\right)180^\circ=\left(Z-0.5\right)180^\circ=-90^\circ$ Thus, $Z=0$

The closed-loop system is stable.

Nyquist Plot of $L(jw)$:



H-2 (a) $L(s)=\dfrac{K(s-2)}{s\left(s^2-1\right)}$ $P_w=1$ $P=1$

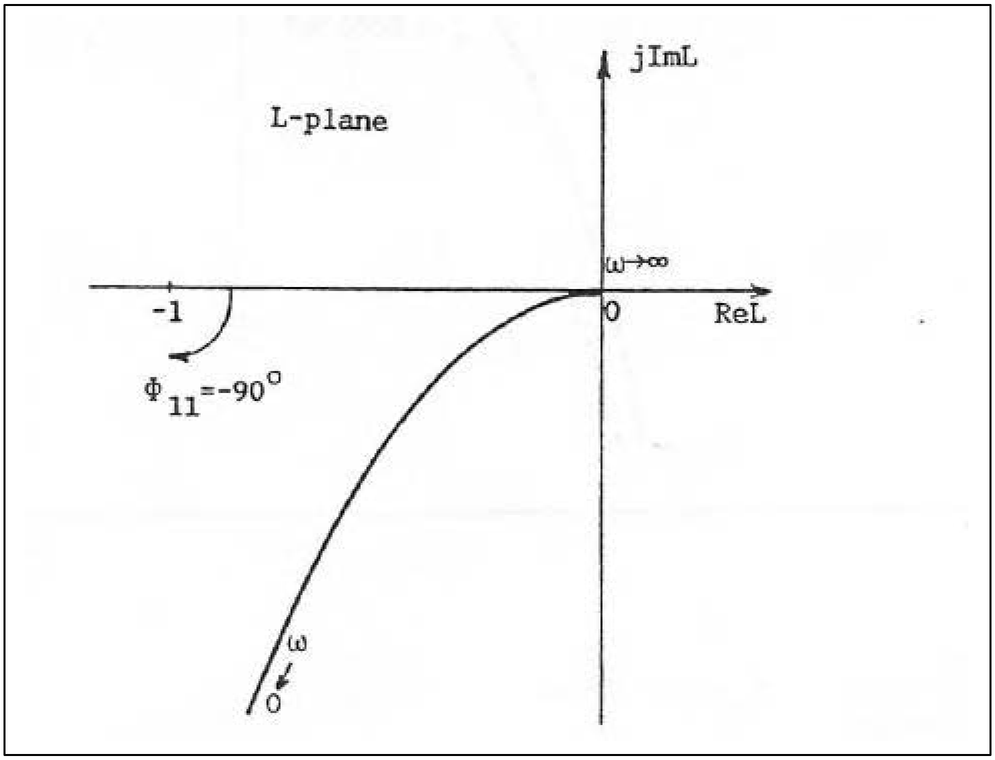
For stability, $Z=0$. $\Phi_{11}=-\left(0.5P_w+P\right)180^\circ=-270^\circ$

For $K>0$, $\Phi_{11}=-90^o$. **The system is unstable.**

For $K<0$, $\Phi_{11}=+90^o\neq-270^o$ **The system is unstable.**

Thus the system is unstable for all *K*.

Nyquist plot



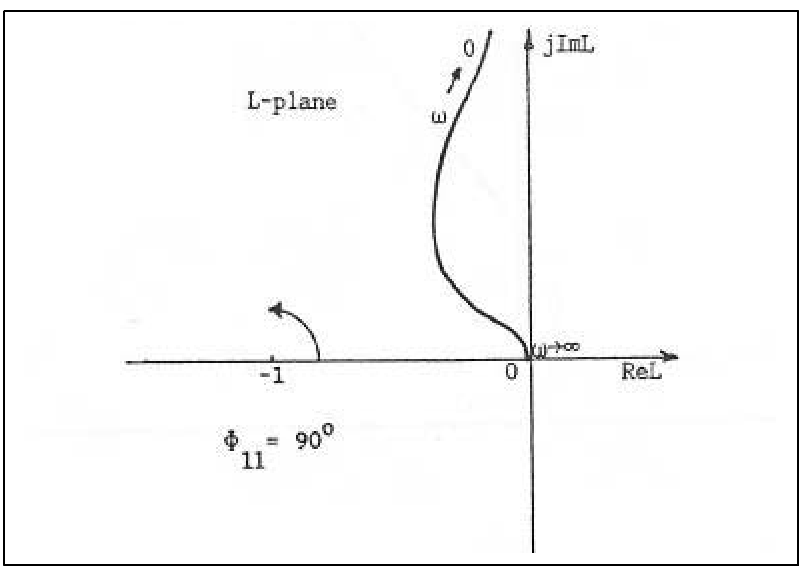
(b) $L(s)=\dfrac{K}{s(s+10)(s-2)}$ $\quad P_w=1 \quad P=1$

For stability, $Z=0$. $\Phi_{11}=-\left(0.5P_w+P\right)180^\circ=-1.5\times180^\circ=-270^\circ$

For $K>0$, $\Phi_{11}=90^o$. **The system is unstable.**

For $K<0$, $\Phi_{11}=-90^o\neq-270^o$. **The system is unstable for all values of *K*.**

Nyquist Plot of $L(jw)$:



(c) $L(s) = \dfrac{K(s+1)}{s\left(s^3 + 3s + 1\right)}$ $\qquad P_w = 1 \qquad P = 2$

For stability, $Z = 0$. $\Phi_{11} = -\left(0.5P_w + P_{-1}\right)180^o = -450^o$

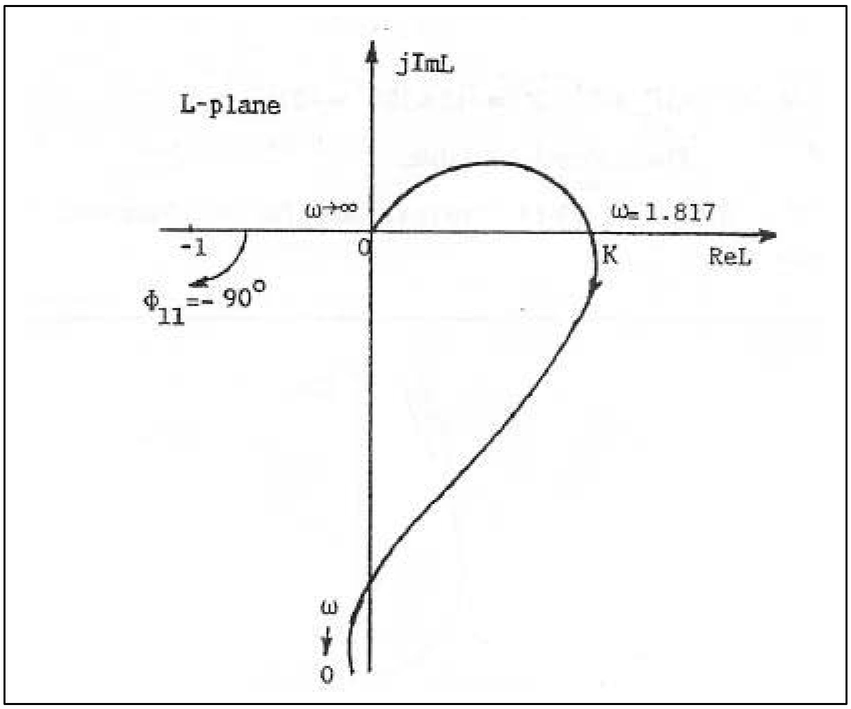
For $K > 0$, $\Phi_{11} = -90^o$. **The system is unstable.**

For $K < 0$, $\Phi_{11} = +90^o$ when $K > -1$. $\Phi_{11} = -270^o$ when $K < -1$.

The Nyquist plot of $L(jw)$ crosses the real axis at K, and the phase crossover frequency is 1.817 rad/sec.

The system is unstable for all values of *K*.

Nyquist Plot of $L(jw)$:



(d) $L(s) = \dfrac{K\left(s^2 - 5s + 2\right)}{s\left(s^3 + 2s^2 + 2s + 10\right)}$ $\qquad P_w = 1 \qquad P = 2$

For stability, $Z = 0$. $\Phi_{11} = -\left(0.5P_w + P\right)180^o = -2.5 \times 180^o = -450^o$

The Nyquist plot of $L(jw)$ intersects the real axis at the following points:

$w = \infty$: $\quad L(j0) = 0$

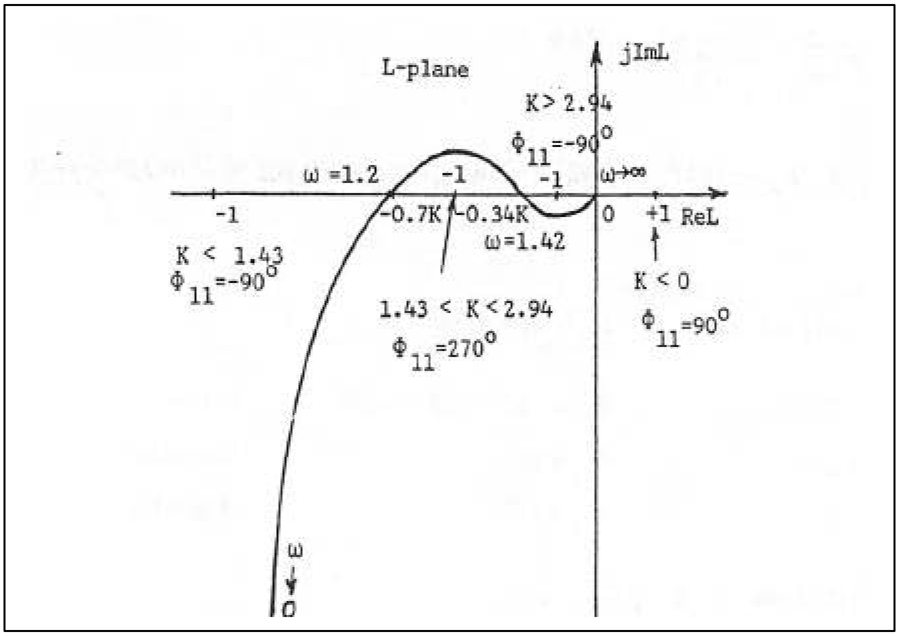
$w = 1.42$ rad / sec: $\quad L(j1.42) = -0.34K$

$w = 1.2$ rad / sec: $\quad L(j1.4) = -0.7K$

$0 < K < 1.43$	$\Phi_{11} = -90^o$
$1.43 < K < 2.94$	$\Phi_{11} = 270^o$
$2.94 < K$	$\Phi_{11} = -90^o$
$K < 0$	$\Phi_{11} = 90^o$

Since Φ_{11} does not equal to -450^o for any K, **the system is unstable for all values of K.**

Nyquist Plot of $L(jw)$:



(e) $$L(s) = \frac{K\left(s^2 - 1\right)\left(s + 2\right)}{s\left(s^2 + s + 1\right)} \qquad P_w = 1 \qquad P = 0$$

For stability, $Z = 0$. $\Phi_{11} = -\left(0.5P_w + P\right)180^o = -90^o$

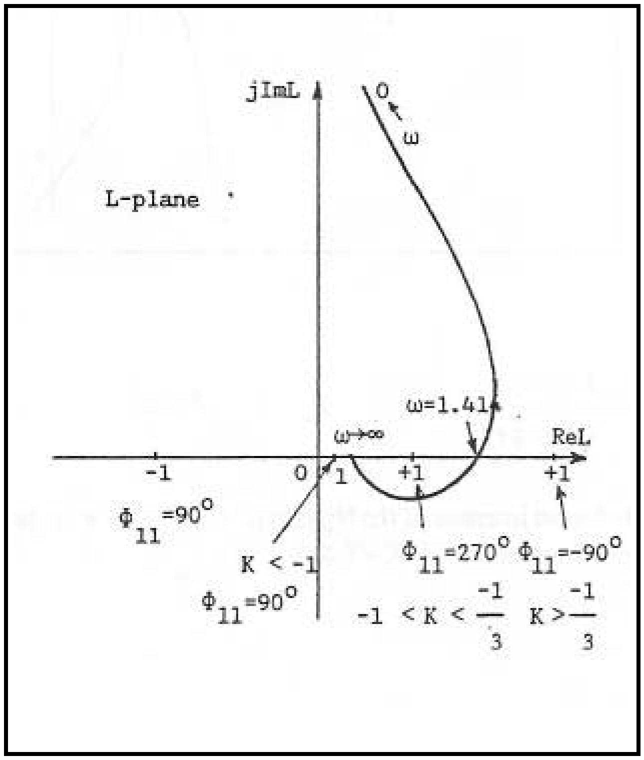
The Nyquist plot of $L(jw)$ intersects the real axis at:

$w = \infty$: $\qquad L(j\infty) = K$

$w = \sqrt{2}$ rad / sec: $\qquad L(j\sqrt{2}) = 3K$

$K > 0$	$\Phi_{11} = 90^o$	**Unstable**
$K < -1$	$\Phi_{11} = 90^o$	**Unstable**
$-1 < K < -\frac{1}{3}$	$\Phi_{11} = 270^o$	**Unstable**
$-\frac{1}{3} < K < 0$	$\Phi_{11} = -90^o$	**Stable**

The system is stable for $-\frac{1}{3} < K < 0$**.**



(f) $\qquad L(s) = \dfrac{K\left(s^2 - 5s + 1\right)}{s(s+1)\left(s^2+4\right)} \qquad P_w = 3 \qquad P = 0$

For stability, $Z = 0$. $\quad \Phi_{11} = -\left(0.5P_w + P\right)180^\circ = -270^\circ$ The Nyquist plot of $L(jw)$ plot intersects the real axis at

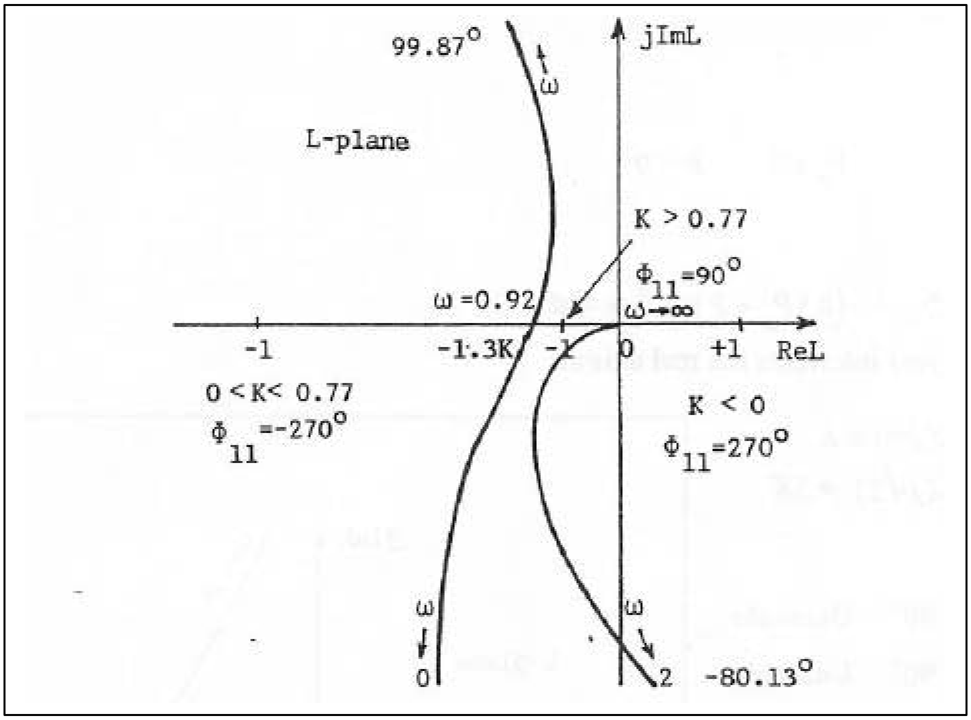
$w = \infty$: $\qquad L(j\infty) = 0$

$w = 0.92$ rad / sec: $\qquad L(j0.92) = -1.3K$

$0 < K < 0.77$	$\Phi_{11} = -90^o - 180^o = -270^o$	**Stable**
$K > 0.77$	$\Phi_{11} = 90^o$	**Unstable**
$K < 0$	$\Phi_{11} = 270^o$	**Unstable**

The system is stable for $0 < K < 0.77$.

Nyquist Plot:



H-3 **(a)** $\Phi_{11} = (Z - 0.5P_w - P)180^\circ = (Z - 1.5)180^\circ = 90^\circ$ Thus, $Z = 2$

The closed-loop system is unstable. The characteristic equation has two roots in the right-half s-plane.

(b) $\Phi_{11} = (Z - 0.5P_w - P)180^\circ = (Z - 1)180^\circ = -180^\circ$ Thus, $Z = 0$.

The closed-loop system is stable.

(c) $\Phi_{11} = (Z - 0.5P_w - P)180^\circ = (Z - 2)180^\circ = -360^\circ$ Thus, $Z = 0$.

The closed-loop system is stable.

(d) $\Phi_{11} = (Z - 0.5P_w - P)180^\circ = Z \times 180^\circ = 360^\circ$ Thus, $Z = 2$

The closed-loop system is unstable. The characteristic equation has two roots in the right-half s-plane.

(e) $\Phi_{11} = (Z - 0.5P_w - P)180^\circ = (Z - 1)180^\circ = -162^\circ - 18^\circ$ Thus, $Z = 0$.

The closed-loop system is stable.

(f) $\Phi_{11} = (Z - 0.5P_w - P)180^\circ = (Z - 0.5)180^\circ = -90^\circ$ Thus, $Z = 0$

The closed-loop system is stable.

H-4 **(a)** The stability criterion for $1/L(s)$ is the same as that for $L(s)$. Thus,

$$\Phi_{11} = (Z - 0.5P_w - P)180^\circ = (Z - 1.5)180^\circ$$

From Fig. HP-4 (a), $\Phi_{11} = -270^o$. Thus, $Z = 0$. **The closed-loop system is stable.**

(b) The stability criterion for $1/L(s)$ is the same as that for $L(s)$. Thus,

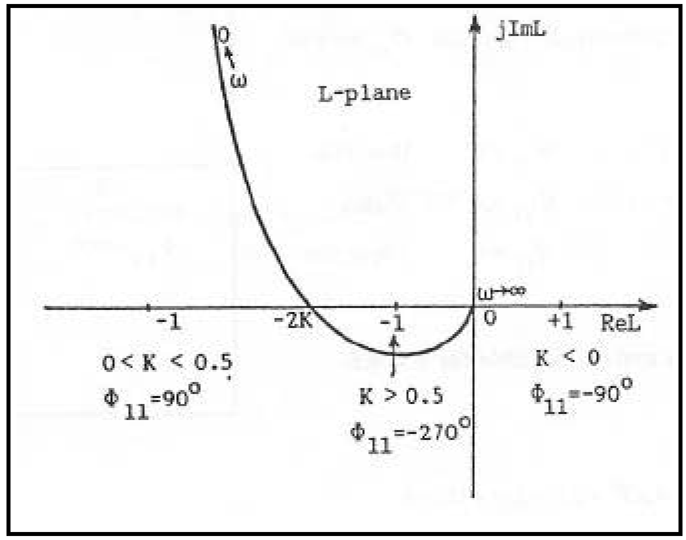
$$\Phi_{11} = (Z - 0.5P_w - P)180^\circ = (Z-1)180^\circ$$

From Fig. HP-4 (b), $\Phi_{11} = -180$. Thus, $Z = 0$. **The closed-loop system is stable.**

H-5 (a) $\Phi_{11} = (Z - 0.5P_w - P)180^\circ = (Z-1.5)180^\circ$ For stability, $Z = 0$. Thus, $\Phi_{11} = -270^o$

$0 < K < 0.5$	$\Phi_{11} = 90^o$	**Unstable**
$K > 0.5$	$\Phi_{11} = -270^o$	**Stable**
$K < 0$	$\Phi_{11} = -90^o$	**Unstable**

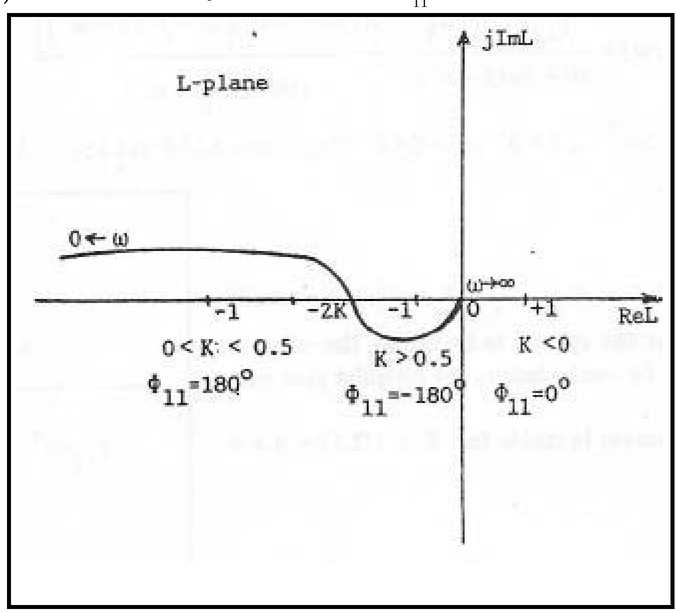
The system is stable for $K > 0.5$.



(b) $\Phi_{11} = (Z - 0.5P_w - P)180^\circ = (Z-1)180^\circ$ For stability, $Z = 0$. Thus, $\Phi_{11} = -180^o$

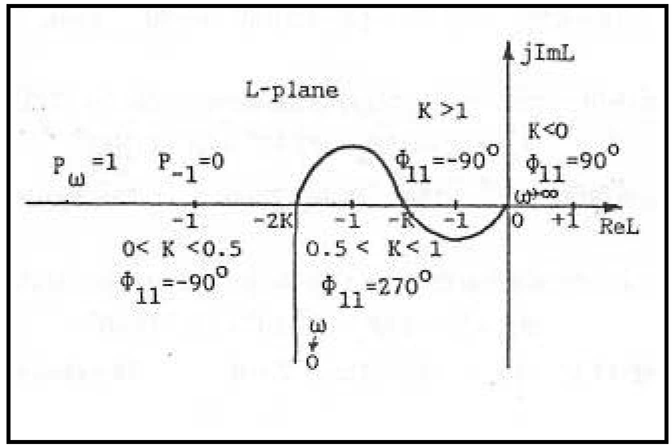
$0 < K < 0.5$	$\Phi_{11} = 90^o$	**Unstable**
$K > 0.5$	$\Phi_{11} = -270^o$	**Stable**
$K < 0$	$\Phi_{11} = -90^o$	**Unstable**

The system is stable for $K > 0.5$.



(c) $\Phi_{11} = \left(Z - 0.5P_w - P\right)180^o = \left(Z - 0.5\right)180^o$

For stability, $Z = 0$. Thus, $\Phi_{11} = -90^o$

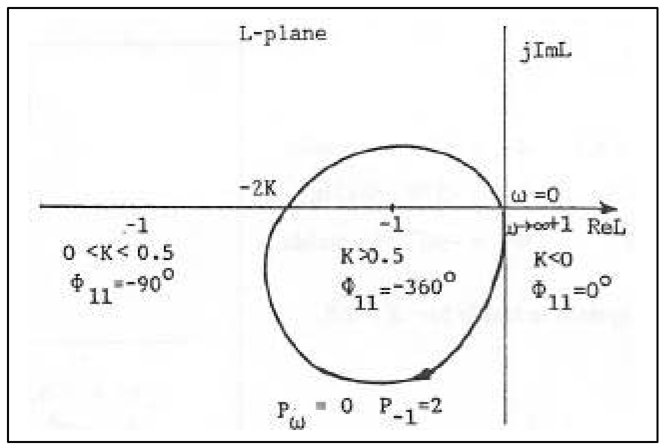


$0 < K < 0.5$	$\Phi_{11} = -90^o$	**Stable**
$0.5 < K < 1$	$\Phi_{11} = 270^o$	**Unstable**
$K > 1$	$\Phi_{11} = -90^o$	**Stable**
$K < 0$	$\Phi_{11} = 90^o$	**Unstable**

The system is stable for $K > 1$.

(d) $\Phi_{11} = \left(Z - 0.5P_w - P\right)180^o = \left(Z - 2\right)180^o$

For stability, $Z = 0$. Thus, $\Phi_{11} = -360^o$



$0 < K < 0.5$	$\Phi_{11} = 0^o$	**Unstable**
$K > 0.5$	$\Phi_{11} = -360^o$	**Stable**
$K < 0$	$\Phi_{11} = 0^o$	**Unstable**

The system is stable for $K > 0.5$.

H-6 **(a)** $s^3 + 4Ks^2 + (K+5)s + 10 = 0$

$$L_{eq}(s) = \frac{Ks(4s+1)}{s^3 + 5s + 10} \qquad P_w = 0 \qquad P = 2$$

When $w = 0$: $\angle L_{eq}(j0) = 90^o$ $\quad \left|L_{eq}(j0)\right| = 0$

When $w = \infty$: $\angle L_{eq}(j\infty) = -90^o$ $\quad \left|L_{eq}(j\infty)\right| = 0$

$$L_{eq}(jw) = \frac{K(jw - 4w^2)}{10 + jw(5 - w^2)} = \frac{K(jw - 4w^2)\left[10 - jw\left(5 - w^2\right)\right]}{100 + w^2\left(5 - w^2\right)^2} \qquad \text{Setting } \operatorname{Im}\left[L_{eq}(jw)\right] = 0$$

$w^4 - 5w^2 - 2.5 = 0 \quad w^2 = 5.46$ Thus, $w = \pm 2.34$ rad/sec. $\quad L_{eq}(j2.34) = -2.18K$

For stability, $\Phi_{11} = -\left(0.5P_w + P\right)180^o = -360^o$

Thus for the system to be stable, the -1 point should be encircled by the Nyquist plot once.

The system is stable for $K > 1/2.18 = 0.458$.

Routh Tabulation:

s^3	1	$K+5$	
s^2	$4K$	10	$K>0$
s^1	$\dfrac{4K^2+20K-10}{4K}$		$K^2+5K-2.5>0$
s^o	10		$(K+5.45)(K-0.458)>0$

For system stability, $K > 0.458$.

(b) $s^3 + K\left(s^3 + 2s^2 + 1\right) = 0$

$$L_{eq}(s) = \frac{K\left(s^3 + 2s + 1\right)}{s^3} \qquad P_w = 3 \qquad P = 0$$

When $w = 0$: $\angle L_{eq}(j0) = -270^o$ $\quad \left|L_{eq}(j0)\right| = \infty$

When $w = \infty$: $\angle L_{eq}(j\infty) = 0^o$ $\quad \left|L_{eq}(j\infty)\right| = K$

$$L_{eq}(jw) = \frac{K\left[\left(1-2w^2\right) - jw^3\right]}{-jw^3} = \frac{K\left[w^3 + j\left(1-2w^2\right)\right]}{w^3} \qquad \text{Setting } \operatorname{Im}\left[L_{eq}(jw)\right] = 0$$

$$\frac{1-2w^2}{w^3} = 0 \qquad w = \infty, \quad w = \pm 0.707 \text{ rad/sec} \qquad L_{eq}(j0.707) = K$$

The Nyquist plot is a straight line, since the equation of $L_{eq}(jw)$ shows that its real part is always equal to K for all values of w.

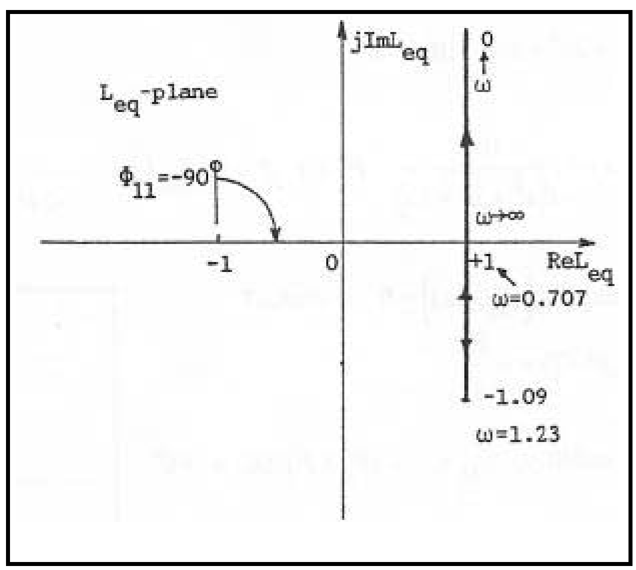
For stability,

$\Phi_{11} = -\left(0.5P_w + P\right)180^\circ = -270^\circ$

$K > 0$ $\qquad \Phi_{11} = -90^o$ **Unstable**

$K < 0$ $\quad \Phi_{11} = 90^o$ **Unstable**

The system is unstable for all values of *K*.

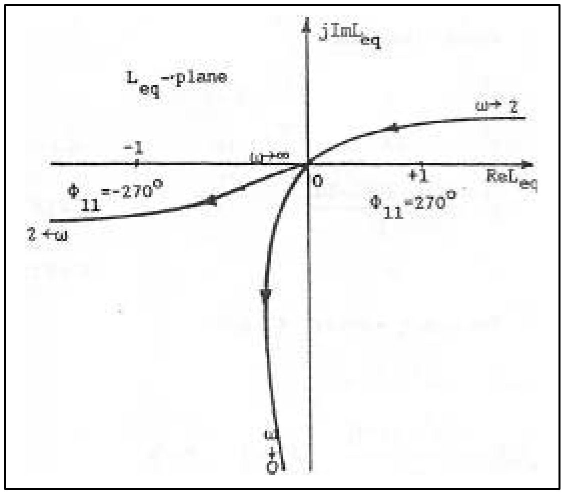


Routh Tabulation:

s^3	$1+K$	0	
s^2	$2K$	K	$K>0$
s^1	$\dfrac{-1-K}{2K}$		$K<-1$
s^o	K		$K>0$

The system is unstable for all values of *K*.

(c) $s(s+1)(s^2+4)+K(s^2+1)=0$



$$L_{eq}(s)=\frac{K(s^2+1)}{s(s+1)(s^2+4)}\quad P_w=3\quad P=0$$

For stability,

$$\Phi_{11}=-(0.5P_w+P)180^o=-270^o$$

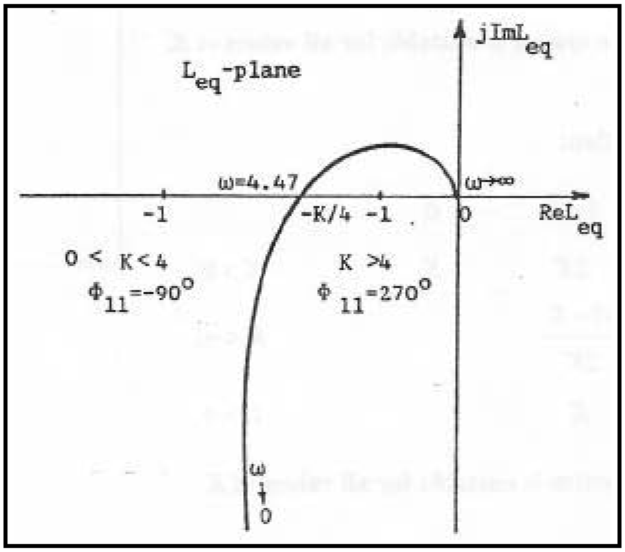
$0<K<\infty$	$\Phi_{11}=-270^o$	**Stable**
$K<0$	$\Phi_{11}=270^o$	**Unstable**

(d) $L_{eq}(s)=\dfrac{10K}{s(s^2+2s+20)}\quad P_w=1\quad P=0$

$$L_{eq}(jw)=\frac{10K}{-2w^2+jw(20-w^2)}=\frac{10K\left[-2w^2-jw(20-w^2)\right]}{4w^4+w^2(20-w^2)^2}$$

Setting $\operatorname{Im}\left[L_{eq}(jw)\right]=0\quad w=\pm 4.47$

$$L_{eq}(j4.47)=-\frac{K}{4}$$



For stability, $\Phi_{11}=-(0.5P_w+P)180^o=-90^o$

$K>4$	$\Phi_{11}=270^o$	**Unstable**
$0<K<4$	$\Phi_{11}=-90^o$	**Stable**

The system is stable for $0 < K < 4$.

Routh Tabulation

s^3	1	20	
s^2	2	$10K$	
s^1	$\dfrac{40-10K}{2}$		$K<4$
s^o	$10K$		$K>0$

For stability, $0 < K < 4$

(e) $s\left(s^3+2s^2+s+1\right)+K\left(s^2+s+1\right)=0$

$$L_{eq}(s)=\frac{K(s+2)}{s\left(s^3+3s+3\right)} \qquad P_w=1 \quad P=2 \qquad L_{eq}(j0)=\infty\angle-90^o \quad L_{eq}(j\infty)=0\angle-270^o$$

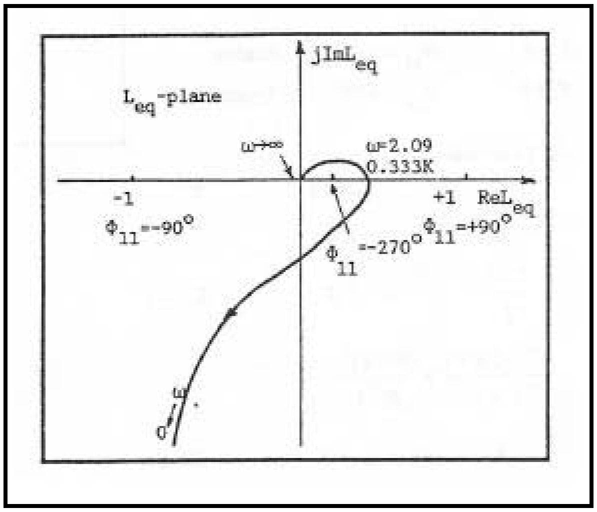
$$L_{eq}(jw)=\frac{K(jw+2)}{\left(w^4-3w^2\right)+j3w}=\frac{K\left[\left(2w^4-3w^2\right)+jw\left(w^4-3w^2-6\right)\right]}{\left(w^4-3w^2\right)^2+4w^2}$$

Setting $\text{Im}\left[L_{eq}(jw)\right]=0 \qquad w^4-3w^2-6=0$

The real positive solution is $w=2.09$ rad/sec

$L_{eq}(j2.09)=0.333K$

For stability, $\Phi_{11}=-\left(0.5P_w+P\right)180^\circ=-450^\circ$



$K>0$	$\Phi_{11}=-90^o$	**Unstable**
$K<-3$	$\Phi_{11}=+90^o$	**Unstable**
$0>K>-3$	$\Phi_{11}=-270^o$	**Unstable**

The system is unstable for all values of *K*.

Routh Tabulation

s^4	1	3	$2K$	
s^3	e	$3+K$		
s^2	$\dfrac{3e-3-K}{e}\cong\dfrac{-3-K}{e}$	$2K$		$K<-3$
s^1	$\dfrac{-(3+K)-2Ke^2}{-(3+K)}\cong 1$			
s^o	$2K$			$K>0$

The conditions contradict. The system is unstable for all values of *K*.

Appendix I DISCRETE-DATA CONTROL SYSTEMS

I-1 **(a)**

$$F(z)=\frac{ze^{-3}}{\left(z-e^{-3}\right)^2}$$

(b)

$$F(z)=\frac{1}{2j}\left[\frac{ze^{2j}}{\left(z-e^{2j}\right)^2}-\frac{ze^{-2j}}{\left(z-e^{-2j}\right)^2}\right]$$

(c)

$$F(z)=\frac{1}{2j}\left[\frac{z}{z-e^{-(2-jw)}}-\frac{z}{z-e^{-(2+jw)}}\right]$$

(d)

$$F(z)=\frac{ze^{-2}\left(z+e^{-2}\right)}{\left(z-e^{-2}\right)^3}$$

I-2 **(a)**

$$F(z)=\frac{T}{2j}\left[\frac{ze^{j2T}}{\left(z-e^{j2T}\right)^2}-\frac{ze^{-j2T}}{\left(z-e^{-j2T}\right)^2}\right]$$

(b)

$$F(z)=1-z^{-1}+z^{-2}-z^{-3}+z^{-4}-\cdots \qquad (1) \qquad z^{-1}F(z)=z^{-1}-z^{-2}+z^{-3}-z^{-4}+\cdots \qquad (2)$$

Add Eqs. (1) and (2). We have $\left(1+z^{-1}\right)F(z)=1$ Or, $F(z)=z/(z+1)$

I-3 **(a)**

$$\mathcal{Z}\left[\frac{1}{(s+5)^3}\right]=\frac{(-1)^2}{2!}\frac{\P^2}{\P a^2}\left[\frac{z}{z-e^{-aT}}\right]\Bigg|_{a=5}=\frac{T^2e^{-5T}z}{2\left(z-e^{-5T}\right)^2}+\frac{T^2e^{-10T}z}{\left(z-e^{-5T}\right)^3}$$

(b)

$$F(s)=\frac{1}{s^3(s+1)}=\frac{1}{s}-\frac{1}{s^2}+\frac{1}{s^3}-\frac{1}{s+1}$$

$$F(z)=\frac{z}{z-1}-\frac{Tz}{(z-1)^2}+\frac{T^2z(z+1)}{2(z-1)^3}-\frac{z}{z-e^{-T}}$$

(c)

$$F(s)=\frac{10}{s(s+5)^2}=\frac{10}{25s}-\frac{10}{25(s+5)}-\frac{2}{(s+5)^2}$$

$$F(z)=\frac{10z}{25(z-1)}-\frac{10z}{25\left(z-e^{-5T}\right)}-\frac{2Tze^{-5T}}{\left(z-e^{-5T}\right)^2}$$

(d)

$$F(s)=\frac{5}{s\left(s^2+2\right)}=\frac{2.5}{s}-\frac{2.5s}{s^2+2}$$

$$F(z)=\frac{2.5z}{z-1}-\frac{2.5z(z-\cos wT)}{z^2-2z\cos wT+1}$$

I-4 (a)

$$F(z) = \frac{10z}{(z-1)(z-0.2)} = \frac{12.5z}{z-1} - \frac{12.5z}{z-0.2} \qquad f(k) = 12.5\left[1-(0.2)^k\right] \qquad k = 0, 1, 2, \cdots$$

(b)

$$F(z) = \frac{z}{(z-1)(z^2+z+1)} = \frac{z}{3(z-1)} - \frac{(0.1667 + j0.2887)z}{(z+0.5+j0.866)} - \frac{(0.1667 - j0.2887)z}{(z+0.5-j0.866)}$$

$$f(k) = 0.333 - (0.1667 + j0.2887)e^{-j2kp/3} - (0.1667 - j0.2887)e^{j2kp/3}$$

$$= 0.333 - 0.333\cos\left(\frac{2kp}{3}\right) - 0.576\sin\left(\frac{2kp}{3}\right) = 0.333 + 0.666\cos\left(\frac{2kp}{3} + 240^0\right) \quad k = 0, 1, 2, \cdots$$

(c)

$$F(z) = \frac{z}{(z-1)(z+0.85)} = \frac{0.541z}{z-1} - \frac{0.541z}{z+0.85} \qquad f(k) = 0.541\left[1-(-0.85)^k\right] \qquad k = 0, 1, 2, \cdots$$

(d)

$$F(z) = \frac{10}{(z-1)(z-0.5)} = \frac{20}{z-1} - \frac{20}{z-0.5} \qquad f(0) = 0 \qquad f(k) = 20\left[1-(0.5)^{k-1}\right] \qquad k = 0, 1, 2, \cdots$$

I-5 (a)

$$\lim_{k\to\infty} f(k) = \lim_{z\to 1}\left(1-z^{-1}\right)F(z) = \lim_{z\to 1}\frac{0.368}{z^2 - 1.364z + 0.732} = 1$$

Expand $F(z)$ into a power series of z^{-1}.

k	$f(k)$	k	$f(k)$
1	0.000000	16	1.109545
2	0.000000	17	1.089310
3	0.368000	18	1.041630
4	0.869952	19	0.991406
5	1.285238	20	0.957803
6	1.484260	21	0.948732
7	1.451736	22	0.960956
8	1.261688	23	0.984269
9	1.026271	24	1.007121
10	0.844277	25	1.021225
11	0.768362	26	1.023736
12	0.798033	27	1.016836
13	0.894075	28	1.005586
14	1.003357	29	0.995292
15	1.082114	30	0.989487

(b)

$F(z)=\dfrac{10z}{(z-1)(z+1)}$ The function $\left(1-z^{-1}\right)F(z)$ has a pole at $z=-1$, so the final-value theorem cannot be applied. The response $f(k)$ oscillates between 0 and 10 as shown below.

k	$f(k)$	k	$f(k)$
1	0.000000	11	0.000000
2	10.000000	12	10.000000
3	0.000000	13	0.000000
4	10.000000	14	10.000000
5	0.000000	15	0.000000
6	10.000000	16	10.000000
7	0.000000	17	0.000000
8	10.000000	18	10.000000
9	0.000000	19	0.000000
10	10.000000	20	10.000000

(c)

$$\lim_{k\to\infty} f(k)=\lim_{z\to1}\left(1-z^{-1}\right)F(z)=\lim_{z\to1}\frac{z}{(z-0.5)}=2$$

F(z) is expanded into a power series of z^{-1} by long division.

k	$f(k)$	k	$f(k)$
1	1.000000	11	1.999023
2	1.500000	12	1.999512
3	1.750000	13	1.999756
4	1.875000	14	1.999878
5	1.937500	15	1.999939
6	1.968750	16	1.999969
7	1.984375	17	1.999985
8	1.992188	18	1.999992
9	1.996094	19	1.999996
10	1.998047	20	1.999998

(d)

$F(z)=\dfrac{z}{(z-1)(z-1.5)}$ $F(z)$ has a pole outside the unit circle. $f(k)$ is unbounded.

The final-value theorem cannot be applied to $F(z)$. $F(z)$ is expanded into a power series of z^{-1}.

k	$f(k)$	k	$f(k)$
1	0.000000	11	113.33078
2	1.000000	12	170.995117
3	2.500000	13	257.492676
4	4.750000	14	387.239014
5	8.125000	15	581.858521
6	13.187500	16	873.787781

7	20.781250	17	1311.681641
8	32.171875	18	1968.522461
9	49.257813	19	2953.783691
10	74.886719	20	4431.675781

I-6 (a) $x(k+2)-x(k+1)+0.1x(k)=u_s(k) \qquad x(0)=x(1)=0$. Taking the z-transform on both sides,

$$z^2X(z)-z^2x(0)-zx(1)-zX(z)+zx(0)+0.1X(z)=\frac{z}{z-1}$$

$$X(z)=\frac{z}{(z-1)\left(z^2-z+0.1\right)}=\frac{10z}{z-1}+\frac{1.455z}{z-0.1127}-\frac{11.455z}{z-0.8873}$$ Taking the inverse z-transform,

$$x(k)=10+1.455(0.1127)^k-11.455(0.8873)^k \qquad k=0,1,2,\cdots$$

(b) $x(k+2)-x(k)=0 \qquad x(0)=1, \quad x(1)=0$ Taking the z-transform on both sides, we have

$$z^2X(z)-z^2x(0)-zx(1)-X(z)=0 \qquad X(z)=\frac{z^2}{z^2-1} \qquad x(k)=\cos\left(\frac{kp}{2}\right) \qquad k=0,1,2,\cdots$$

I-7 (a)

$P(1)=(1+r)P(0)-u$ where $P(1)=$ amount owed after the first period.

$P(2)=(1+r)P(1)-u \qquad P(0)=P_0=$ amount borrowed initially

$\vdots$

$P(k+1)=(1+r)P(k)-u \qquad u=$ amount paid each period including principal and interest.

(b) By direct substitution,

$$P(2)=(1+r)P(1)-u=(1+r)^2P(0)-(1+r)u-u$$

$$P(3)=(1+r)P(2)-u=(1+r)^3P(0)-(1+r)^2u-(1+r)u-u$$

$$\vdots$$

$$P(N)=(1+r)^NP(0)-u\left[(1+r)^{N-1}+(1+r)^{N-2}+\cdots+(1+r)+1\right]=0$$

Solving for u from the last equation, we have

$$u=\frac{(1+r)^NP_0}{\sum_{k=0}^{N-1}(1+r)^k}=\frac{(1+r)^NP_0r}{(1+r)^N-1}$$

(c) Taking the z-transform on both sides of the difference equation, $P(k+1)=(1+r)P(k)-u$, we have

$$\sum_{k=0}^{\infty}P(k+1)z^{-k}=(1+r)\sum_{k=0}^{\infty}P(k)z^{-k}-\frac{uz}{z-1}$$

Or, $zP(z) - zP_0 = (1+r)P(z) - \dfrac{uz}{z-1}$ Thus, $P(z) = \dfrac{zP_0}{z-(1+r)} - \dfrac{uz}{(z-1)\left[z-(1+r)\right]}$

Taking the inverse z-transform and setting $k = N$, we have

$$P(N) = P_0(1+r)^N + \frac{u}{r} - \frac{u}{r}(1+r)^N = 0$$

Solving for u from the last equation, we have

$$u = \frac{(1+r)^N P_0 r}{(1+r)^N - 1}$$

(d) For $P_0 = \$15{,}000$, $r = 0.01$, and $N = 48$ months,

$$u = \frac{(1+r)^N P_0 r}{(1+r)^N - 1} = \frac{(1.01)^{48}(15000)(0.01)}{(1.01)^{48} - 1} = \frac{(1.612)(15000)(0.01)}{1.612 - 1} = \$395.15$$

I-8 **(a)**

$$G(z) = \frac{5.556}{z-1} - \frac{5.556}{z-0.1}$$

(b)

$$G(z) = \frac{20z}{z-0.5} - \frac{20z}{z-0.8} + \frac{80z}{z-1}$$

(c)

$$G(z) = \frac{4z}{z-1} - \frac{4z}{z-0.5} - \frac{2z}{(z-0.5)^2}$$

(d)

$$G(z) = \frac{2z}{z-1} - \frac{2z^2}{(z-0.5)^2}$$

I-9 $\lim\limits_{k\to\infty} f(k) = \lim\limits_{z\to 1} \left[(1 - z^{-1})F(z)\right] = \lim\limits_{z\to 1} \dfrac{z}{(z-0.5)} = 2$

F(z) is expanded into a power series of z^{-1} by long division.

k	$f(k)$	k	$f(k)$
1	1.000000	11	1.999023
2	1.500000	12	1.999512
3	1.750000	13	1.999756
4	1.875000	14	1.999878
5	1.937500	15	1.999939
6	1.968750	16	1.999969
7	1.984375	17	1.999985
8	1.992188	18	1.999992
9	1.996094	19	1.999996
10	1.998047	20	1.999998

(d)

$$F(z) = \frac{z}{(z-1)(z-1.5)}$$ $F(z)$ has a pole outside the unit circle. $f(k)$ is unbounded.

The final-value theorem cannot be applied to $F(z)$. $F(z)$ is expanded into a power series of z^{-1}.

k	$f(k)$	k	$f(k)$
1	0.000000	11	113.33078
2	1.000000	12	170.995117
3	2.500000	13	257.492676
4	4.750000	14	387.239014
5	8.125000	15	581.858521
6	13.187500	16	873.787781
7	20.781250	17	1311.681641
8	32.171875	18	1968.522461
9	49.257813	19	2953.783691
10	74.886719	20	4431.675781

2-18 (a) $x(k+2) - x(k+1) + 0.1x(k) = u_s(k)$ $\quad x(0) = x(1) = 0$. Taking the z-transform on both sides,

$$z^2 X(z) - z^2 x(0) - zx(1) - zX(z) + zx(0) + 0.1X(z) = \frac{z}{z-1}$$

$$X(z) = \frac{z}{(z-1)(z^2 - z + 0.1)} = \frac{10z}{z-1} + \frac{1.455z}{z-0.1127} - \frac{11.455z}{z-0.8873}$$ Taking the inverse z-transform,

$$x(k) = 10 + 1.455(0.1127)^k - 11.455(0.8873)^k \qquad k = 0, 1, 2, \cdots$$

(b) $x(k+2) - x(k) = 0$ $\quad x(0) = 1, \quad x(1) = 0$ Taking the z-transform on both sides, we have

$$z^2 X(z) - z^2 x(0) - zx(1) - X(z) = 0 \qquad X(z) = \frac{z^2}{z^2 - 1} \qquad x(k) = \cos\left(\frac{\pi}{2}k\right) \quad k = 0, 1, 2, \cdots$$

2-19 (a)

$P(1) = (1+r)P(0) - u$ where $P(1) =$ amount owed after the first period.

$P(2) = (1+r)P(1) - u$ $\quad P(0) = P_0 =$ amount borrowed initially

$\vdots$

$P(k+1) = (1+r)P(k) - u$ $\quad u =$ amount paid each period including principal and interest.

(b) By direct substitution,

$$P(2) = (1+r)P(1) - u = (1+r)^2 P(0) - (1+r)u - u$$

$$P(3) = (1+r)P(2) - u = (1+r)^3 P(0) - (1+r)^2 u - (1+r)u - u$$

$$\vdots$$

$$P(N) = (1+r)^N P(0) - u\left[(1+r)^{N-1} + (1+r)^{N-2} + \cdots + (1+r) + 1\right] = 0$$

Solving for u from the last equation, we have

$$u = \frac{(1+r)^N P_0}{\sum_{k=0}^{N-1}(1+r)^k} = \frac{(1+r)^N P_0 r}{(1+r)^N - 1}$$

(c) Taking the z-transform on both sides of the difference equation, $P(k+1) = (1+r)P(k) - u$, we have

$$\sum_{k=0}^{\infty} P(k+1) z^{-k} = (1+r)\sum_{k=0}^{\infty} P(k) z^{-k} - \frac{uz}{z-1}$$

Or,

$$zP(z) - zP_0 = (1+r)P(z) - \frac{uz}{z-1} \qquad \text{Thus,} \quad P(z) = \frac{zP_0}{z-(1+r)} - \frac{uz}{(z-1)\left[z-(1+r)\right]}$$

Taking the inverse z-transform and setting $k = N$, we have

$$P(N) = P_0(1+r)^N + \frac{u}{r} - \frac{u}{r}(1+r)^N = 0$$

Solving for u from the last equation, we have

$$u = \frac{(1+r)^N P_0 r}{(1+r)^N - 1}$$

(d) For $P_0 = \$15{,}000$, $r = 0.01$, and $N = 48$ months,

$$u = \frac{(1+r)^N P_0 r}{(1+r)^N - 1} = \frac{(1.01)^{48}(15000)(0.01)}{(1.01)^{48} - 1} = \frac{(1.612)(15000)(0.01)}{1.612 - 1} = \$395.15$$

I-9

Taking the z-transform of $y(kT)$, we have

$$Y(z) = \sum_{k=0}^{\infty}\left(1-e^{-2kT}\right)z^{-k} = \frac{z}{z-1} - \frac{z}{z-e^{-2T}} = \frac{z\left(1-e^{-2T}\right)}{(z-1)\left(z-e^{-2T}\right)} \qquad R(z) = \frac{z}{z-1} \qquad G(z) = \frac{Y(z)}{R(z)} = \frac{1-e^{-2T}}{z-e^{-2T}}$$

I-10

(a)

$$\frac{Y(z)}{R(z)} = \mathcal{Z}\left(\frac{1}{s(s+2)}\right) = \mathcal{Z}\left(\frac{1}{2s} - \frac{1}{2(s+2)}\right) = \frac{0.5z}{z-1} - \frac{0.5z}{z-e^{-1}} = \frac{0.316z}{(z-1)(z-0.368)}$$

(b)

$$\frac{Y(z)}{R(z)} = \mathcal{Z}\left(\frac{10}{(s+1)(s+2)}\right) = \mathcal{Z}\left(\frac{10}{s+1} - \frac{10}{s+2}\right) = \frac{10z}{z-0.607} - \frac{10z}{z-0.368} = \frac{2.387z}{(z-0.607)(z-0.368)}$$

(c)

$$\frac{Y(z)}{R(z)} = \mathcal{Z}\left(\frac{1}{s+1}\right)\mathcal{Z}\left(\frac{1}{s+2}\right) = \frac{z}{z-1} - \frac{10z}{z-0.368} = \frac{10z^2}{(z-1)(z-0.368)}$$

(d)

$$\frac{Y(z)}{R(z)} = \left(1-z^{-1}\right)\mathcal{Z}\left(\frac{5}{s^2(s+2)}\right) = \left(1-z^{-1}\right)\mathcal{Z}\left(\frac{1.25z}{(z-1)^2} - \frac{1.25z}{z-1} + \frac{1.25z}{z-0.368}\right) = \frac{0.46z+0.33}{(z-1)(z-0.368)}$$

(e) From Part (d),

$$G(z) = \frac{Y(z)}{E(z)} = \left(1-z^{-1}\right)\mathcal{Z}\left(\frac{5}{s^2(s+2)}\right) = \frac{0.46z+0.33}{(z-1)(z-0.368)}$$

$$\frac{Y(z)}{R(z)} = \frac{G(z)}{1+G(z)} = \frac{0.46z+0.33}{z^2-0.908z+0.6982}$$

(f)

$$G(z) = \left(1-z^{-1}\right)\mathcal{Z}\left(\frac{5}{s^2(s+1)(s+2)}\right) = \left(1-z^{-1}\right)\mathcal{Z}\left(\frac{2.5}{s^2} - \frac{3.75}{s} + \frac{5}{s+1} - \frac{1.25}{s+2}\right)$$

$$G(z) = \frac{0.0728z^2+0.2037z+0.0344}{z^3-1.9744z^2+1.1975z-0.2232} = \frac{0.0728z^2+0.2037z+0.0344}{(z-1)(z-0.607)(z-0.368)}$$

$$\frac{Y(z)}{R(z)} = \frac{0.0728z^2+0.2037z+0.0344}{z^3-1.9z^2+1.4z-0.189}$$

I-11

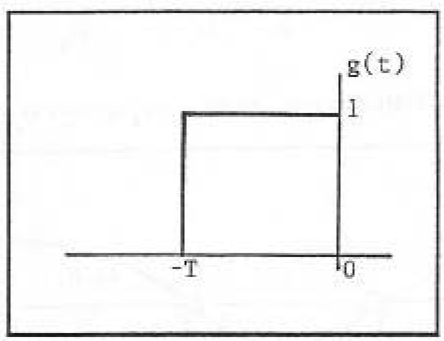
(a) $y(kT) = y[(k-1)T] + Tx(kT)$ Taking the z-transform on both sides of the equation, we have

$$\sum_{k=0}^{\infty} y(kT)z^{-k} = \sum_{k=0}^{\infty} y[(k-1)T]z^{-k} + T\sum_{k=0}^{\infty} x(kT)z^{-k}$$

$$Y(z) = z^{-1}Y(z) + TX(z) \qquad Y(z) = \frac{T}{1-z^{-1}}X(z) \qquad G(z) = \frac{Y(z)}{X(z)} = \frac{Tz}{z-1}$$

(b) The impulse response of the backward rectangular hold is shown below. The impulse response is written $g(t) = u_s(t-T) - u_s(t)$. The transfer function is

$$G(s) = \frac{e^{Ts}-1}{s}$$



Then,

$$Y(z) = \mathcal{Z}\left(\frac{e^{Ts}-1}{s}\frac{1}{s}\right) = (z-1)\mathcal{Z}\left(\frac{1}{s^2}\right) = \frac{Tz}{z-1}$$

(c)

$$Y(z) = \left(1-z^{-1}\right)\mathcal{Z}\left(\frac{1}{s^2}\right) = \left(1-z^{-1}\right)\frac{Tz}{(z-1)^2} = \frac{T}{z-1}$$

I-12 **(a)** **Discrete state equations:**

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \left(s\mathbf{I} - \mathbf{A}\right) = \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix} \qquad \left(s\mathbf{I} - \mathbf{A}\right)^{-1} = \frac{1}{(s+1)(s+2)} \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}$$

$$f(t) = \mathcal{L}^{-1}\left[\left(s\mathbf{I} - \mathbf{A}\right)^{-1}\right] = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix} \qquad f(T) = \begin{bmatrix} 2e^{-T} - e^{-2T} & e^{-T} - e^{-2T} \\ -2e^{-T} + 2e^{-2T} & -e^{-T} + 2e^{-2T} \end{bmatrix}$$

$T = 1$ sec. $\qquad f(T) = \begin{bmatrix} 0.6 & 0.2325 \\ -0.465 & -0.0972 \end{bmatrix}$

$$q(t) = \mathcal{L}^{-1}\left[\left(s\mathbf{I} - \mathbf{A}\right)^{-1}\mathbf{B}\right] = \mathcal{L}^{-1}\left[\frac{1}{(s+1)(s+2)}\begin{bmatrix} 1 \\ s \end{bmatrix}\right] = \begin{bmatrix} e^{-t} - e^{-2t} \\ -e^{-t} + 2e^{-2t} \end{bmatrix} \qquad q(T) = \begin{bmatrix} e^{-T} - e^{-2T} \\ -e^{-T} + 2e^{-2T} \end{bmatrix}$$

$T = 1$ sec. $\qquad q(T) = \begin{bmatrix} 0.2325 \\ -0.0972 \end{bmatrix}$

(b)

$$f(NT) = \begin{bmatrix} 2e^{-NT} - e^{-2NT} & e^{-NT} - e^{-2NT} \\ -2e^{-NT} + e^{-2NT} & -e^{-NT} + 2e^{-2NT} \end{bmatrix} = \begin{bmatrix} 2e^{-N} - e^{-2N} & e^{-N} - e^{-2N} \\ -2e^{-N} + e^{-2N} & -e^{-N} + 2e^{-2N} \end{bmatrix}$$

$$\mathbf{x}(NT) = f(NT)\mathbf{x}(0) + \sum_{k=0}^{N-1} f\left[\left(N-k-1\right)T\right]q(T)u(kT)$$

I-13 **(a)** **Discrete state equations:**

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \left(s\mathbf{I} - \mathbf{A}\right) = \begin{bmatrix} s & -1 \\ 0 & s \end{bmatrix} \qquad \left(s\mathbf{I} - \mathbf{A}\right)^{-1} = \frac{1}{s}\begin{bmatrix} s & 1 \\ 0 & s \end{bmatrix} = \begin{bmatrix} \frac{1}{s} & \frac{1}{s^2} \\ 0 & \frac{1}{s} \end{bmatrix}$$

$$f(t) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \qquad f(T) = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0.001 \\ 0 & 1 \end{bmatrix}$$

$$q(t) = \mathcal{L}^{-1}\left[\left(s\mathbf{I} - \mathbf{A}\right)^{-1}\mathbf{B}\right] = \mathcal{L}^{-1}\begin{bmatrix} \frac{1}{s^2} \\ \frac{1}{s} \end{bmatrix} = \begin{bmatrix} t \\ 1 \end{bmatrix} \qquad q(T) = \begin{bmatrix} T \\ 1 \end{bmatrix} = \begin{bmatrix} 0.001 \\ 1 \end{bmatrix}$$

(b) $\qquad f(NT) = \begin{bmatrix} 1 & NT \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0.001N \\ 0 & 1 \end{bmatrix}$

$$\mathbf{x}(NT) = f(NT)\mathbf{x}(0) + \sum_{k=0}^{N-1} f\left[\left(N-k-1\right)T\right]q(T)u(kT)$$

$$f\left[(N-k-1)T\right]=\begin{bmatrix}1 & (N-k-1)T\\ 0 & 1\end{bmatrix}=\begin{bmatrix}1 & 0.001(N-k-1)\\ 0 & 1\end{bmatrix}$$

$$f\left[(N-k-1)T\right]q(T)=\begin{bmatrix}0.001(N-k)\\ 1\end{bmatrix}$$

I-14 **(a)** **Transfer function:**

$$\frac{\mathbf{X}(z)}{U(z)}=\left[z\mathbf{I}-f(T)\right]^{-1}q(T)=\begin{bmatrix}z-0.6 & -0.2325\\ 0.465 & z+0.0972\end{bmatrix}^{-1}\begin{bmatrix}0.2325\\ -0.0972\end{bmatrix}=\frac{1}{\Delta(z)}\begin{bmatrix}z+0.0972 & 0.2325\\ -0.465 & z-0.6\end{bmatrix}\begin{bmatrix}0.2325\\ -0.0972\end{bmatrix}$$

$$\frac{\mathbf{X}(z)}{U(z)}=\frac{1}{\Delta(z)}\begin{bmatrix}0.2325z\\ -0.0972(z+0.6673)\end{bmatrix}\qquad \Delta(z)=z^2-0.5028z+0.04978$$

(b) **Characteristic equation:**

$$\Delta(z)=z^2-0.5028z+0.04978=0$$

I-15 **(a)** **Transfer function:**

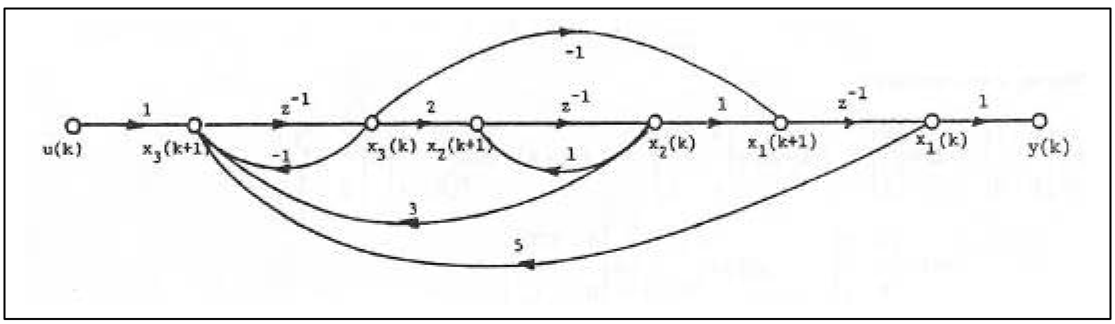
$$\frac{\mathbf{X}(z)}{U(z)}=\left[z\mathbf{I}-f(T)\right]^{-1}q(T)=\frac{1}{\Delta(z)}\begin{bmatrix}z-1 & 0.001\\ 0 & z-1\end{bmatrix}\begin{bmatrix}0.001\\ 1\end{bmatrix}=\frac{1}{\Delta(z)}\begin{bmatrix}0.001z\\ z-1\end{bmatrix}$$

$$\Delta(z)=(z-1)^2$$

(b) **Characteristic equation:**

$$\Delta(z)=(z-1)^2=0$$

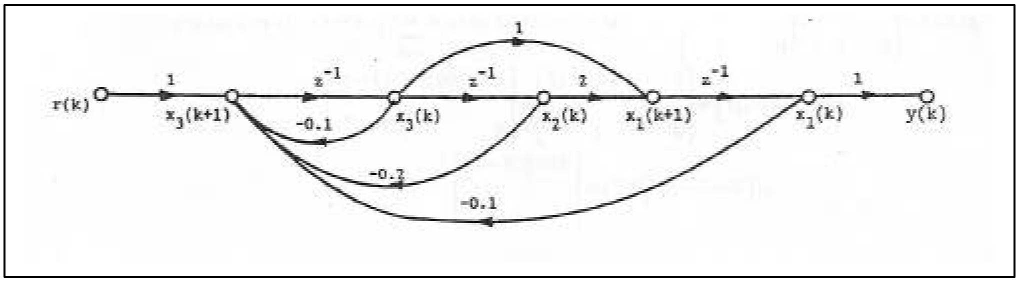
I-16 **State diagram:**



$$\Delta(z)=z^{-1}-z^{-1}-6z^{-2}-z^{-2}+5z^{-2}-10z^{-3}-5z^{-3}=-2z^{-2}-15z^{-3}=0$$

Characteristic equation: $2z+15=0$

I-17 State diagram:



State equations:

$x_1(k+1) = 2x_2(k) + x_3(k)$

$x_2(k+1) = x_3(k)$

$x_3(k+1) = -0.1x_1(k) - 0.2x_2(k) - 0.1x_3(k) + r(k)$

Output equation:

$y(k) = x_1(k)$

Transfer function:

$$\frac{Y(z)}{R(z)} = \frac{z^{-2} + z^{-3}}{1 + 0.1z^{-1} + 0.2z^{-2} + 0.2z^{-3}} = \frac{z+2}{z^3 + 0.1z^2 + 0.2z + 0.2}$$

I-18

Open-loop transfer function:

$$G(z) = \frac{Y(z)}{E(z)} = \left(1 - z^{-1}\right)\mathcal{Z}\left(\frac{1}{s(s+1)}\right) = \frac{0.632}{z - 0.368}$$

Closed-loop transfer function:

$$\frac{Y(z)}{R(z)} = \frac{G(z)}{1 + G(z)} = \frac{0.632}{z - 0.264}$$

Discrete-data state equation:

$$x[(k+1)T] = 0.264\, x(kT) + 0.632\, r(kT)$$

I-19 **(a)** $F(z) = z^2 + 1.5z - 1 = 0$ This is a second-order system, $n = 2$.

$F(1) = 1.5 > 0 \quad F(-1) = -1.5 < 0$

Thus, for $n = 2$ = even, $F(-1) < 0$. The system is unstable.
The characteristic equation roots are at $z = 0.5$ and $z = -2$.
$F(z)$ has one root outside the unit circle.

Let $z = \frac{w+1}{w-1}$. The characteristic equation becomes $\left(\frac{w+1}{w-1}\right)^2 + 1.5\left(\frac{w+1}{w-1}\right) - 1 = 0$

Or $\left(w+1\right)^2 + 1.5\left(w^2 - 1\right) - \left(w-1\right)^2 = 0$ or $1.5w^2 + 4w - 1.5 = 0$

The last coefficient of the last equation is negative. Thus, the system is unstable.

(b) $F(z) = z^3 + z^2 + 3z + 0.2 = 0$ This is a third-order system, $n = 3$.

Let $z = \dfrac{w+1}{w-1}$. The characteristic equation becomes

$$\left(\frac{w+1}{w-1}\right)^3 + \left(\frac{w+1}{w-1}\right)^2 + 3\left(\frac{w+1}{w-1}\right) + 0.2 = 0$$

Or

$(w+1)^3 + (w+1)^2(w-1) + 3(w+1)(w-1)^2 + 0.2(w-1)^3 = 0$ or $5.2w^3 + 0.4w^2 - 0.4w + 2.8 = 0$

Since there is a negative sign in the characteristic equation in w, the equation has at least one root in the right-half w-plane.

Routh Tabulation:

w^3	5.2	-0.4
w^2	0.4	2.8
w^1	-36.8	
w^0	2.8	

Since there are two sign changes in the first column of the Routh tabulation, $F(z)$ has two roots outside the unit circle. The three roots are at: –0.0681, –0.466 + j1.649 and –0.466 – j1.649.

(c) $F(z) = z^3 - 1.2z^2 - 2z + 3 = 0$

Let $z = \dfrac{w+1}{w-1}$. The characteristic equation becomes

$$\left(\frac{w+1}{w-1}\right)^3 - 1.2\left(\frac{w+1}{w-1}\right)^2 - 2\left(\frac{w+1}{w-1}\right) + 3 = 0$$

Or $\quad 0.8w^3 - 5.2w^2 + 15.2w - 2.8 = 0$

Routh Tabulation:

w^3	0.8	15.2
w^2	-5.2	-2.8
w^1	14.77	
w^0	-2.8	

There are three sign changes in the first column of the Routh tabulation. Thus, $F(z)$ has three roots outside the unit circle. The three roots are at –1.491, 1.3455 + j0.4492, and 1.3455 – j0.4492.

(d) $F(z) = z^3 - z^2 - 2z + 0.5 = 0$

Let $z = \dfrac{w+1}{w-1}$. The characteristic equation becomes

$$\left(\frac{w+1}{w-1}\right)^3 - \left(\frac{w+1}{w-1}\right)^2 - 2\left(\frac{w+1}{w-1}\right) + 0.5 = 0$$

Or $\quad -1.5w^3 + 2.5w^2 + 7.5w - 0.5 = 0$

Routh Tabulation:

w^3	-1.5	7.5
w^2	2.5	-0.5
w^1	7.2	
w^0	-0.5	

Since there are two sign changes in the first column of the Routh tabulation, $F(z)$ has three zeros outside the unit circle. The three roots are at z = 1.91, –1.1397, and 0.2297.

I-20 Taking the z-transform of the state equation, we have

$$zX(z) = (0.368 - 0.632K)X(z) + KR(z)$$

Or,
$$\frac{X(z)}{R(z)} = \frac{K}{z - 0.368 + 0.632K}$$

The characteristic equation is $z - 0.368 + 0.632K = 0$ The root is $z = 0.368 - 0.632K$

Stability Condition:

$$|0.368 - 0.632K| < 1 \quad \text{or} \quad -1 < K < 2.165$$

I-21 $z^3 + z^2 + 1.5Kz - (K + 0.5) = 0$

Let $z = \dfrac{w+1}{w-1}$. The characteristic equation becomes

$$\left(\frac{w+1}{w-1}\right)^3 + \left(\frac{w+1}{w-1}\right)^2 + 1.5K\left(\frac{w+1}{w-1}\right) - (K + 0.5) = 0$$

Or $(1.5 + 0.5K)w^3 + (5.5 + 1.5K)w^2 + (0.5 - 4.5K)w + (2.5K + 0.5) = 0$

Routh Tabulation:

w^3	$1.5 + 0.5K$	$0.5 - 4.5K$	
w^2	$5.5 + 1.5K$	$2.5K + 0.5$	
w^1	$\dfrac{1 - 14K - 4K^2}{5.5 + 1.5K}$		$(K - 0.07)(K + 3.57) < 0$
w^0	$2.5K + 0.5$		$2.5K + 0.5 > 0$

Stability Condition: $-0.2 < K < 0.07$

I-22 **(a)** **Forward-path transfer function:**

$$G_{ho}G_p(z) = \left(1 - z^{-1}\right)\mathcal{Z}\left(\frac{K}{s^2(s+1.5)}\right) = \left(1 - z^{-1}\right)K\mathcal{Z}\left(\frac{0.667}{s^2} - \frac{0.444}{s} + \frac{0.444}{s+1.5}\right) = \frac{K(0.00476z + 0.004527)}{z^2 - 1.8607z + 0.8607}$$

Characteristic equation:

$$z^2 - 1.8607z + 0.8607 + 0.00476Kz + 0.004527K = 0$$

For T = 0.1 sec. $z^2 + (0.00476K - 1.8607)z + 0.8607 + 0.004527K = 0$

Let $z = \dfrac{w+1}{w-1}$. The characteristic equation becomes

$$0.009287 w^2 + (0.2786 - 0.009054 K)w + 3.7214 - 0.000233 K = 0$$

For stability, all the coefficients of the characteristic equation must be of the same sign. Thus, the conditons for stability are:

$0.2786 - 0.009054K > 0$ or $K < 30.77$

$3.7214 - 0.000233K > 0$ or $K < 15971.67$

Thus, for stability, $0 < K < 30.77$

(b) ***T* = 0.5 sec. Forward-path transfer function:**

$$G_{ho}G_p(z) = \frac{K(0.09883 z + 0.07705)}{z^2 - 1.4724 z + 0.47237}$$

Characteristic equation:

$$z^2 + (0.09883 K - 1.4724)z + (0.07705 K + 0.47237) = 0$$

Let $z = \dfrac{w+1}{w-1}$. The characteristic equation becomes

$$0.17588 Kw^2 + (1.05526 - 0.1541 K)w + 2.94477 - 0.02178 K = 0$$

For stability, all the coefficients of the characteristic equation must be of the same sign.

Thus, the conditions for stability are: $1.05526 - 0.1541 K > 0$ or $K < 6.8479$

$2.9447 - 0.02178 K > 0$ or $K < 135.2$

Stability condition: $0 < K < 6.8479$

(c) ***T* = 1 sec. Forward-path transfer function:**

$$G_{ho}G_p(z) = \frac{K(0.3214 z + 0.19652)}{z^2 - 1.2231 z + 0.22313}$$

Characteristic equation:

$$z^2 + (0.3214 K - 1.2231)z + (0.22313 + 0.19652 K) = 0$$

Let $z = \dfrac{w+1}{w-1}$. The characteristic equation becomes

$$0.5179 Kw^2 + (1.55374 - 0.393 K)w + 2.4462 - 0.12488 K = 0$$

For stability, all the coefficients of the characteristic equation must be of the same sign.

Thus, the conditions for stability are : $1.55374 - 0.393 K > 0$ or $K < 3.9535$

$2.4462 - 0.12488 K > 0$ or $K < 19.588$

Stability condition: $0 < K < 3.9535$

I-23 **(a)** **Roots:** -1.397 -0.3136 + j0.5167 -0.3136 – j0.5167

Unstable System

(b) **Roots:** 0.3425 -0.6712 + j1.0046 -0.06712 – j1.0046

Unstable System.

(c) **Roots:** -0.4302 -0.7849 + j1.307 -0.7849 – j1.307
Unstable System.

(d) **Roots:** 0.5 -0.8115 -0.0992 + j0.7708 -0.0992 – j0.7708
Stable System.

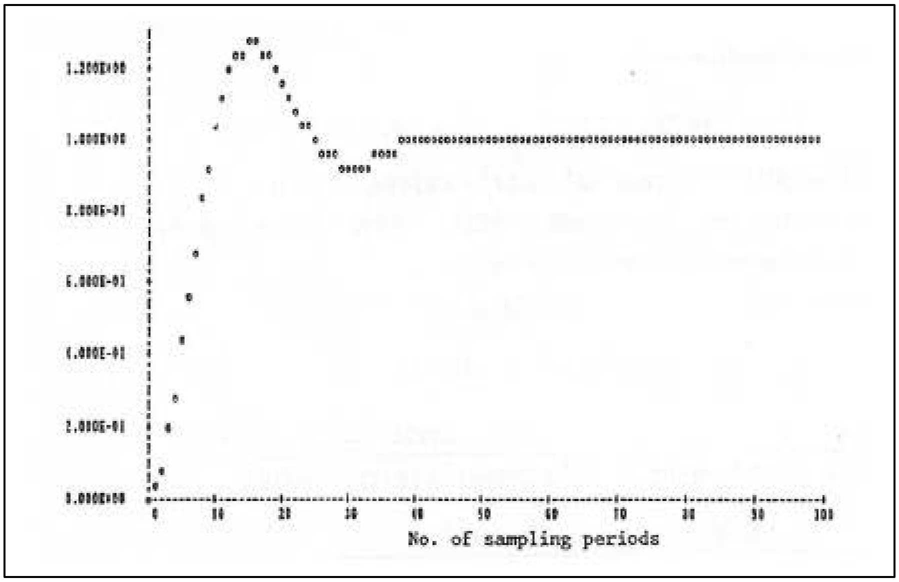
I-24 **(a)** **Forward-path Transfer Function: T = 0.1 sec.**

$$G(z) = \left(1 - z^{-1}\right) \mathcal{Z}\left(\frac{5}{s^2(s+2)}\right) = \left(1 - z^{-1}\right) \mathcal{Z}\left(\frac{2.5}{s^2} - \frac{1.25}{s} + \frac{1.25}{s+2.5}\right) = \frac{0.02341z + 0.021904}{z^2 - 1.8187z + 0.8187}$$

Closed-loop Transfer Function:

$$\frac{Y(z)}{R(z)} = \frac{G(z)}{1 + G(z)} = \frac{0.0234z + 0.021904}{z^2 - 1.7953z + 0.8406}$$

(b) **Unit-Step Response $y(kT)$**



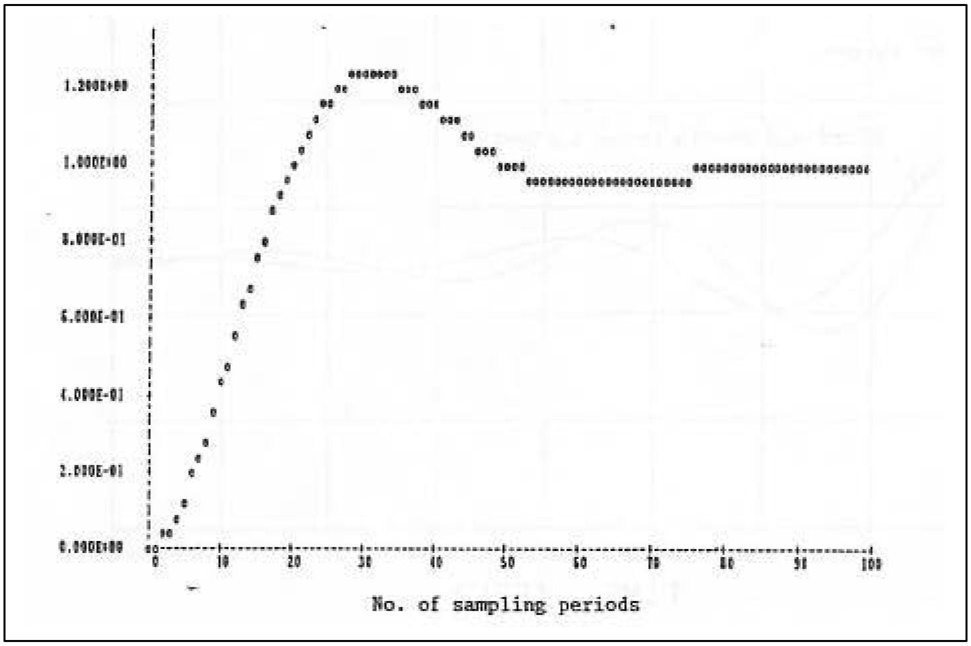
(c) **Forward-path Transfer Function: T = 0.05 sec.**

$$G(z) = \left(1 - z^{-1}\right) \mathcal{Z}\left(\frac{5}{s^2(s+2)}\right) = \frac{0.0060468z + 0.0058484}{z^2 - 1.9048z + 0.9048}$$

Closed-loop Transfer Function:

$$\frac{Y(z)}{R(z)} = \frac{0.006046z + 0.0058484}{z^2 - 1.8988z + 0.91069}$$

Unit-step Response $y(kT)$



I-25 **(a)**

$$Y(z) = \left(1 - z^{-1}\right) \mathcal{Z}\left(\frac{1}{s^3}\right) U(z) \qquad U(z) = 10E(z) - K_t\left(1 - z^{-1}\right)\mathcal{Z}\left(\frac{1}{s^2}\right)U(z)$$

$$U(z) = 10E(z) - \frac{K_t T}{z-1}U(z) \qquad \text{Thus} \quad U(z) = \frac{10(z-1)}{z - 1 + K_t T}E(z)$$

$$Y(z) = \left(1 - z^{-1}\right)\frac{T^2 z(z+1)}{2(z-1)^3}\frac{10(z-1)}{z-1+K_t T}E(z)$$

$$G(z) = \frac{Y(z)}{E(z)} = \frac{5T^2(z+1)}{(z-1)(z-1+K_t T)}$$

Error Constants:

$$K_p^* = \lim_{z\to 1} G(z) = \infty \qquad K_v^* = \frac{1}{T}\lim_{z\to 1}(z-1)G(z) = \frac{10T^2}{K_t T^2} = \frac{10}{K_t}$$

$$K_a^* = \frac{1}{T^2}\lim_{z\to 1}(z-1)^2 G(z) = 0$$

(b) **Forward-path Transfer Function:** **Closed-loop Transfer Function:**

$$G(z) = \frac{5T^2(z+1)}{(z-1)(z-1+K_t T)} \qquad \frac{Y(z)}{R(z)} = \frac{5T^2(z+1)}{z^2 + \left(K_t T + 5T^2 - 2\right)z + \left(5T^2 - K_t T + 1\right)}$$

(c) **Characteristic Equation:** $T = 0.1$ **sec.**

$$F(z) = z^2 + \left(K_t T + 5T^2 - 2\right)z + \left(5T^2 - K_t T + 1\right) = z^2 + \left(0.1K_t - 1.95\right)z + \left(1.05 - 0.1K_t\right) = 0$$

For stability, from Eq. (6-62), $F(1) > 0, \quad F(-1) > 0, \quad |a_0| < a_2$

where $a_0 = 1.05 - 0.1K_t \qquad a_1 = 0.1K_t - 1.95$

Thus, $F(1) = 0.1 > 0 \qquad F(-1) = 4 - 0.2K_t > 0 \quad \text{or} \quad K_t < 20$

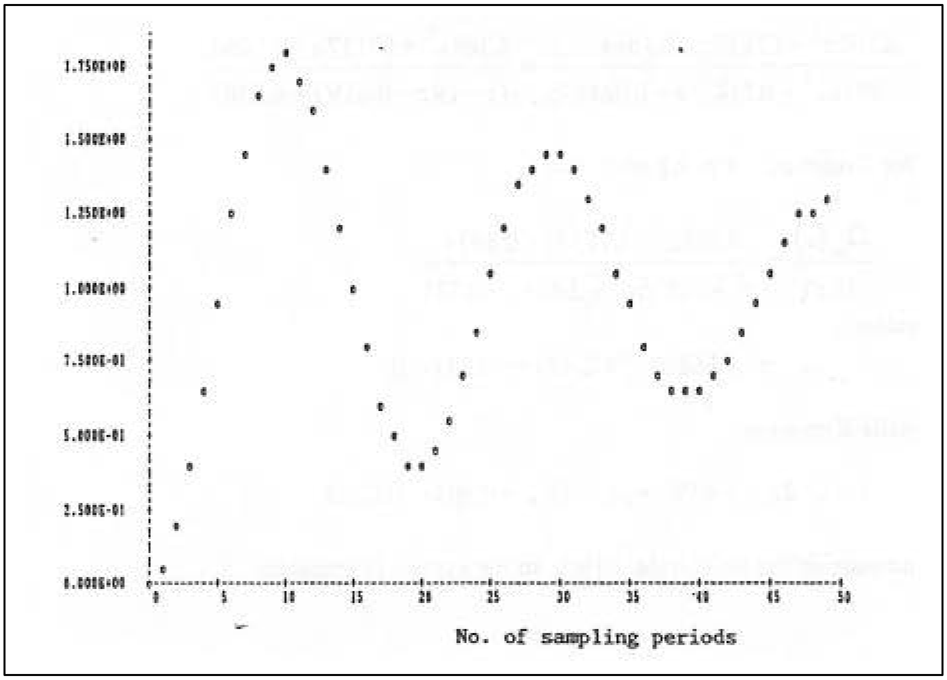
$$|a_0| = |1.05 - 0.1K_t| < a_1 = 1 \quad \text{or} \quad 0.5 < K_t < 20.5$$

Stability Condition:

$$0.5 < K_t < 20$$

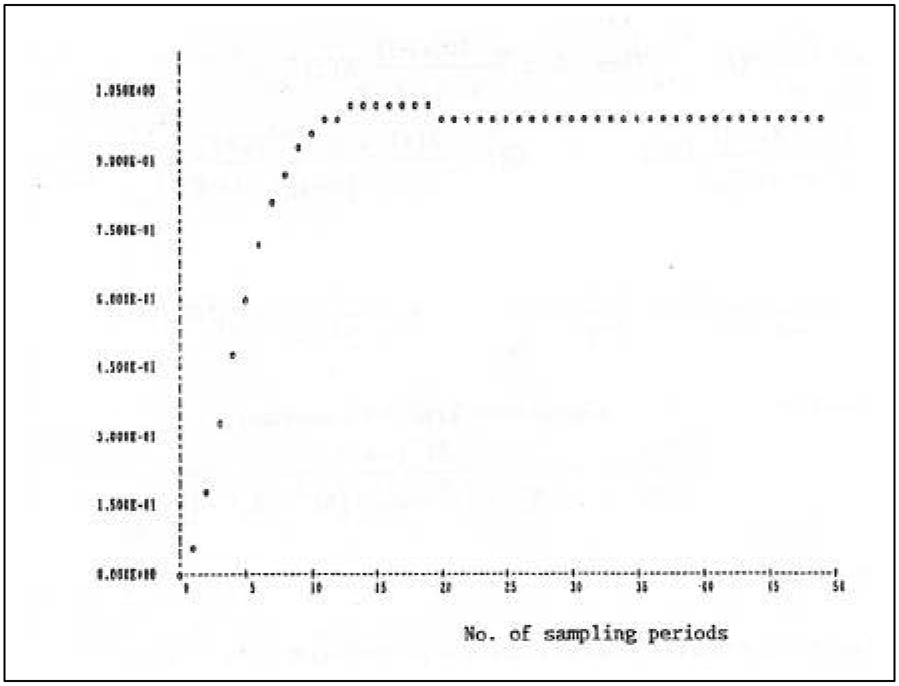
(d) Unit-step Response: $K_t = 5$

$$\frac{Y(z)}{R(z)} = \frac{0.05(z+1)}{z^2 - 1.45z + 0.55}$$



(e) Unit-step Response $K_t = 1$

$$\frac{Y(z)}{R(z)} = \frac{0.05(z+1)}{z^2 - 1.85z + 0.95}$$



I-26 (a) ***T* = 0.1 sec.**

$$G_p(s) = \frac{1000(z+20)}{s^2 + 37.06s + 141.2} \qquad \frac{\Omega_m(z)}{E(z)} = \left(1 - z^{-1}\right) \mathcal{Z}\left(\frac{100(s+20)}{s\left(s^2 + 37.06z + 141.2\right)}\right)$$

$$\frac{\Omega_m(z)}{E(z)} = \left(1 - z^{-1}\right) \mathcal{Z}\left(\frac{-30}{s} + \frac{14.2}{s^2} + \frac{29.7}{s + 4.31} + \frac{0.418}{s + 32.7}\right)$$

$$= \left(1 - z^{-1}\right)\left(\frac{-30z}{z-1} + \frac{14.2z}{(z-1)^2} + \frac{29.7z}{z - 0.65} + \frac{0.418z}{z - 0.038}\right)$$

$$= \frac{3.368z^2 + 1.7117z - 0.3064}{z^3 - 1.6876z^2 + 0.71215z - 0.024576} = \frac{3.368z^2 + 1.7117z - 0.3064}{(z-1)(z-0.65)(z-0.038)}$$

(b) **Closed-loop Transfer Function: *T* = 0.1 sec.**

$$\frac{\Omega_m(z)}{R(z)} = \frac{3.368z^2 + 1.7117z - 0.3074}{z^3 + 1.6805z^2 + 2.424z - 0.331}$$

Characteristic Equation:

$$z^3 + 1.6805z^2 + 2.424z - 0.331 = 0$$

Roots of Characteristic Equation:

$$z = 0.125, \quad -0.903 + j1.3545, \quad -0.903 - j1.3545$$

The complex roots are outside the unit circle $|z| = 1$, so the system is unstable.

(c) ***T* = 0.01 sec.**

Forward-path Transfer Function $$\frac{\Omega_m(z)}{E(z)} = \frac{0.04735z^2 + 0.0059743z - 0.03663}{z^3 - 2.6785z^2 + 2.3689z - 0.69032}$$

Closed-loop Transfer Function $$\frac{\Omega_m(z)}{R(z)} = \frac{0.047354z^2 + 0.0058753z - 0.03663}{z^3 - 2.6312z^2 + 2.3748z - 0.72695}$$

Characteristic Equation:

$$z^3 - 2.6312z^2 + 2.3748z - 0.72695 = 0$$

Characteristic Equation Roots:

$$z = 0.789, \quad 0.921 + j0.27, \quad 0.921 - j0.27$$

***T* = 0.001 sec.**

Forward-path Transfer Function

$$\frac{\Omega_m(z)}{E(z)} = \frac{-1.43\times10^{-6}z^3 + 4.98\times10^{-4}z^2 - 2.384\times10^{-7}z - 4.89\times10^{-4}}{z^3 - 2.9635z^2 + 2.9271z - 0.9636}$$

Closed-loop Transfer Function

$$\frac{\Omega_m(z)}{R(z)} = \frac{-1.43\times10^{-6}z^3 + 4.98\times10^{-4}z^2 - 2.384\times10^{-7}z - 4.89\times10^{-4}}{z^3 - 2.963z^2 + 2.9271z - 0.9641}$$

Characteristic Equation:

$$z^3 - 2.963z^2 + 2.9271z - 0.9641 = 0$$

Characteristic Equation Roots:

$$z = 0.99213, \quad 0.98543 + j0.02625, \quad 0.98543 - j0.02625$$

(d) **Error Constants:**

$$K_p^* = \lim_{z\to1} G_{ho}G_p(z) = \lim_{z\to1}\frac{3.368z^2 + 1.7117z - 0.3064}{(z-1)(z-0.65)(z-0.038)} = \infty$$

$$K_v^* = \frac{1}{T}\lim_{z\to1}\left[(z-1)^2 G_{ho}G_p(z)\right] = -\lim_{z\to1}\frac{3.368z^2 + 1.7117z - 0.3064}{(z-0.65)(z-0.038)} = \frac{14.177}{T}$$

$$K_a^* = \frac{1}{T^2}\lim_{z\to1}\left[(z-1)^2 G_{ho}G_p(z)\right] = \frac{1}{T^2}\lim_{z\to1}\frac{\left(3.368z^2 + 1.7117z - 0.3064\right)(z-1)}{(z-1)(z-0.038)} = 0$$

Steady-state Errors:

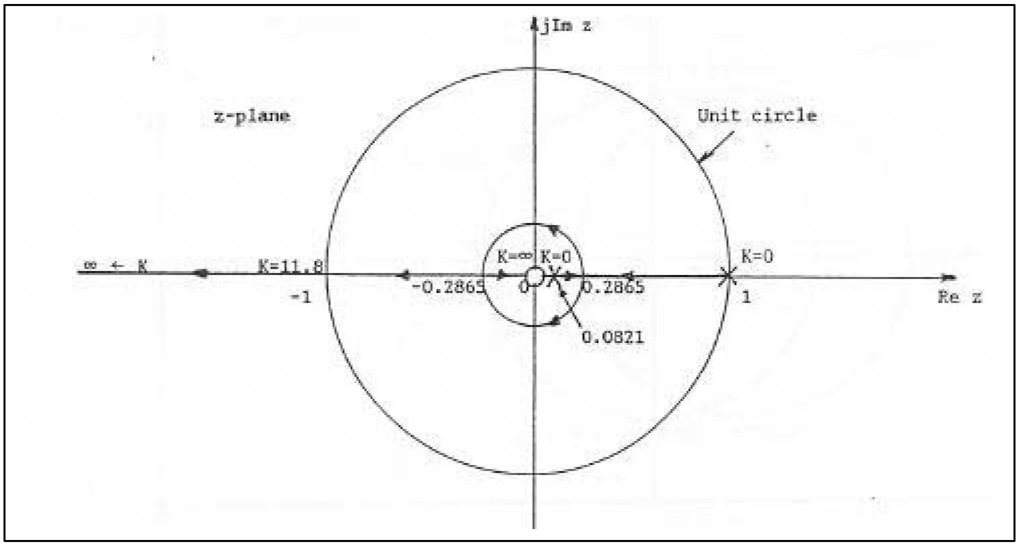
Step Input: $$e_{ss}^* = \frac{1}{1+K_p^*} = 0$$

Ramp Input: $$e_{ss}^* = \frac{1}{K_v^*} = \frac{T}{14.177}$$

Parabolic Input: $$e_{ss}^* = \frac{1}{K_a^*} = \infty$$

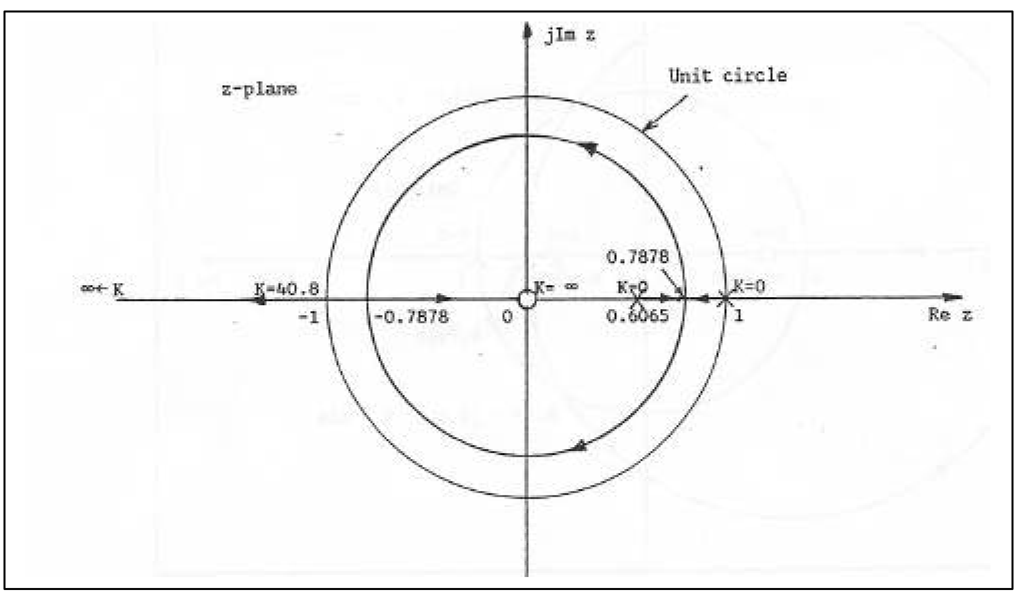
I-27 **(a)** **Forward-path Transfer Function: (no zero-order hold) T = 0.5 sec.**

$$G(z) = \frac{0.1836\,zK}{z^2 - 1.0821\,z + 0.0821} = \frac{0.1836\,zK}{(z-1)(z-0.0821)}$$



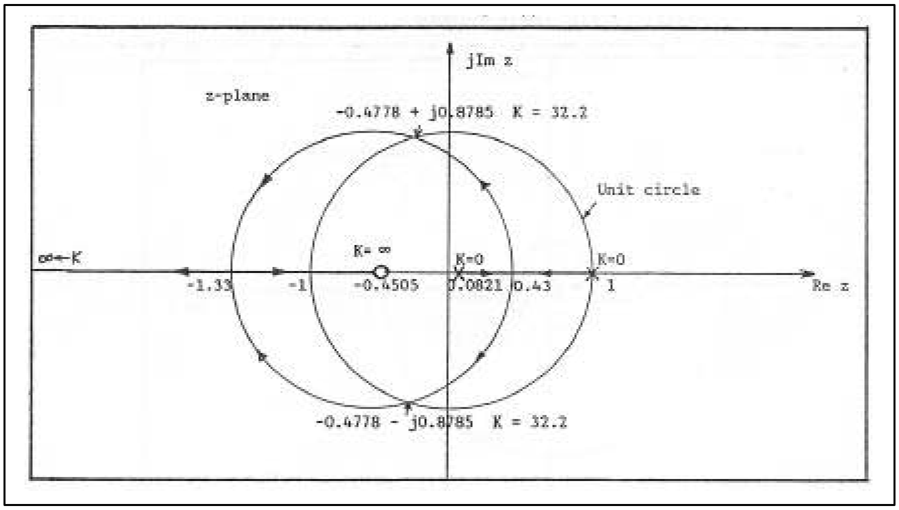
T = 0.1 sec.

$$G(z) = \frac{0.0787\,zK}{z^2 - 1.6065\,z + 0.6065} = \frac{0.0787\,zK}{(z-1)(z-0.6065)}$$



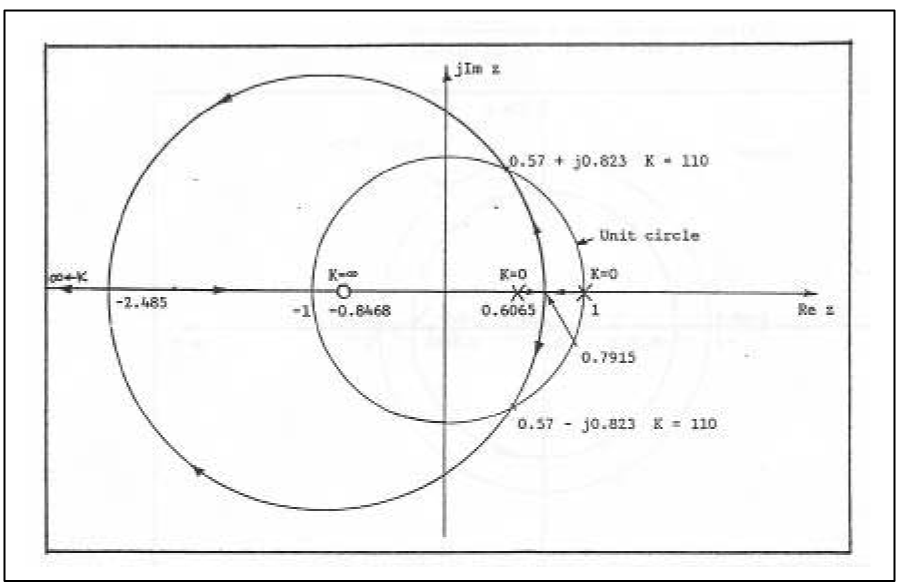
(b) **Open-loop Transfer Function: (with zero-order hold) T = 0.5 sec.**

$$G(z) = \frac{K(0.06328z + 0.02851)}{z^2 - 1.0821z + 0.08021} = \frac{0.06328K(z+0.4505)}{(z-1)(z-0.0821)}$$



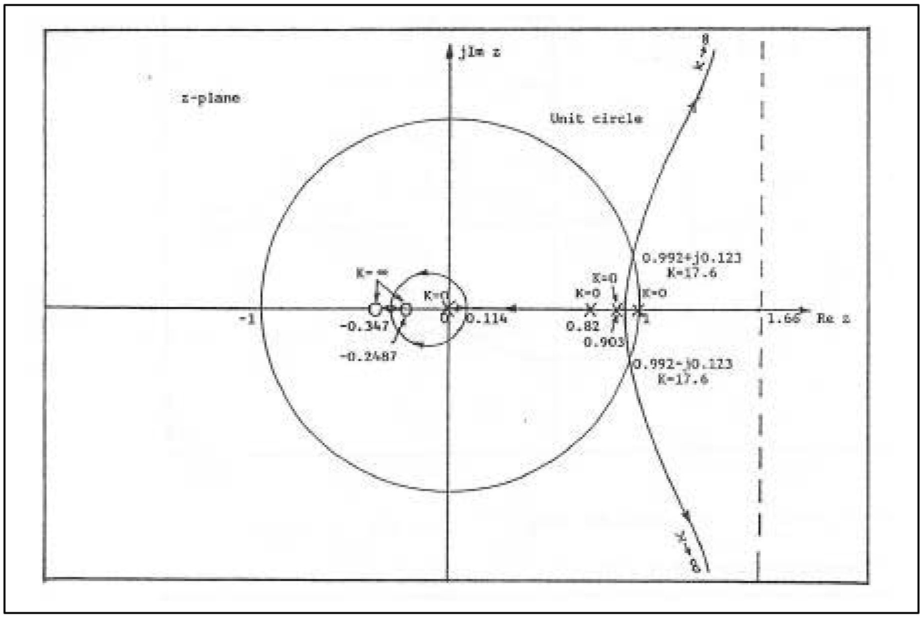
***T* = 0.1 sec.**

$$G(z) = \frac{K(0.00426z + 0.003608)}{z^2 - 1.6065z + 0.6065} = \frac{0.00426K(z+0.8468)}{(z-1)(z-0.6065)}$$



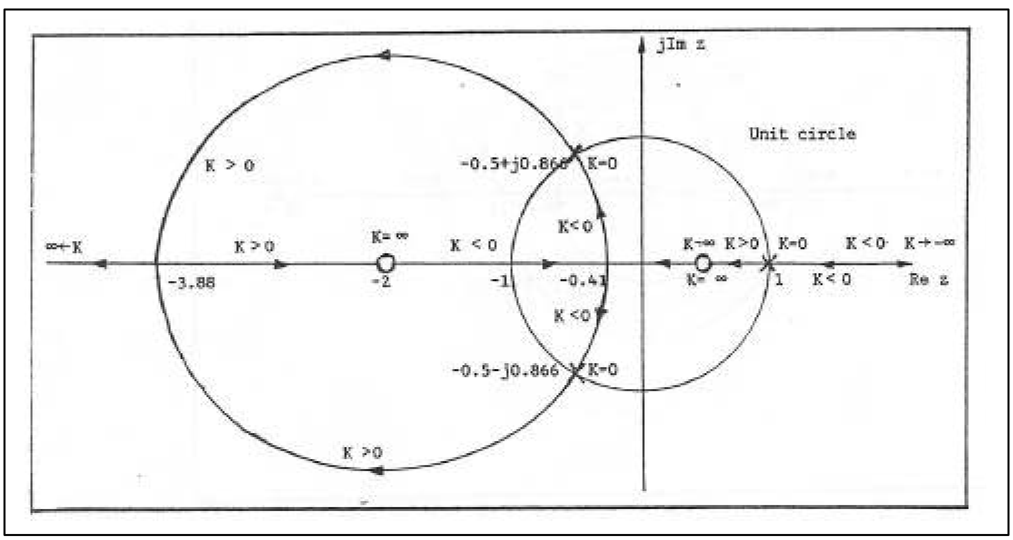
I-28 **Forward-path Transfer Function:**

$$G(z) = \frac{0.0001546K\left(z^2 + 3.7154z + 0.8622\right)}{z\left(z^3 - 2.7236z^2 + 2.4644z - 0.7408\right)}$$

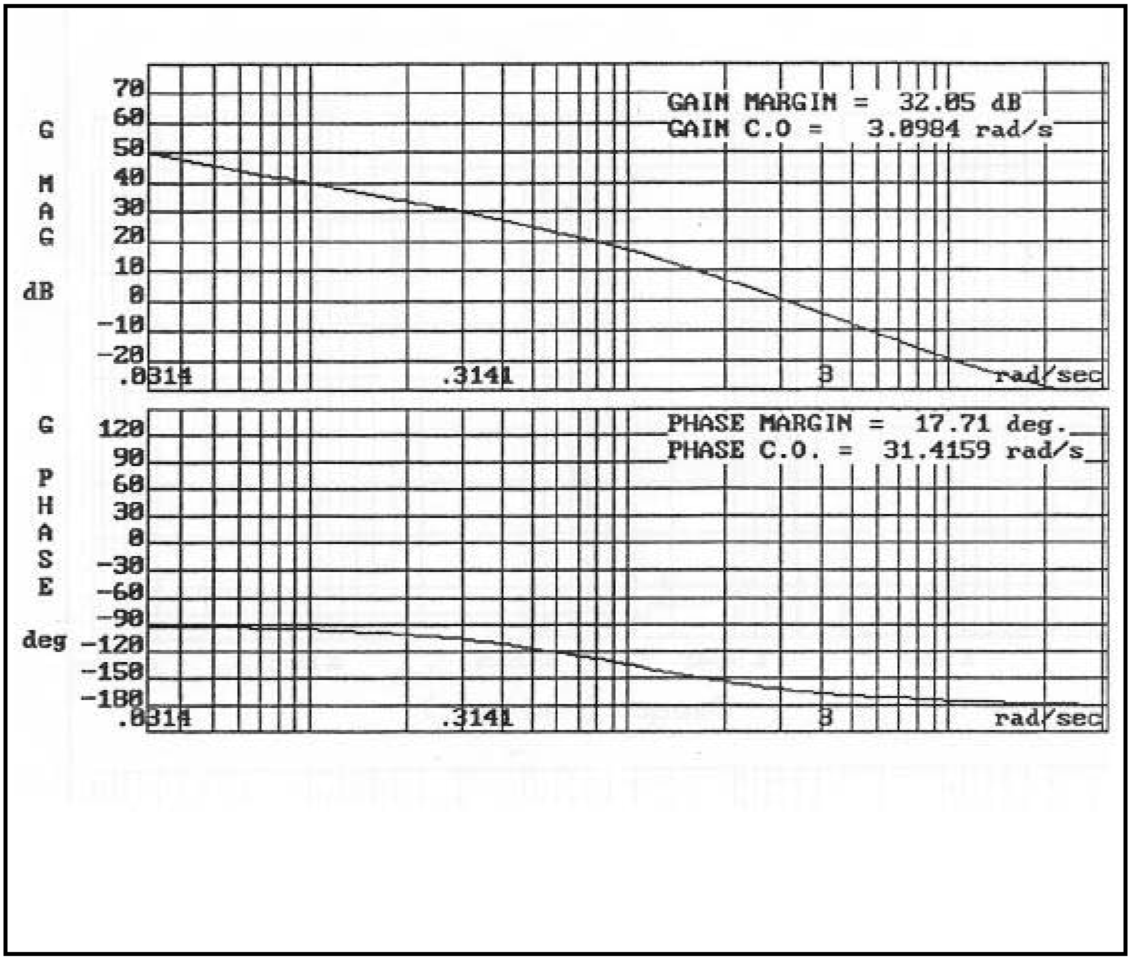


I-29 (a) $P(z) = z^3 - 1 \qquad Q(z) = z^2 + 1.5z - 1 = (z - 0.5)(z + 2)$

The system is unstable for all values of K.



I-30 **(a)** **Bode Plot:**



The system is stable.

(b) Apply *w*-transformation,

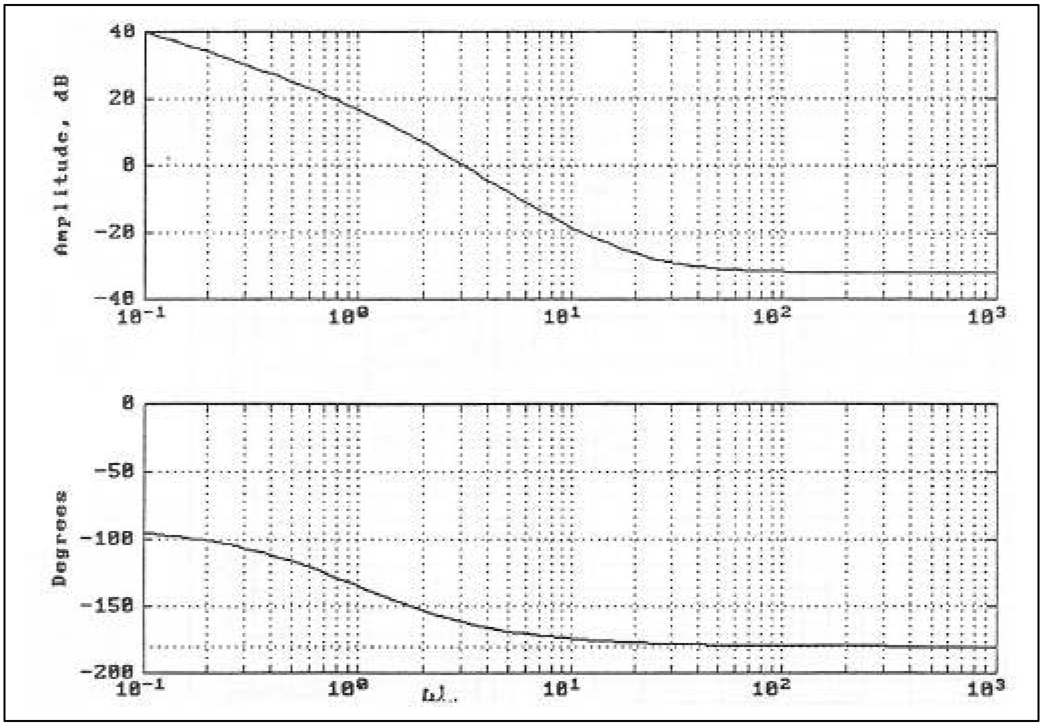
$$z = \frac{2 + wT}{2 - wT}$$

Then $$G_{ho}G(w) = G_{ho}G(z)\Big|_{z=\frac{2+wT}{2-wT}} = \frac{10(1 - 0.0025w^2)}{w(1+w)}$$

The Bode diagram of $G_{ho}G(w)$ is plotted as shown below.
The gain and phase margins are determined as follows:

GM = 32 dB PM = 17.7 deg.

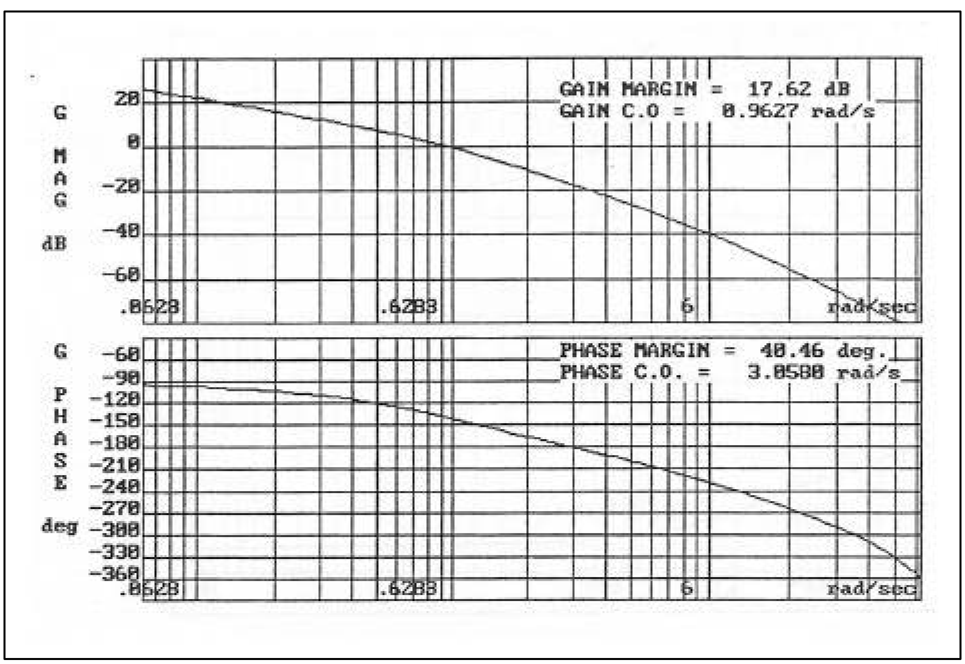
Bode plot of $G_{ho}G(w)$**:**



I-31 $G_{ho}G(z) = \mathcal{Z}\left[\frac{1-e^{-Ts}}{s}\left(\frac{16.67N}{s(s+1)(s+12.5)}\right)\right] = \frac{0.000295\left(z^2+3.39z+0.714\right)}{(z-1)(z-0.9486)(z-0.5354)}$

The Bode plot of $G_{ho}G(z)$ is plotted as follows. The gain margin is 17.62 dB, or 7.6. Thus selecting an integral value for N, the maximum number for N for a stable system is 7.

Bode Plot of $G_{ho}G(z)$



I-32 **(a)** $G_c(s) = 2 + \dfrac{200}{s}$

Backward-rectangular Integration Rule:

$$G_c(z) = 2 + \frac{200T}{z-1} = \frac{2z - 2 + 200T}{z-1} = \frac{2 + \left(200T - 2\right)z^{-1}}{1 - z^{-1}}$$

Forward-rectangular Integration Rule:

$$G_c(z) = 2 + \frac{200Tz}{z-1} = \frac{\left(2 + 200T\right)z - 2}{z-1} = \frac{\left(2 + 200T\right) - 2z^{-1}}{1 - z^{-1}}$$

Trapazoidal Integration Rule:

$$G_c(z) = 2 + \frac{200T\left(z+1\right)}{2\left(z-1\right)} = \frac{\left(4 + 200T\right)z + 200T - 2}{2\left(z-1\right)} = \frac{\left(4 + 200T\right) + \left(200T - 2\right)z^{-1}}{2\left(1 - z^{-1}\right)}$$

(b) $G_c(s) = 10 + 0.1s$

The controller transfer function does not have any integration term.
The differentiator is realized
by backward difference rule.

$$G_c(z) = 10 + \frac{0.1\left(z-1\right)}{Tz} = \frac{\left(10T + 0.1\right)z - 0.1}{z} = \left(10T + 0.1\right) - 0.1z^{-1}$$

(c) $G_c(s) = 1 + 0.2s + \dfrac{5}{s}$

Backward-rectangular Integration Rule:

$$G_c(z) = 1 + \frac{0.2\left(z-1\right)}{Tz} + \frac{5T}{\left(z-1\right)} = \frac{\left(T + 0.2\right) - 0.2z^{-1}}{T} + \frac{5Tz^{-1}}{1 - z^{-1}}$$

Forward-rectangular Integration Rule:

$$G_c(z) = 1 + \frac{0.2\left(z-1\right)}{Tz} + \frac{5Tz}{z-1} = \frac{\left(T + 0.2\right) - 0.2z^{-1}}{T} + \frac{5T}{1 - z^{-1}}$$

Trapezoidal Integration Rule:

$$G_c(z) = 1 + \frac{0.2\left(z-1\right)}{Tz} + \frac{5T\left(z+1\right)}{2\left(z-1\right)} = \frac{\left(T + 0.2\right) - 0.2z^{-1}}{T} + \frac{5T\left(1 + z^{-1}\right)}{2\left(1 - z^{-1}\right)}$$

I-33 **(a)** $G_c(s) = \dfrac{10}{s+12}$ $\quad T = 0.1$ sec

$$G_c(z) = \left(1 - z^{-1}\right) \mathcal{Z}\left[\frac{10}{s\left(s+12\right)}\right] = 0.8333\left(1 - z^{-1}\right)\mathcal{Z}\left(\frac{1}{s} - \frac{1}{s+12}\right)$$

$$= 0.8333\left(1 - z^{-1}\right)\left[\frac{z}{z-1} - \frac{z}{z - e^{-1.2}}\right] = \frac{0.5825}{z - 0.301}$$

(b) $G_c(s) = \dfrac{10\left(s+1.5\right)}{s+10}$ $\quad T = 1$ sec

$$G_c(z) = \left(1 - z^{-1}\right)\mathcal{Z}\left[\frac{10\left(s+1.5\right)}{s\left(s+10\right)}\right] = \left(1 - z^{-1}\right)\mathcal{Z}\left(\frac{1.5}{s} + \frac{8.5}{s+10}\right)$$

$$= \left(1 - z^{-1}\right)\left(\frac{1.5z}{z-1} + \frac{8.5}{z - e^{-1}}\right) = \frac{10\left(z - 0.9052\right)}{z - 0.368}$$

(c) $G_c(s) = \dfrac{s}{s+1.55}$ $\quad T = 0.1$ sec

$$G_c(z) = \left(1 - z^{-1}\right)\mathcal{Z}\left(\frac{1}{s+1.55}\right) = \left(1 - z^{-1}\right)\left(\frac{z}{z - e^{-0.155}}\right) = \frac{z-1}{z - 0.8564}$$

(d) $G_c(z) = \dfrac{1 + 0.4s}{1 + 0.01s}$

$$G_c(z) = \left(1 - z^{-1}\right)\mathcal{Z}\left[\frac{1 + 0.4s}{s\left(1 + 0.01s\right)}\right] = 40\left(1 - z^{-1}\right)\mathcal{Z}\left[\frac{0.025z}{z-1} + \frac{0.975z}{z - e^{-10}}\right] = 40\left(\frac{z - 0.975}{z - 0.0000454}\right)$$

I-34 **(a)** Not physically realizable, since according to the form of Eq. (11-18), $b_0 \neq 0$ but $a_0 = 0$.

(b) Physically realizable.

(c) Physically realizable.

(d) Physically realizable.

(e) Not physically realizable, since the leading term is 0.1*z*.

(f) Physically realizable.

I-35 **(a)** $G_c(s) = 1 + 10s$ $\quad K_P = 1 \quad K_D = 10$ $\quad$ Thus $\quad G_c(z) = \dfrac{\left(T+10\right)z - 10}{Tz}$

$$G_{ho}G_p(z) = \left(1 - z^{-1}\right) \mathcal{Z}\left(\frac{4}{s^3}\right) = \frac{2T^2\left(z+1\right)}{\left(z-1\right)^2}$$

$$G(z) = G_c(z)G_{ho}G_p(z) = \frac{2T\left(z+1\right)\left[\left(T+10\right)z-10\right]}{z\left(z-1\right)^2}$$

By trial and error, when $T = 0.01$ sec, the maximum overshoot of $y(kT)$ is less than 1 percent. When $T = 0.01$ sec,

$$G(z) = \frac{0.02\left(z+1\right)\left(10.01z-10\right)}{z\left(z-1\right)^2} \qquad \frac{Y(z)}{R(z)} = \frac{0.02\left(z+1\right)\left(10.01z-10\right)}{z^3 - 1.7998z^2 + 1.0002z - 0.2}$$

When the input is a unit-step function, the output response $y(kT)$ is computed and tabulated in the following for 40 sampling periods. The maximum overshoot is 0.68%, and the final value is 1.

Sampling Periods k	$y(kT)$				
1	0.0000E+00	15	1.0025E+00	29	1.0025E+00
2	2.0020E-01	16	1.0025E+00	30	1.0025E+00
3	5.6072E-01	17	1.0025E+00	31	1.0025E+00
4	8.0934E-01	18	1.0025E+00	32	1.0025E+00
5	9.3626E-01	19	1.0025E+00	33	1.0025E+00
6	9.8813E-01	20	1.0025E+00	34	1.0025E+00
7	1.0042E+00	21	1.0025E+00	35	1.0025E+00
8	1.0068E+00	22	1.0025E+00	36	1.0025E+00
9	1.0056E+00	23	1.0025E+00	37	1.0025E+00
10	1.0041E+00	24	1.0025E+00	38	1.0025E+00
11	1.0032E+00	25	1.0025E+00	39	1.0025E+00
12	1.0027E+00	26	1.0025E+00	40	1.0025E+00
13	1.0025E+00	27	1.0025E+00		
14	1.0025E+00	28	1.0025E+00		

I-36 **(a)** $G_{ho}G_p(z) = \left(1 - z^{-1}\right) \mathcal{Z}\left(\frac{1}{s^3}\right) = \frac{2T^2\left(z+1\right)}{\left(z-1\right)^2}$

$$G(z) = G_c(z)G_{ho}G_p(z) = \frac{K_PTz + K_D\left(z-1\right)}{Tz}\frac{2T^2\left(z+1\right)}{\left(z-1\right)^2}$$

$$= \frac{2T\left[\left(K_PT + K_D\right)z^2 + K_PTz - K_D\right]}{z\left(z-1\right)^2}$$

Characteristic Equation: $z^3 + 2\left(K_PT^2 + K_DT - 1\right)z^2 + \left(2K_PT^2 + 1\right)z - 2K_DT = 0$

For two roots to be at $z = 0.5$ and 0.5, the characteristic equation should have $z^2 - z + 0.25$ as a factor. Dividing the characteristic equation by $z^2 - z + 0.25$ and solving for zero remainder, we get

$$4K_P T^2 + 2K_D T - 0.25 = 0 \quad \text{and} \quad -0.5K_P T^2 - 2.5K_D T + 0.25 = 0$$

Solving for K_P and K_D from these two equations, we have

$$K_P = \frac{0.0139}{T^2} \qquad K_D = \frac{0.0972}{T}$$

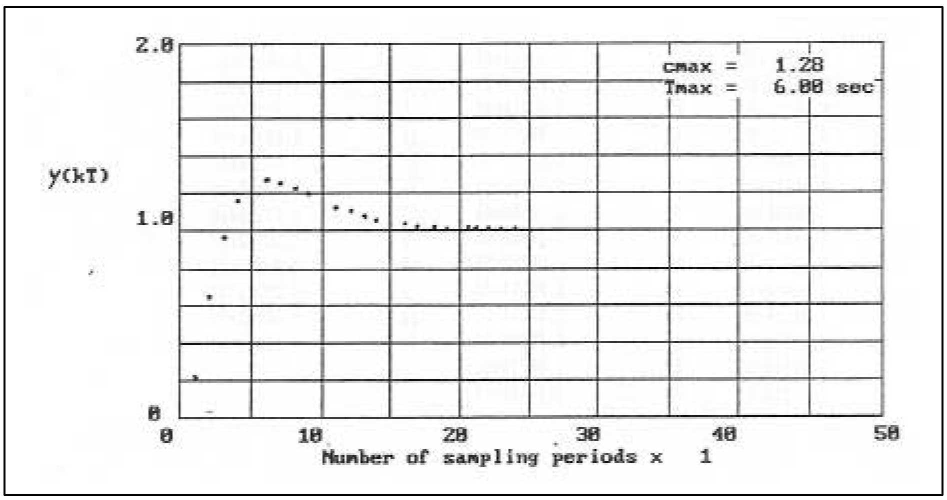
The third root is at $z = 1 - 2K_P T^2 - 2K_D T = 0.7778$ for $T = 0.01$ sec

The forward-path transfer function is

$$G(z) = \frac{0.2222(z+1)(z-0.8749)}{z(z-1)^2}$$

$$\frac{Y(z)}{R(z)} = \frac{0.2222z^2 + 0.0278z - 0.1944}{z^3 - 1.7778z^2 + 1.0278z - 0.1944}$$

Unit-step Response:



(b) **(b)** $K_P = 1$, $T = 0.01$ **sec**

$$G(z) = G_c(z)G_{ho}G_p(z) = \frac{2T\left[(T+K_D)z^2 + Tz - K_D\right]}{z(z-1)^2}$$

$$= \frac{0.02\left[(0.01+K_D)z^2 + 0.01z - K_D\right]}{z(z-1)^2}$$

The unit-step response of the system is computed for various values of K_D. The results are tabulated below to show the values of the maximum overshoot.

K_D	1.0	5.0	6.0	7.0	8.0	9.0	9.1	9.3	9.5	10.0
Max overshoot (%)	14	0.9	0.67	0.5	0.38	0.31	0.31	0.32	0.37	0.68

I-37 (a) Phase-lead Controller Design:

$$G(z) = G_{ho}G_p(z) = \left(1 - z^{-1}\right)\mathcal{Z}\left(\frac{4}{s^3}\right) = \frac{0.02\left(z+1\right)}{z\left(z-1\right)^2} \qquad T = 0.1 \text{ sec}$$

Closed-loop Transfer Function:

$$\frac{Y(z)}{R(z)} = \frac{G_{ho}G_p(z)}{1 + G_{ho}G_p(z)} = \frac{0.02\left(z+1\right)}{z^2 - 1.98z + 1.02}$$

The system is unstable.

With the w-transformation, $z = \dfrac{\dfrac{2}{T} + w}{\dfrac{2}{T} - w} = \dfrac{20 + w}{20 - w}$ $\qquad G(w) = \dfrac{4\left(1 - 0.05w\right)}{w^2}$

From the Bode plot of $G(jw_w)$ the phase margin is found to be –5.73 degrees.
For a phase margin of 60 degrees, the phase-lead controller is

$$G_c(w) = \frac{1 + aTw}{1 + Tw} = \frac{1 + 1.4286\, w}{1 + 0.0197\, w}$$

The Bode plot is show below. The frequency-domain characteristics are:

PM = 60 deg GM = 10.76 dB $M_r = 1.114$

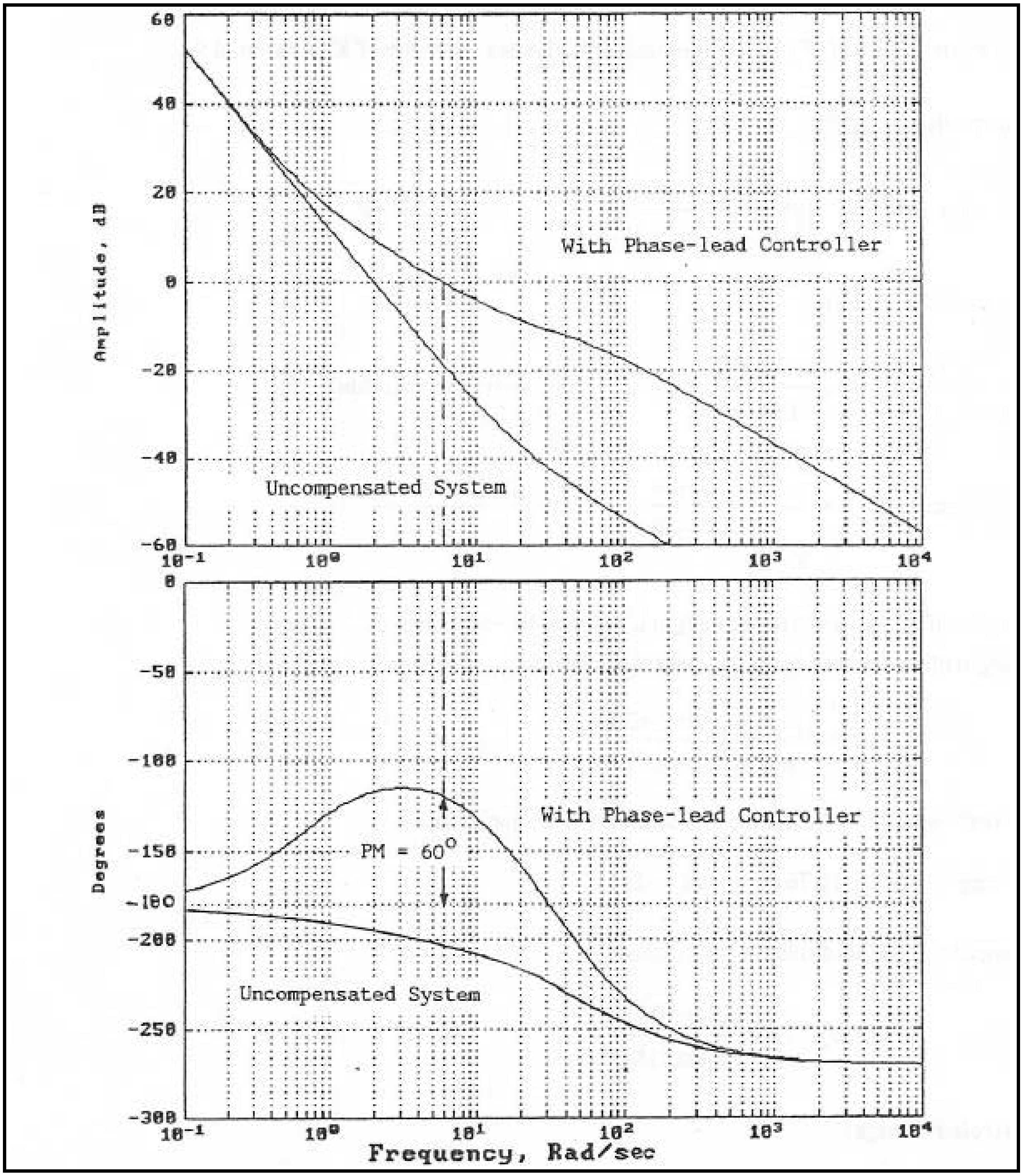
The transfer function of the controller in the z-domain is

$$G_c(z) = \frac{21.21\left(z - 0.9222\right)}{\left(z + 0.4344\right)}$$

(b) Phase-lag Controller Design:

Since the phase curve of the Bode plot of $G(jw_w)$ is always below –180 degrees, we cannot design a phase-lag controller for this system in the usual manner.

Bode Plots for Part (a):

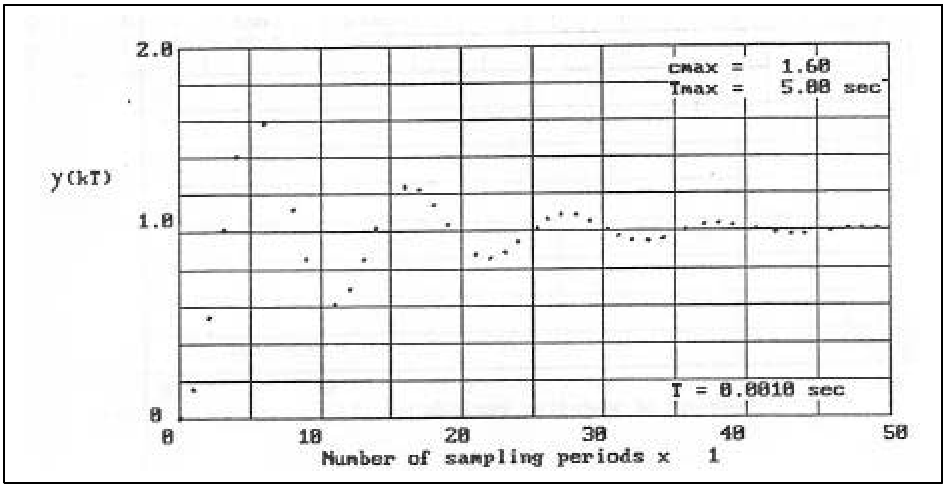


$$G_{ho}G_p(z) = \left(1 - z^{-1}\right)\mathcal{Z}\left[\frac{4500K}{s^2(s+361.2)}\right] = \frac{K(0.002008z + 0.001775)}{(z-1)(z-0.697)}$$

I-38 **(a)** **Forward-path Transfer Function:**

$$K_v^* = \frac{1}{T}\lim_{z\to 1}\left[(z-1)\,G_{ho}G_p(z)\right] = \lim_{z\to 1}\frac{K(2.008z+1.775)}{z-0.697} = 1000 \qquad \text{Thus} \quad K_v^* = 80.1$$

(b) **Unit-step Response:**



Maximum overshoot = 60 percent.

(c) **Deadbeat-response Controller Design: (*K* = 80.1)**

$$G_{ho}G_p(z) = \frac{0.16034z + 0.14217}{(z-1)(z-0.697)}$$

$$G_{ho}G_p(z^{-1}) = \frac{Q(z^{-1})}{P(z^{-1})} = \frac{0.16034\,z^{-1} + 0.14217\,z^{-2}}{1 - 1.697\,z^{-1} + 0.697\,z^{-2}} \qquad Q(1) = 0.3025$$

Digital Controller:

$$G_c(z) = \frac{P(z^{-1})}{Q(1) - Q(z^{-1})} = \frac{1 - 1.697z^{-1} + 0.697z^{-2}}{0.3025 - 0.16034z^{-1} - 0.14217z^{-2}}$$

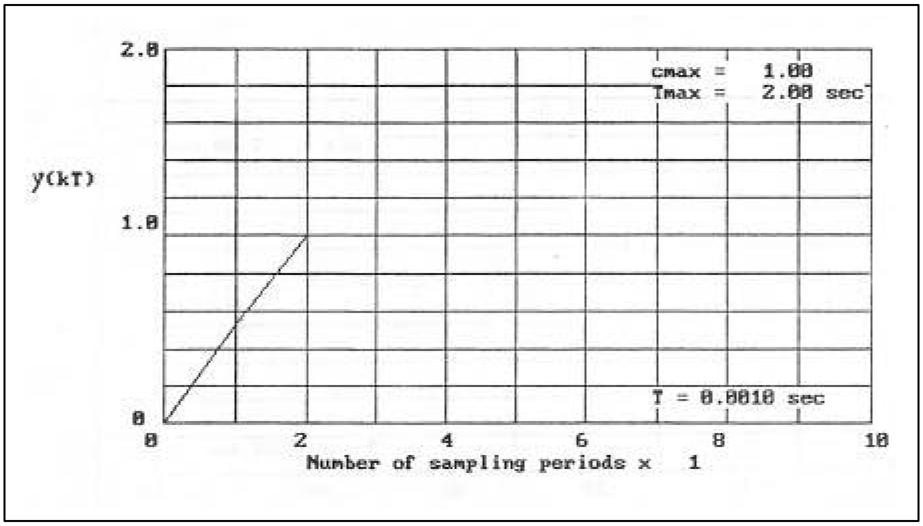
$$= \frac{3.3057(z-1)(z-0.697)}{z^2 - 0.53z - 0.47}$$

Forward-path transfer function: $$G(z) = G_c(z)G_{ho}G_p(z) = \frac{3.3057(z-1)(z-0.697)}{z^2 - 0.53z - 0.47}$$

Closed-loop system transfer function: $$M(z) = \frac{0.53z + 0.47}{z^2}$$

Unit-step response: $$Y(z) = 0.53z^{-1} + z^{-2} + z^{-3} + \cdots$$

Deadbeat Response:



I-39 $G_p(s) = \dfrac{2500}{s(s+25)}$ $\qquad T = 0.05$ sec

$$G_{ho}G(z) = \left(1 - z^{-1}\right)\mathcal{Z}\left(\frac{2500}{s^2(s+25)}\right) = \frac{2.146z + 1.4215}{(z-1)(z-0.2865)}$$

$$G_{ho}G\left(z^{-1}\right) = \frac{Q\left(z^{-1}\right)}{P\left(z^{-1}\right)} = \frac{2.146z^{-1} + 1.4215z^{-2}}{1 - 1.2865z^{-1} + 0.2865z^{-2}} \qquad Q(1) = 3.5675$$

Deadbeat Response Controller Transfer Function:

$$G_c\left(z^{-1}\right) = \frac{P\left(z^{-1}\right)}{Q(1) - Q\left(z^{-1}\right)} = \frac{1 - 2.865z^{-1} + 0.2865z^{-2}}{3.5675 - 2.146z^{-1} - 1.4215z^{-2}} \qquad G_c(z) = \frac{(z-1)(z-0.285)}{3.5675z^2 - 2.146z - 1.4215}$$

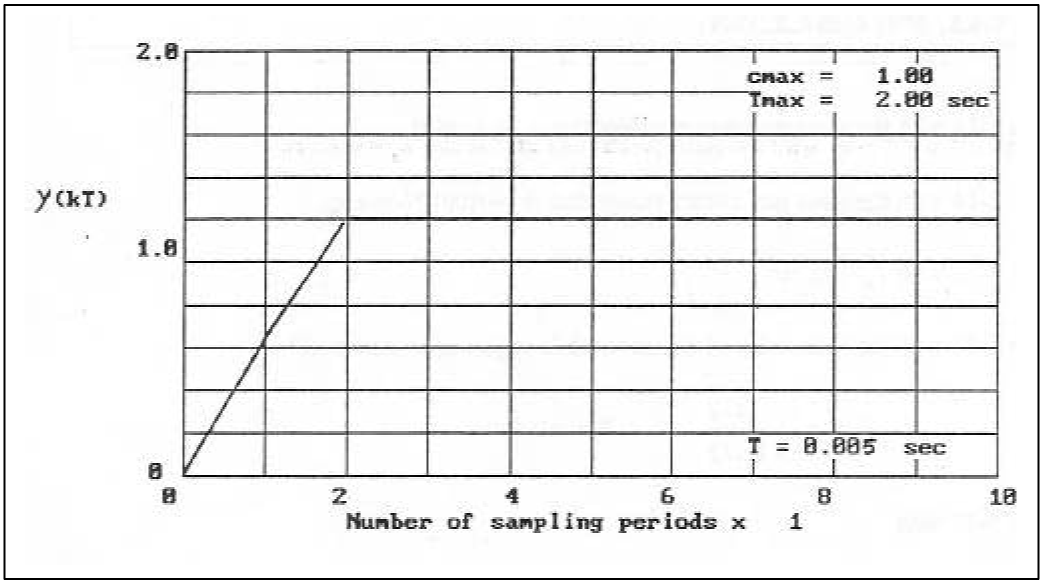
Forward-path Transfer Function:

$$G(z) = G_c(z)G_{ho}G_p(z) = \frac{2.146z + 1.4215}{3.5675z^2 - 2.146z - 1.4215}$$

Closed-loop System Transfer Function:

$$M(z) = \frac{0.6015z + 0.3985}{z^2}$$

Unit-step response: $\qquad Y(z) = 1 + 0.6015z^{-1} + z^{-2} + z^{-3} + \cdots$



I-40 The characteristic equation is

$$z^2 + (-1.7788 + 0.1152k_1 + 22.12k_2)z + 0.7788 + 4.8032k_1 - 22.12k_2 = 0$$

For the characteristic equation roots to be at 0.5 and 0.5, the equation should be

$$z^2 - z + 0.25 = 0$$

Equating like coefficients in the last two equations, we have

$$-1.7788 + 0.1152k_1 + 22.12k_2 = -1$$

$$0.7788 + 4.8032k_1 - 22.12k_2 = 0.25$$

Solving for the value of k_1 and k_2 from the last two equations, we have $k_1 = 0.058$ and $k_2 = 0.035$.